Engineering Preview

AN INTRODUCTION TO ENGINEERING INCLUDING THE NECESSARY REVIEW OF SCIENCE AND MATHEMATICS

L. E. GRINTER — HARRY N. HOLMES — H. C. SPENCER
RUFUS OLDENBURGER — CHARLES HARRIS
R. G. KLOEFFLER — V. M. FAIRES

1947
THE MACMILLAN COMPANY — NEW YORK

PRINTED IN THE UNITED STATES OF AMERICA

Reprinted January 1947

PREFACE

Engineering Preview is designed to open the field of engineering udy to anyone who has the determination to master the subject. ne book can be studied most effectively in the senior year of gh school or the freshman year of college, at which time the reparation of the student in mathematics is still fresh. However, nce the minimum essentials of mathematics are included as one ection of the book, those who have been away from formal contact with mathematics for several years may study the book profitbly by giving particular attention to the refresher section on athematics.

The sections of this book treat the background sciences of en-,ineering which are mathematics, chemistry and physics; the two anguages of engineering which are mathematics and technical rawing; the three tools for making quantitative calculations which re mathematics, technical drawing, and the use of the slide rule; and some of the basic engineering applications of physics such as illumination, electric power, electronics, mechanics, heat engines, and refrigeration. The applications in these latter sections are from the engineering point of view rather than that of the physicist.

Engineering Preview is not a collection of unrelated chapters but it is an organized presentation of background material needed for the ultimate study of any specialized field of engineering. Even in the chapters on science, the problems and examples are drawn wherever possible from engineering practice. Since each author has written widely in his own field, the book brings the mature experience of seven technical writers together to bear upon the reader's problem of developing a proper background for professional engineering study. It represents the best available technical advice on the subject of what to study and how to study it with he objective of preparation for engineering practice.

The book does not, of course, complete any subject discussed. It takes four years of college study to produce an engineer. No doubt the reader will notice that a part of the material recalls ome of his high school work and he will find the rest of it treated

again in somewhat different form in college courses. This is as it should be. We learn by repetition and by seeing a subject presented from more than one point of view. For example, the study of forces may be brought up in a dozen college courses in mechanical engineering. Only by such repetition can we be certain that the mechanical engineer will develop a complete understanding of forces and of their effects upon structures and machines.

Engineering Preview serves to orient the reader in the fields of science and engineering. Almost every person has some question as to his bent in life. An effective way to learn whether you really should become an engineer or scientist is to read and to work simple problems in these fields. If you find this book tiresome and confusing, it is unlikely that you would want to become a technologist; but if it seems clear and stimulating, you are probably destined to use a slide rule. The first section of this book is devoted entirely to orientation, and it even includes a simple self-scoring test giving you a rough inventory of your technological traits. A careful reading of this section will help you decide for or against technology as a career and will lift into focus for you the many subdivisions of science and engineering.

The authors do not recommend this book for a rapid survey unless the reader has covered much of the material once before and wishes to scan it as a review. Probably the equivalent material could not be found elsewhere in less than a half-dozen volumes, so deliberate attention to details and to the solution of problems is recommended. However, it is surprising how fast one progresses if a full hour each day is set aside for study. Such preparation may well change the study of a specialized engineering curriculum from a difficult task to a pleasing experience.

It is a pleasure to acknowledge the assistance of Professor Walter Peterhans who selected many of the photographs and of Professor L. Hilberseimer who made the designs and drawings for the endpapers. The excellence of the drawings is due largely to the careful supervision and personal work of Professor H. C. Spencer. A few illustrations are from the authors' previous books: *Technical Drawing* by Giesecke, Mitchell, and Spencer; *General Chemistry* by Harry Holmes; *Applied Thermodynamics* by Virgil Faires, all published by Macmillan; and *Principles of Electronics* by R. G. Kloeffler, published by John Wiley & Sons.

L. E. GRINTER

THE AUTHORS

LINTON E. GRINTER, Ph.D., C.E., Vice President and Dean of the Graduate School, Illinois Institute of Technology. Author of *Theory of Modern Steel Structures*, Macmillan, 1936; *Automatic Design of Continuous Frames in Steel and Reinforced Concrete*, Macmillan, 1939; *Design of Modern Steel Structures*, Macmillan, 1941; *Elementary Structural Analysis and Design*, Macmillan, 1942. Author of the section on Outlines in Science and Engineering in *Engineering Preview*.

HARRY N. HOLMES, Ph.D., LL.D., Head of the Severance Chemical Laboratory at Oberlin College. Author of the following books: *Introductory College Chemistry*, Macmillan, 1939; *Introductory Colloid Chemistry*, Wiley, 1934; *General Chemistry*, Macmillan, 1941; *Out of a Test Tube*, Emerson Books, 1943; *Strategic Materials and National Strength*, Macmillan, 1942. Author of the section on Chemistry in *Engineering Preview*.

HENRY CECIL SPENCER, A.B., M.S., Chairman of the Department of Technical Drawing, Illinois Institute of Technology. Co-author of the following books: *Technical Drawing for High Schools*, Macmillan, 1936; *Technical Drawing Problems*, Macmillan, 1936; *Lettering Exercises*, Macmillan, 1939; *Technical Drawing*, Macmillan, 1940. Author of the section on Technical Drawing in *Engineering Preview*.

RUFUS OLDENBURGER, A.B., Ph.D., Professor of Mathematics at Illinois Institute of Technology. Author of numerous papers in the mathematical journals and author of the section on Mathematics in *Engineering Preview*.

CHARLES O. HARRIS, B.S., D.Sc., Associate Professor of Mechanics at Illinois Institute of Technology. Author of *Slide Rule Simplified*, American Technical Society, 1943. Author of the section on The Slide Rule in *Engineering Preview*.

ROYCE G. KLOEFFLER, B.S., S.M., Head of the Department of Electrical Engineering at Kansas State College. Co-author of *Direct Current Machinery*, Macmillan, 1934; author of *Telephone Communication Systems*, Macmillan, 1925; *Principles of Electronics*, Wiley, 1942. Author of the sections on Illumination, Electricity and Electronics in *Engineering Preview*.

VIRGIL M. FAIRES, M.S., M.E., Professor of Mechanical Engineering and Head of the Department of Management Engineering at the A. & M. College of Texas. Author of *Applied Thermodynamics*, Macmillan, 1938; *Elementary Thermodynamics*, Macmillan, 1938; *Design of Machine Elements*, Macmillan, 1941; co-author of *Mechanism*, McGraw-Hill, 1939, and *Analytic Mechanics*, Macmillan, 1943. Author of the sections on Mechanics and Thermodynamics in *Engineering Preview*.

CONTENTS

Engineering Preview

Engineers and scientists devote their lives to improving the physical well-being of man. From the field engineer, every report shows how some difficulty of construction or operation was overcome. Each product of the designer is somehow better than the old. Through research new methods, new products, new materials are sought and found. Greater production at lower cost is the industrial engineer's creed. A literature filling whole libraries has been required to record our scientific advancement. With clarity, the engineer sees the changes to come.

Together from field and factory we proclaim progress for the new world.

Outlines in Science and Engineering

Reading and writing

It is a wise precaution for any author to think a bit about the reader before he starts to write. We believe that you will be a hardy reader rather than the "gentle" type. You are probably interested in science and engineering either because you expect to become an engineer or scientist or because you really want to know what makes this technological world revolve around you. In either instance we authors should be able to depend upon your fortitude which will be needed when the digging gets down past the soft topsoil into the hardpan beneath. But, truly, we expect to use simple language and not to strain your resolution unduly. It is said of good engineers that they are not unusually brilliant people but that they are tough-fibered and mentally tenacious. If you had lacked those characteristics completely, you would not likely have become interested in this book, nor would we be interested in inducing you to read it.

Science refresher

Engineering is applied science. Technology is a more impressive word with about the same meaning. No one needs to be told that he must learn some science before he can apply it. You have probably studied mathematics and perhaps some chemistry or physics in high school, and you will restudy all of these subjects in college or at home by "burning the midnight oil" if you want to become an engineer. But there is an in-between field of refreshing whatever science you may have studied once before and of advancing a little way into the modern concepts of science and engineering that this book intends to cover. Such a book could be made quite imposing by using long technical words and longer sentences. However, like the preacher who admitted that he had probably never saved any souls after the first twenty minutes of his sermon, we doubt our ability to create many engineers by that method. We are, therefore, inclined here to rely upon the dignity of science itself to impress the reader and to let the words fall where they may.

Technological evolution

Year by year, a continually increasing percentage of the populace becomes intimately associated with engineering developments. As we look back, it is easy to trace the growth of popular awareness

of the engineer and his works. During the twenties, our everyday life gradually became associated with the automobile, the radio, and the electric refrigerator. In the thirties, we found that we were face to face with the economic necessity of invention and new technological developments to keep our industries alive. Finally, when the plastics industry loomed over the horizon, it was greeted with great enthusiasm, as much because of its economic significance as because of its contribution to our life and culture. Beginning with the forties, we entered a totally technological war and found our very existence depending upon the speed of our technical progress. With sweethearts, wives, and mothers running machines, making blueprints, and driving rivets, it soon became trite to say "this has become a technological world." We realized, at last, that it had been a technological world for a generation and that the engineer had become important to our way of life.

Culture and technology

A great deal of time has been wasted in discussions of the relative merits of a "cultural" versus a "technological" education. Actually, any education may be cultural, but only a few students in any field obtain a cultural education. The mere study of languages, literature, and history is no assurance that the student has achieved culture. But neither is it true that specialization in one branch of engineering alone will produce a cultured person. Culture is a deep understanding of man and his environment. At the time of Christ, culture was achieved through a casual knowledge of history, local geography, the Mediterranean languages, and the religious and social customs of the people. Of limited usefulness was any knowledge of mathematics, and science was mainly philosophical speculation. Law, philosophy and art, reaching high plateaus, contributed the most enduring influence.

Depth of culture

The situation now has changed so greatly that a cultured man of the Roman Empire would appear pathetically unlearned in many common fields of knowledge today. Without a reasonable understanding of mathematics, science, and technology — an understanding appreciation of their influence now and probable future influence upon man — how can anyone be truly cultured? Then, it seems that one other requirement of culture is often overlooked. No one can develop real understanding in many fields unless he is a thorough student of one. In fact, any deeply cultured person will have been interested in making some contribution to man's ultimate spiritual, social, intellectual, or physical welfare.

SCIENTIST, ENGINEER, OR TECHNICIAN

When you find that your inner interests direct or lead you into technical fields — that is, when you get a thrill out of machinery and blueprints and laboratories — the question still should be raised: Are you to educate yourself as a technician, an engineer, or a scientist? No one can answer this question but yourself although there is some help available through placement examinations that will be mentioned later. However, if a man wants anything strongly enough, he often finds ways of getting around a lack of natural ability. Determination almost always wins. Lack of determination means failure in engineering or in anything else.

Technician

But, within reason, it is wise to aim at the field for which your aptitudes best fit you. Several times as many technicians are needed by industry as engineers, and several times as many engineers are needed as scientists. A technician is commonly one who conditions complicated machinery or who acts as troubleshooter or detail draftsman or laboratory assistant. He usually works under an engineer or scientist or production superintendent. He does not need much knowledge of mathematics or of basic science. His knowledge is practical and is composed of "how to do something" rather than "why it is done." The scientific information in this book might be adequate for the technician while it is just an introduction for the engineer.

Spectrum analogy

The scientist is usually looked upon as more theoretically inclined than the engineer, and he is often a better mathematician. If we think of the spectrum as symbolizing the whole field of technology, the red, yellow, and blue primary colors with their secondary colors of orange, green, and violet might be compared to six major fields of engineering (civil, mechanical, electrical, chemical, industrial, and aeronautical). From one end of the visible spectrum extends the well-known infrared or heat band which we will think of as analogous to the broad field of the technician's work, while from the other end of the spectrum we step off into the cold but strangely dazzling field of the ultraviolet or black light radiation in which the scientist primarily operates.

Engineer

The analogy may be carried further. The warm belt of technician service is a comfortable, satisfying, steady state of life for which reliability is more important than imagination. It has many compensations for its workers. The color bands of engineering also

are stable and in their fundamental aspects unchanging, but they are brilliant and challenging to the imagination. They can be mixed to produce new shades or tints of color — new fields of technological accomplishment. The possibilities for new creations in engineering technology are as limitless as the paintings that may come from the artist's brush; we both have the entire spectrum at our command.

Scientist

Finally, science is much like the black light of the ultraviolet radiation which is the source of fluorescence. Impartially, yes coldly analytical, the scientist works largely with the unknown. Nonetheless, there are times when the invisible light of his scientifically trained mind falls upon a fluorescent situation and suddenly there lights up before him a scene of startling beauty, his own intellectual creation. The scientist and the research engineer will often experience this thrill of personal creation; meaningless to some, it is the whole of life to others.

Your career

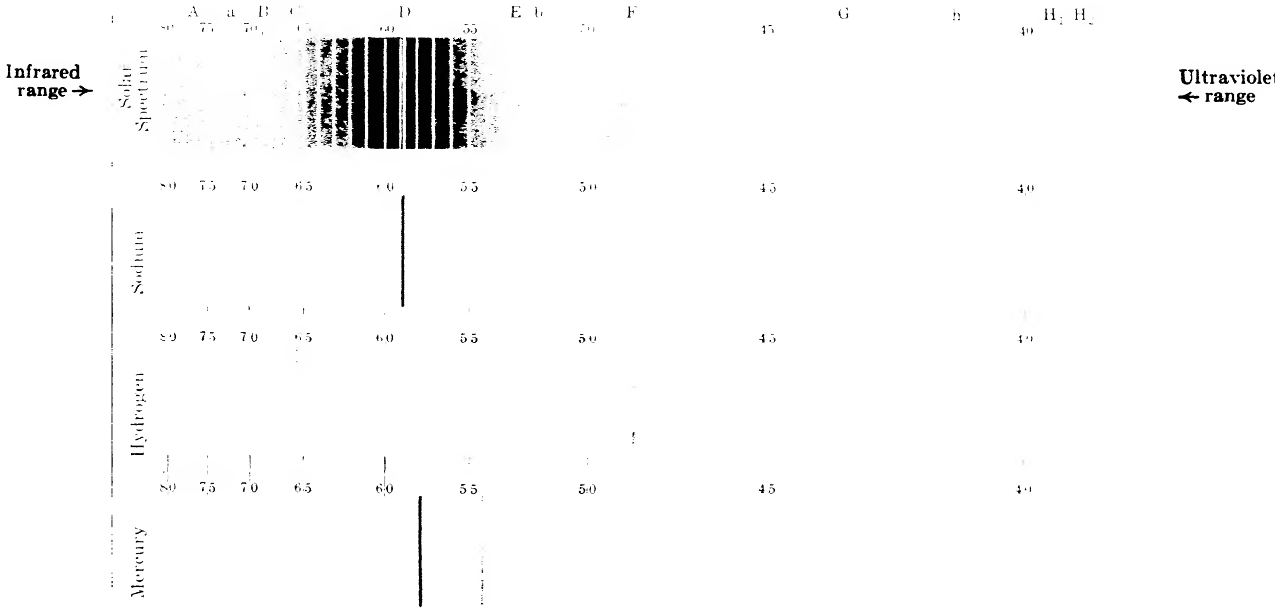
It is your career that you would build and so you must choose it for yourself. If my analogy will help you, go over it again:

Warmth	→	Light	→	Fluorescence
Technician	→	*Engineer*	→	*Scientist*
Comfort	→	Vision	→	Creation
Worker	→	Supervisor	→	Researcher
Freedom of mind		Mental responsibility		Incessant mental activity
Respect of associates		Recognition for accomplishments		Possible public applause

Of course, each classification flows into the adjacent ones. Still, if we pull together all of the corresponding word descriptions, we find for the technician a comfortable worker's job, with freedom from heavy responsibility or mental worry and with the certain respect of his associates. The engineer has a job requiring the light of clear vision, usually carrying the inevitable responsibility of the supervisor either for success or failure and offering the reward of probable recognition for accomplishments. The scientist's job is the brilliant one of creating new things through research which requires a particular kind of mental activity and imagination and offers the uncertain reward of possible public recognition reserved for statesmen, military heroes, and occasionally scientists.

Invention

At all levels of technology there is opportunity for invention. The simplest gadget may still be improved. Look for improvements and you may surprise yourself by becoming an inventor.



Adapted from Erdmann's "Lehrbuch der Anorganischen Chemie"

FIG. 1. SPECTRUM CHART

The *solar* spectrum with Fraunhofer lines; the bright-line spectrum of *sodium;* the spectra of *hydrogen* gas and *mercury* vapor. The numbers refer to wavelengths expressed in hundredths of a micron. For comparison, 1/100 micron = 0.00001 mm. The dark lines on the solar spectrum are absorption lines representing wavelengths of light that were absorbed by the gases in the atmosphere of the sun. (Reprinted from Black's *Introductory Course in College Physics*, The Macmillan Company, 1941.)

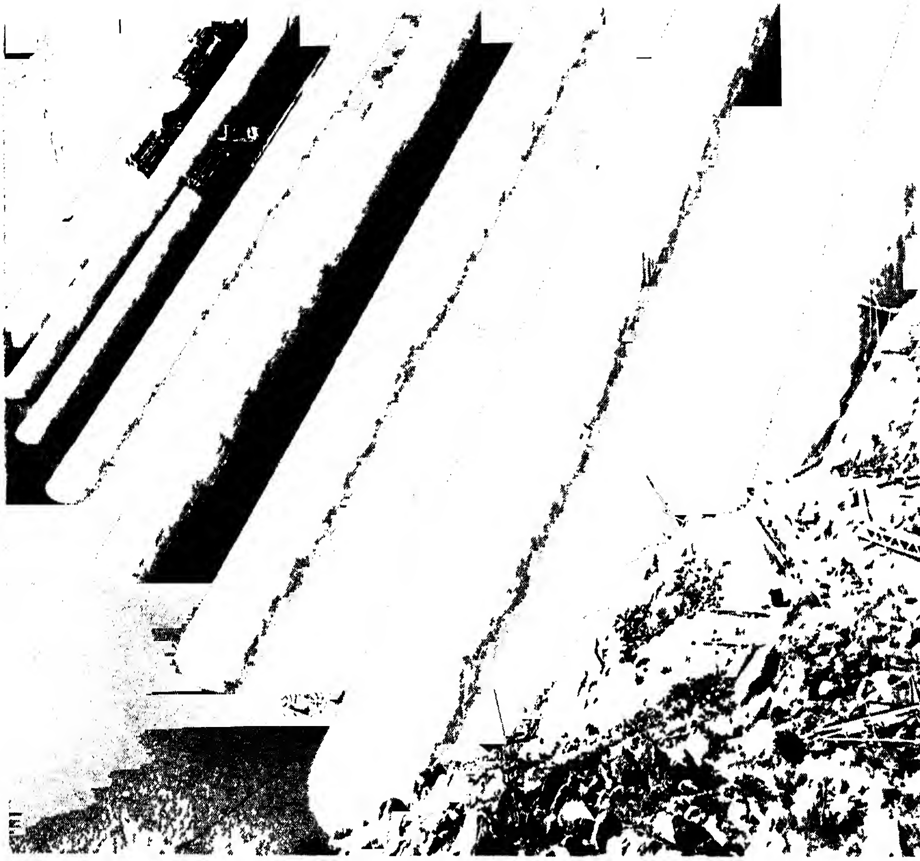


FIG. 2 GREAT ARCHED DAMS PROVIDE IRRIGATION WATER AND HYDRO-ELECTRIC POWER IN THE WEST

THE STORY OF ENGINEERING

I imagine that the first engineer, to our way of thinking, was a savage — a man without tools and almost without clothes. Fire was the symbol of the beginning of civilization and he used it not only for warmth and light and partially to cook his food but also as a tool. He must have crossed streams on fallen logs and then one day he slowly began to realize how fine it would be to have a log bridge across the stream just outside his cave. A tree grew there, leaning across the stream, and, moreover, it was burned partly through by his own carelessness. Why not renew this fire and keep it tended until the tree fell across the stream?

Primitive technology

In his rude mind this simple plan must have been slow to evolve for it has in it all the elements of engineering. We say that engineering is the art and science of bringing the forces of nature to serve the use and convenience of man. Our primitive engineer felt the need of a bridge and his plan for producing one made use of two of the forces of nature — *fire* to burn the tree trunk through and *gravity* to fell it across the stream. When precivilized man first walked across a stream on a log bridge of his own creation, engineering had started its course toward the ultimate production of the steel mills, the airplane, the radio, and the mechanized world yet to come.

The dugout

From the log bridge it was but a step to produce the dugout canoe. Fire burned away the inside of the dugout so that only the crudest scraping tools were needed to speed up the work. The dugout provided transportation for a whole family and its few possessions. With it, man began his attack upon distance, and, with the invention of outriggers, he even used it to cross from island to island over the sea. From this stage the development of sailing ships and eventually great steamships followed in natural course.

The wheel

Perhaps the greatest single engineering achievement of all time was the invention of the wheel and axle. We can picture this invention as mainly accidental. A flat round stone just happened to have a hole in the center; a small boy stuck a stick through the hole and with his hands on either side of the stone leaned over like an animated wheelbarrow and annoyed everybody by running the contraption over their toes! But one man of the tribe could not drop the matter from his mind by cuffing the boy, and at long last the idea of the one-wheeled vehicle had evolved. In due time

the improvement of adding another wheel to produce the cart and then two more to balance the load was inevitable. Man had at last started his long journeys over the land as well as the water. As engineering science progressed, railroads and automobiles came as logical developments using the all-important wheel.

Irrigation systems

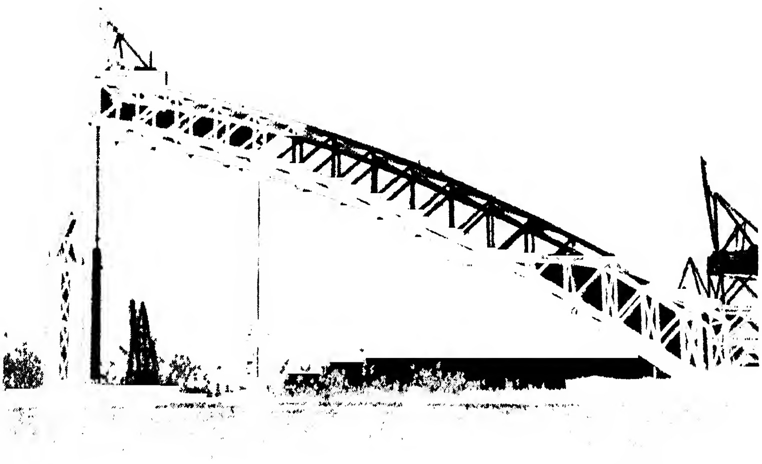
One of the most civilizing influences, and also a great engineering achievement, was the irrigation development of the ancient world. The civilizations of Babylon and Egypt in particular were built upon the agriculture of irrigated lands. As compared to the great concrete arched dams of today (Fig. 2) with their hydroelectric power developments, the irrigation systems of the ancient world would seem simple indeed, but they brought fertility to the land, nevertheless. Today around the site of Babylon there is nothing much but wasteland due to the shifting of rivers and the destruction of dams, canals, aqueducts, and other devices of the sixth century B.C. In Egypt we know that it was necessary, except in floodtime, to raise irrigation water from the level of the Nile to the level of the farm land and that several types of hand lifts were used. By this laborious process, Egypt attained the most powerful civilization of the pre-Alexandrian world. Engineering and agriculture still go hand in hand and still remain great civilizing influences upon mankind.

The column and lintel

Architectural construction followed the growth of organized government and religion. Monumental buildings were not needed until government or religion developed to the point where the protection of the fort or the prestige of the temple became important. For several thousand years through the civilizations of Egypt and Greece, architectural construction was limited to the structural possibilities of the column and lintel. Extraordinary structures in stone were built under these restrictions.

Temple of Amen

An interesting example is the Temple of Amen at Karnak in ancient Egypt, completed about 980 B.C., which was under construction for seven centuries. This building was 1200 ft. long and 350 ft. across its greatest width. It was larger than the great airship dock at Akron, Ohio, which is 1175 ft. long and 325 ft. wide. In sheer mass of stone, the Temple of Karnak exceeds anything in the nature of shelter ever constructed by man. Built without modern tools, the great central stone columns are 66 ft. high and 10 ft., 7 in. in diameter. The lintel slabs of stone spanning from column to column are 36 ft. long and 4 ft. thick. It is believed that the



Courtesy American Bridge Company

FIG. 3 ERECTION OF AN ARCHED TRUSS BRIDGE

This structure combines the action of the arch and the truss. When finished it will be a gracefully curved arch made up of steel bars put together to form triangles which is the principle of the truss. One temporary vertical bent is supporting the arch during construction and another is being erected at the left. Both will be removed when the arch is completed.

building was filled with sand as it was constructed so that these monumental stones could be dragged up a sloping ramp and into place without cranes. The building then was emptied of sand as the final step of construction.

The arch and dome

It remained for Roman engineers to make the first major use of the arch. They built long aqueducts on multiple stone arches and also hundred-foot bridge spans of arched construction. Many still exist. For roofing one magnificent building, the Pantheon at Rome, the world's greatest masonry dome was completed in 120 A.D. The diameter of this impressive domed structure is 144 ft. which is also the height of the wall from which the dome springs. Only in the church of Hagia Sophia in Constantinople, 563 A.D., was this work exceeded. This building has a nave 243 ft. long and 107 ft. wide covered by a central dome and two half-domes at the ends, in part supported upon stone arches of about 100-ft. span. Artistically, its interior is one of the supreme masterpieces of all time.

The truss

We should not leave the subject of architectural construction without mentioning the late development of the truss. It was not

until 1758 that Ulric Grubenmann, a Swiss carpenter, constructed a great wooden truss bridge over the Rhine. Two hundred years earlier, Andrea Palladio (1518–1580), an Italian architect, invented the truss and may have used it in building construction, but his idea was lost with his death. It is strange that the truss, constructed with straight bars joined to form a system of adjacent triangles, which is not a very complex construction, should have developed several thousand years after the lintel and the arch. Soon after its first use, however, it became well understood and evolved rapidly. First built as a wood structure, the truss soon was made with cast-iron and wrought-iron members and then after the Civil War its construction in steel revolutionized bridge building.

THE MACHINE AGE

Iron and steel

Before machines could really change man's daily life, there had to be a cheap strong material for their construction. Simple machines such as the potter's wheel, the spinning wheel, and the hand loom had been built of wood since the dawn of history, but iron was needed for steam engines, power tools, or even farm machines. Iron from meteorites had certainly been known for many thousands of years, but its production from ores was not common until a thousand years before the time of Christ. Bronze was the common metal of early historical times because it could be produced (from copper and tin) at a temperature below 2000° F. compared with 2800° F. at which iron must be cast. Also, it really takes machines to make iron because its great weight presents a handling problem only solved by the use of power. Hence, the machine waited upon the cheap production of iron in large quantities and iron production required the machine.

Bessemer steel

The development of iron foundries had progressed in England to such an extent that a cast-iron arch bridge of 100-ft. span was constructed at Coalbrookdale in 1776. From then until Bessemer developed his converter to make iron into steel (1856), we find a steady increase in the production of cast iron and wrought iron accompanied by an extraordinary development of the machine and the industrialization of England, America, and other parts of the world. The greater strength and reduced cost of steel after Bessemer's invention simply speeded up the process of industrialization, which by that time had become clearly defined.

Age of power

James Watt's development of the steam engine (1765) and

Michael Faraday's explanation of electromagnetic induction, the basic concept of the electric generator (1831), have a special significance because they symbolize the introduction of two forms of power that were to revolutionize civilization. Industrialization through the nineteenth century was accomplished by steam power; the evident symbols were the factory steam engine with its belts, shafts, and pulleys, the steam locomotive, and the steamship.

Electricity

Toward the end of that period, steam was beginning to be generated for the purpose of producing electric power, and the transformation of industry to the age of power began. By 1943, our over-all power load of the electric utilities of the country had reached 36,900,000 kilowatts or nearly 50 million horsepower. The rapid production of airplanes indicates that the total horsepower of airplane motors will soon exceed this figure while the total potential power of American automobiles is in excess of a billion horsepower. Given assurance of peace, industry could obviously remove the main load of heavy labor from the shoulders of man, at the same time making possible new and higher standards of living for all.

Electronics

Steam and electric power released man from the need for slave labor, and then it almost made him its slave by pinning him down to monotonous watching, feeding, and tending of machines. As machines have become more and more nearly automatic (but not entirely so), our task of watching machines has become more and more dull. There is a cultural influence in fine craftsmanship, but machine tending is deadening rather than stimulating and at best monotonous.

Now, clearly before us, we see the coming emancipation of man from his self-imposed servitude. Through technology has come a new development — electronic controls for machines. The vacuum tube of radio has been put to new uses such as the electric eye, the electronic ear, the electromagnetic arm. So far, man's senses of taste and smell have defied electronic duplication, but the functions of sight, sound, and touch can be performed more exactly by electronic devices than by human beings, although as yet this field of technology has been only touched by the scientist and the engineer.

Automatic machines

In this new age of electronics that we visualize for the last half of the twentieth century, machines will become entirely automatic and vastly more complex than anything we now know. A recent

development is a machine nearly 200 ft. long that takes airplane cylinder-head castings into its maw at one end, moves them along automatically past cutting and grinding and polishing tools, to deliver the finished parts at the rate of more than one per minute. The machine automatically shuts itself down if anything goes wrong. Instead of many workmen tending the individual machine tools, one man only is needed and, in fact, he may eventually be freed by a few additional gadgets. There seems little doubt but that such machines will soon be provided with automatic inspection devices which will discard pieces with flaws that would not be seen by the human eye. The shadow on the X-ray screen, the whine of the cutting tool, the reflection of light from a polished surface, the ring of a part when it drops on a table may each be used with electronic devices to discard imperfect parts.

Robot tools

Technologic leisure

Some persons wonder what we can do or will do with the leisure that the development of more automatic machines will provide. Some envision calamitous unemployment. One answer is that no unemployment need result if man's standard of living is permitted to rise — but this is more a problem of economics and the distribution of goods than it is a problem of technology. If more leisure can be provided, there is much use to be made of it. We begin to see the need for a large group of persons in all countries to be educated for world citizenship. No such education would be complete without world travel and the ability to communicate with all peoples through a common language. If technologic leisure should be used to train such world citizens who would then return to become productive citizens of their own countries, it might be that technology would have developed a force able to cope with the problem of "peace in our time."

THE PROFESSION OF ENGINEERING

The professional man

We will not take time here to go into the fine distinctions between the professions and those callings that are not professions. According to common understanding, engineering is a profession and the engineer is a professional man. So is the scientist. The first recognized professions were the ministry, the law, and medicine. These are sometimes called the *closed professions* because the one way of entering them is by education in a recognized institution followed ordinarily by an examination for a state license to practice in that profession.

Registration

Engineering is still an open profession since it is possible for anyone to prepare himself to serve as an engineer. Of course, there are state registration law for engineers, but the examinations are open to anyone who has demonstrated reasonable competence in his work as an engineer, registration being required only for those who approve plans. State registration is desirable because engineering works affect the health and safety of the public which deserves the protection given by the registration law. An incompetent engineer might build an unsafe building or design a water-supply system that would permit pollution of our drinking water. Since we cannot take chances with the public safety, undoubtedly all states will eventually pass registration laws for engineers.

Professional societies

Each branch of engineering is represented by a professional society to which leading engineers in that field belong. The first of these (1856) was the American Society of Civil Engineers. Other major societies are the American Society of Mechanical Engineers, the American Institute of Electrical Engineers, the American Institute of Mining and Metallurgical Engineers, and the American Institute of Chemical Engineers. Engineers consider membership in one of these societies to be a mark of professional distinction because only those who have attained a certain level of professional experience and ability are accepted for membership.

Society membership

Each society has more than one grade of membership and, as the young engineer gains in professional experience, he is advanced from *junior* member to *associate* member and finally to *full* member. For this reason the grade of member in one of these societies is a badge of professional distinction. The engineer often belongs to other technical societies devoted to the study of problems in which he has a special interest, but he should always maintain his membership in one of the professional engineering societies mentioned. Student engineers have a chance to join these societies before graduation through student chapters that exist at most of the major schools of engineering. If you are a student engineer, do not miss this opportunity; you are eligible as soon as you become a classified student in engineering.

Obligations of the professional man

When you become a member of a profession, you thereby accept certain responsibilities and obligations. Perhaps the greatest responsibility is that you must undertake to look after the interests of the public; and this assumes, of course, that you will be honest in the strictest definition of the term. It may help if we cite a few examples

of professional conduct. There are some callings where the trading of an opinion or a vote is openly condoned. An engineer will not long remain a member of one of the professional engineering societies if he permits his personal gain or loss to influence his professional opinion on technical problems.

Professional conduct

If a bridge is weak, the engineer's report must say so. If he feels that a proposal to build an industrial plant in a certain location is contrary to the public interest, it is his obligation to render an adverse opinion. And, if his study of a proposed manufacturing project convinces him that the product could not be produced and sold profitably, he has no choice but to advise against the project even though he loses thereby a desirable job of designing and building the plant. This, in brief, is what we mean by the professional honesty of the engineer; this is simply the obligation of the professional man.

Code of ethics for engineers

Long codes of ethics or of proper professional conduct have been adopted by the professional engineering societies for the guidance of their members, but these are too complex for study here. Instead, we shall consider only a few points of ethics as outlined by a committee of the Engineers' Council for Professional Development, a joint council founded by the professional engineering societies:

> *The engineer will avoid conduct and practices likely to discredit the honor and dignity of the engineering profession.*
>
> He will interest himself in the public welfare and be ready to apply his special knowledge, skill, and training for the benefit of mankind.
>
> He will not express publicly an opinion on an engineering subject without being informed as to the facts relating thereto.
>
> He will not lend his name to any questionable enterprise or engage in any occupation contrary to law.
>
> He will carry on his work in a spirit of fairness and loyalty to associates, subordinates, and employees, fidelity to public needs, and devotion to high ideals of courtesy and personal honor.

WHO SHOULD BE AN ENGINEER

Technical competency

There is an old saying that you can't make a silk purse from a sow's ear. It is quite as impracticable to make an engineer of a young man who should be a musician or a poet or perhaps a lawyer. Not that you may not become both an engineer and a musician or lawyer. But if you are to become an engineer, you must have most of the well-known qualities of an engineer, and then, if you also have other useful qualities, so much the better.

It is pretty well understood that a good engineer must be reasonably good in mathematics. Now that does not mean that every engineer is a "math whiz" by any means, but it does mean that, on the average, engineering students make about the same grades in their engineering subjects that they make in mathematics. Good in math, good in engineering; poor in math, poor in engineering is the common rule. I cannot say, however, that most engineers like to study mathematics. They are too impatient, too anxious "to study something practical," but they do stick to their mathematics courses and learn far more about the subject than most liberal arts students. They do so because they realize that mathematics is of daily use in engineering. *Mathematical ability*

Physics is the first of the basic sciences of engineering, which is largely applied physics. The usual subdivisions of physics — mechanics, heat, light, and electricity — are the engineer's theoretical tools. Along with chemistry and some bacteriology, they make up the background sciences of engineering. It is not likely that a good prospective engineer could really dislike physics, and certainly a chemical engineering student should enjoy chemistry. A knowledge of bacteriology is essential to the sanitary engineer whose job it is to guard the public health. *Physical sciences*

There is plenty of chance for self-expression within the profession of engineering. Out in the field, there is the construction man, and in the factory the production man. Construction and production engineers "kick the job along" and see to it that things happen fast. They need less theory and mathematics than the designer who works in the office or the drafting room. The designer's job is making calculations and supervising the production of drawings which transmit design information in the "language of engineering." The designer must be careful, accurate, and sure of himself. The research engineer must be all of these things and be curious and imaginative besides. He must continually find new ways of doing things, new materials, new devices, new ideas. He is responsible for new industries such as plastics, television, air conditioning, aircraft. Research raises our standard of living in peacetime and may be the savior of civilization in time of war. *Engineering practice*

"PARLOR TEST" FOR TECHNOLOGICAL TRAITS

It would be wonderful if psychologists and professional testers could pry into our minds, our personalities, and our characters to *Your bent in life*

determine for each beyond question the direction of his true bent in life. You should understand before you read this discussion that such claims are made only by astrologers and mystics. Some day psychology should be developed to the point where it will produce tests of great reliability, but the human mind-personality is so unbelievably complex that improvements in placement testing naturally develop slowly. In the meantime, you can make use of tests that have been devised, recognizing, of course, that a test is considered quite successful if it predicts correctly the performance of three persons out of four in a random group.

Aptitudes and interests

Tests have been prepared either to measure your knowledge, your interest, or your aptitude for a subject. It seems likely that there will always be some overlapping here because it is difficult to see how you can show an intelligent interest in any field before you know something about it. However, if a test is devised merely to measure vocational interests, it would seem reasonable to assume that a person who has the same general pattern of interests shown by most successful engineers could be a successful engineer himself. At least, we would feel that his chances of success were improved by his fortunate pattern of interests.

Then again, if you take a series of aptitude tests and score high in the technical aptitudes, you should feel encouraged to study engineering. One such test is the "wiggly block" produced by cutting a wooden cube into many pieces by curved saw cuts. Engineers can usually put the parts back together in short order while successful bankers are much slower. You would do well to take such tests before you spend several years studying engineering or science; any college or university can advise you.

Simple self-scoring test

A thoroughgoing test requires considerable time to take and to score. Vocational-interest tests use several hundred questions and aptitude tests may require several hours of your time. The purpose here in providing a test of twenty-five questions that you can answer and score by yourself in a few minutes is merely to interest you in your own abilities and to provide you with a very rough measure of your interests and aptitudes, especially those that would naturally lead you into the technical field. In fact, it is hoped that your experience in taking this simple test may develop your interest in taking a complete vocational test because you should not take the results of this "parlor test" too seriously. When taken by a group the test may be used as an entertaining game.

All tests apply most effectively to groups while any individual may be an exception. You may have a tenacity or keenness of mind that will make it possible for you to succeed in engineering even though you lack some of the usual interests and special traits of the engineer. You may also fail even though you have the desired technical qualifications if the "will to succeed" is lacking in your nature.

BACKGROUND FOR TECHNOLOGY

"Yes" or "no" questions

Answer each question "yes" or "no." Give the first answer that comes to your mind. These are not questions in the main that require careful analysis. You can answer several questions each minute. Answer all questions before you read the instructions on scoring that follow the questions. Be honest with yourself for otherwise your test score will be misleading.

Group 1

(a) Do you like mathematics more than three of these subjects: English, history, art, music, civics?
(b) Were you ordinarily able to work the problems in your mathematics courses without great effort?
(c) Can you attack a problem in mathematical symbols without confusion or displeasure? For example: Given that $\theta = Z$ while $R = \phi$ and $Z = R$: Does $\phi = \theta$?
(d) In high school a mathematics course consumes about one-fourth of one's time. Would you like a lifetime job that consumed a quarter of your time solving mathematical problems?
(e) Does the difficulty that everyone finds in getting the correct answers to mathematical problems challenge you rather than plague you?

Group 2

(a) One circular gear turns between two others, the three being in line. If the center one rotates clockwise, both outside gears rotate counterclockwise. Do you find it confusing to visualize this motion?
(b) A wheel rolling from left to right over the floor turns clockwise, but, if rolled along the ceiling, it turns counterclockwise. Are you uncertain whether this statement is correct?
(c) An object hanging near a corner of a room would cast a triangular shadow on either adjacent wall or a circular shadow on the floor or ceiling. Do you have difficulty visualizing the object as a cone with axis vertical?
(d) Read the following sentence over until you know whether the man returns home. *A man walks north 3 blocks, east 1 block, south 1 block, west 2 blocks, south 2 blocks, and east 1 block.* Did you read the directions more than twice before you were sure that he returns home?

(e) Place a cube on a table. Make two horizontal saw cuts entirely through the block, keeping the pieces together. Then place the saw on the diagonal of the top face and cut vertically downward. Turn the saw 90 degrees and repeat on the other diagonal. Do you fail to see immediately that you have produced 12 similar wedges?

Group 3

(a) Does it bore you to read about molecules, viruses, atoms, protons, and electrons?

(b) Is chemical jargon such as $2\ H_2 + O_2 = 2\ H_2O$ confusing to you?

(c) Could you define the meaning of either *citizenship* or *truth* more readily than a *degree* of longitude or latitude?

(d) Does it seem meaningless to you to associate color with vibration and wave length?

(e) Do you pass up newspaper or magazine articles on plastics, electronics, synthetic materials, or other popular but technical subjects?

Group 4

(a) Did you ever (of your own volition) construct a machine or doorbell or other apparatus that worked?

(b) Would you like to take a clock or an airplane engine apart just to see and feel its parts?

(c) Do you enjoy working with tools, shaping material, or fixing household gadgets?

(d) Would you rather work in a shop or laboratory than listen to good lectures and read interesting books?

(e) Have you on your own initiative visited a machine shop, a welding job, or carefully inspected any mechanical process because of your special interest?

Group 5

(a) Do you speak easily and also understand the meaning of most words such as *squalid*, *condone*, and *furtive*?

(b) Would you prefer being tied down with more work and greater responsibility if it did not involve larger salary?

(c) Do you accept yourself and others without worrying about speech, manners, or character?

(d) Have you been planning your own career instead of waiting for it to develop?

(e) Have you been an officer of your team, class, or club, editor or manager of your school paper, or been elected or appointed to any official position?

Scoring the test

Score each group of questions separately beginning with Group 1. If three or more of the five questions in any group are answered "yes," your score for that group is 1; otherwise, the score is 0. After scoring the five groups, make up your code number by placing the group scores in sequence from the first to the fifth. Your code number would be 10100 if the answer "yes" had predomi-

nated in the first and third groups while the answer "no" had predominated in the second, fourth, and fifth groups. There are thirty-two possible code numbers.

Mathematical aptitude

Engineers are expected to show strong aptitude for mathematics. If you are in this group, your code number should begin with 1. If it begins with 0, you should take a full mathematical aptitude test as a check.

Technical visualization

A trait that seems to appear with great regularity in engineers is the ability and desire to visualize or picture things and motions in space. If you have this ability, your code number will have a 0 in the second place from the left. This capacity will not substitute for mathematical aptitude but does supplement it.

Science interest

It is difficult to see how an engineer can fail to have a greater interest in science than the common person. In fact, it has been found that a high score on science tests usually goes along with mathematical aptitude. We would expect the code number of a potential engineer or scientist to have 0 in the third place or the third digit from the left.

Mechanical bent

Many engineers — particularly mechanical engineers — have the interest and ability to produce things with their hands. This is the special field of the mechanic, but it is shared by laboratory research workers, glass blowers, usually by shop foremen, and often by inventors. If you are mechanically inclined, your code number should have a 1 in the next to the last place. If you find a 0 there, you need not worry, since this trait is not common to all engineers.

Supervisory quality

More than one-half of all engineers are in supervisory work after fifteen or twenty years of experience. Hence, this quality itself points to one type of engineering ability. If you have several qualities of the successful supervisor, your code number should end in 1. If you lack such abilities, you can still be successful as a designing or development engineer and you may later develop the characteristics of the supervisor. These qualities may change gradually through life.

USING TEST RESULTS

High and low scores

This "parlor test" is not complete enough to make each individual score significant. However, if your code number begins with 100 or even 101, you have shown the main characteristics of the engineer, 10011 being the most rounded engineering type. The

person with all negative engineering characteristics would be expected to have a code number 01100. However, any code number beginning with 011 or 010 is a negative score in regard to the most essential engineering traits.

Scoring stars

As an added indication, you may give yourself one star for each group of questions in which you wrote down four "no" or four "yes" answers and two stars for each group for which all five answers were alike. Then, the number of stars is a measure of the definiteness of your interests. If you score at least five stars, you may feel somewhat more confident that the test is indicative of your technological traits.

Why people are successful

Even if professional testing can be developed to the point where it accurately measures our interests and aptitudes, it is doubtful that we shall be able to predict by such tests who will and who will not be successful. While writing that your code number should begin with 100 if you have the proper characteristics of an engineer, the picture of one of my former students came to my mind. He certainly would have scored low in mathematics and science aptitudes; by all the rules he should have "flunked out" of engineering school in his freshman year. When he entered my class as a third-year student, he said, "I know I'm dumb in theoretical subjects: I've had to take several theoretical courses twice, but I'm going to be an engineer. If you don't drop me out of your class in the first few weeks, you will find that before the end I will have all the problems worked correctly even if it takes me half the night every night." He passed the course and those that followed it and graduated as a civil engineer. By directing his energies into the practical field of construction, he has become a successful practicing engineer.

Determination wins

If you are not unchangeably set on being an engineer or scientist and if you score low on the test given here and also on more thorough engineering aptitude tests, you would be well advised to try another field. But if you cannot get technology out of your mind and if you are a determined sort of person, for you the tests may be unimportant. All teachers have known students of only average ability who were able to graduate from an engineering college in four years and hold down another job at the same time. City college deans often find that the "honor man" is supporting not only himself but his mother or his own family by night work. Determination wins in every phase of life.

Another point worth making is that the test you have taken

here is probably more indicative of the ease or difficulty you would have in graduating as an engineer than it is of your ultimate success. If you have only average aptitude for mathematics but speak and write easily, you might have difficulty graduating as an engineer and then find yourself successful as a patent lawyer, technical magazine editor, personnel manager, sales engineer, or a promoter of engineering projects. The practical person who is weak on theory may become successful as a production or maintenance engineer and later advance to chief operations engineer or superintendent. The field of engineering is so broad that it provides outlets for men of varied talents. Nonetheless, it is a good idea to test yourself because your greatest success should come when you get into a job that fits your aptitudes as well as your interests.

Promoters and producers

HOW TO READ BOOKS ON TECHNOLOGY

The thrill of the mystery novel is likely to be lacking in scientific or engineering books. Luckily, there is another pleasure to take its place — the sense of power over physical reactions grasped through the understanding of scientific thought. It does not come to you without effort, or all at once, but the satisfaction rising from this ability is enduring.

Avoid distractions

Since technical reading is not exactly exciting, you ought to prepare for it. There is little profit in picking up a book on electric circuits or airplane design in a room where others are engaged in discussing the cost of living or where a radio is blaring. If your job is to concentrate on electronic circuits, give yourself a chance to succeed. Two may be "company" under certain circumstances, but you study alone or usually not at all — at least not efficiently — and the sense of frustration after an unsuccessful attempt to concentrate may keep you from trying again.

Adjust your surroundings to your own temperament. It is best not to get too comfortable. Most people who slump into a deeply cushioned easy chair want to overrelax and go to sleep. Some persons can study in bed — the habit is often developed during an extended illness — but it is a poor method for most people. A straight chair that is reasonably comfortable, a desk where you need not slump over while making notes, and a steady light without glare are the minimum physical aids to successful study.

Ideas vs. calories

Now, with surrounding distractions reduced as far as reasonable, you must learn to concentrate on the book before you. Persons

who flit back and forth between the kitchen and the study room pick up more calories than ideas. We observe that the great scientists stick very closely to a single field of study through an entire lifetime. You must at least give science an undivided hour of mental concentration if you expect to be rewarded with the clear understanding of a new scientific idea.

Scanning

Then there is the question of whether scanning has any usefulness in technical study. Of course, before you quit, you must read not only for ideas but for details as well. To read rapidly over a half-dozen pages for the main ideas and then to cover the same material at a slower rate, reading for details, is a good plan if it is followed consciously. The danger is that you may get into the habit of scanning and fail to develop the patience necessary for digging through a tough chapter paragraph by paragraph, sentence by sentence, or word by word if necessary. You must accept the fact that a single page of a textbook may require hours of study for a complete understanding. Unfortunately, the author may not have written that page clearly or you may have forgotten some basic principle that he thought you would know.

Technological language

Keep in mind that most books on technology are written logically. This means that ideas and concepts are developed step by step. If you miss the derivation of one formula or fail to grasp one logical development, you probably will not understand the next one. It is impracticable to pick out only interesting topics and skip dull ones. Only children pass up the meat course of a fine dinner and hope to get two desserts.

Probably by now I have made the whole subject seem too formal, too rigorous, and too prosaic for you to look forward with enthusiasm to technical reading. Actually, there is a bit of inspiration in every paragraph and a thrill on every page. Mathematical symbols, chemical formulas, tables, drafts, and charts are simply the language of technology. They are like the words of a novel, individually unimportant but, taken together, they weave the plot. You must recognize the words to read the book. Naturally, you must understand this language of science and engineering if you are really to learn how rubber can be made from alcohol, and plastics from wood fiber, why diesel locomotives are replacing steam engines but television did not immediately start to replace radio, how great bridges sometimes collapse, or why the helicopter after a half-century of neglect now becomes a practical

means of flight. The thrill is there if you prepare yourself to experience it.

TECHNIQUES — GOOD AND BAD

Handbook engineers

Depending upon whether you get started right or wrong, you may become either a real engineer or a "handbook engineer." Now a handbook is a wonderful asset if it is used rightly, but it becomes a snare when it is given too important a place in your library. There are handbooks for every basic field of engineering and then special ones for welding, radio, plastics, and a dozen other limited branches of technology. They give all of the formulas, equations, charts, and tables generally available in the field; in fact, they seem to solve most of the standard problems for you.

Of course, you cannot expect to remember every formula and every accepted rule of engineering. There are just too many. Neither can you take time to derive each equation every time you want to use it. So you keep a handbook handy to pick out the formula, rule, equation, or curve when and as you need it. Now here is the catch question: If you are going to use the handbook anyway, why bother to derive the formulas or plot the curves in the first place? Can you just become familiar with the authoritative sources of material and then assemble and use these data when the need arises? The answer is *No!* That is handbook engineering and it has been proved to be too dangerous to encourage. That is why most states require engineers to prove their competency to design engineering works without the use of handbooks before they can become registered.

High-octane technology

Engineering is not a series of convenient pigeonholes from which you pluck facts and figures that apply directly to the problem before you. The same problem seldom occurs twice. We have been busy for years building plants to produce high-octane gasoline; but no two of these plants are identical. If they had all been cut to a single pattern, there could have been no progress, no improvement, no use of successive inventions. Besides this, the plants must handle different kinds of crude oil, with the result that the by-product oils will be different for plants producing the same high-octane gasoline. Handbook engineering fails when confronted with such complex situations. Judgment based upon a thorough knowledge of theory and familiarity with good practice is the only background that is adequate for the designing engineer. Besides, we

must not forget that the public safety is involved in many engineering projects.

Scientific skepticism

When you first start to study technology, you are likely to feel that anything in print is bound to be right and that every author is an authority. A halo often seems to fall upon the brow of the writer; we are likely to assume that he never makes mistakes. But the fact is that there are very few books without errors, and an error or confused statement every few pages is common in technical books. Most of these errors are not very important, but some of them are completely misleading.

Textbook errors

One textbook, in fact a good one, was found to contain the following mistakes: a misspelled word that changed the meaning; a decimal point misplaced; a table with two column headings reversed; a misprinted formula; a symbol used twice with different meanings, an unexplained numeral that could not be checked; a reference to the wrong specification; an error in a problem answer; a use of the wrong formula; a factor omitted; an error in a table; a square root sign omitted; a slide-rule error; an error carried over from a previous publication; the unit inch-pound given incorrectly as pound; an illustration with vital dimensions omitted; a plus sign changed to minus; the use of an obsolete specification; a garbled table; a wrong heading to a graph; an error in a reference; a few minor typographical misprints; several statements that might be misinterpreted.

Hypotheses and proofs

Undoubtedly, in reading the book the critic did not find all the errors. If you should read the same book, you might find others. The point is that you cannot afford to accept anything you read until you have checked it. Naturally, you must pass over some statements with the mental reservation that you have not time to check them during a first reading and that they will be brought back to mind for restudy later. And then, there are certain hypotheses that are not subject to proof. For example, we usually assume that a liquid is noncompressible. Actually, at room temperature and a pressure of about one hundred thousand pounds per square inch, water will be reduced one-third in volume after having passed through a special form of ice. But for low pressures, liquids remain almost constant in volume, and so this assumption is a convenient one for simplifying our calculations whenever it is applicable.

Hypotheses are not subject to logical or mathematical proof but

are justified by observation and experiment as the starting points for theoretical derivations. Keep in mind, nevertheless, that mathematical transformations applied to a starting hypothesis produce formulas that are no more applicable, exact, or rigorous than the original assumption or hypothesis. *Mathematics adds no authority to the final results* — it merely transforms our stated hypothesis into equations, formulas, or graphs for convenient use.

USING MATHEMATICS

Morals and mathematics

Here is a mathematics problem that teaches an important lesson without being difficult to understand. Two streamliner trains are 100 miles apart and are approaching each other at 100 miles an hour. A pursuit plane traveling 400 miles an hour flies from one to the other and back again and again until the trains meet.

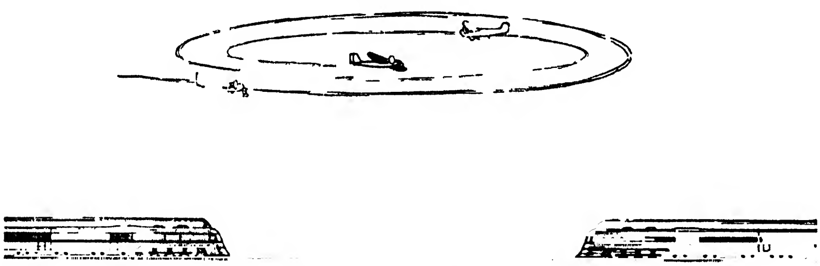


FIG. 4. PLANE FLIES FROM ONE TRAIN TO THE OTHER CONTINUOUSLY

The question to be answered is: What is the length of path that the airplane travels? We will try the complicated solution first. As the trains get closer together, the plane makes the round trip more and more quickly and it passes over the meeting point an unlimited or so-called "infinite" number of times. Its total distance of travel, then, is the sum of an infinite number of round trips that become continuously shorter until they approach zero as a limit. This could be a rather difficult problem in infinite series, but it happens that there is another and much simpler solution. The trains approaching each other and each moving at 100 miles per hour will cover the 100-mile distance between them in 30 minutes. Evidently in 30 minutes the airplane flying at 400 miles per hour would cover 200 miles, which is the answer. If we had been able to sum the infinite number of round trips made by the airplane, the answer would also have been 200 miles.

Simple solutions

The moral of this story is that it is a good thing to look around for a simple solution to any problem. You will remember how

much easier some arithmetic problems became when you learned to call the unknown quantity x and to solve for it by use of an algebraic equation. All the way through your study of mathematics you should keep looking for the simple way to approach a problem. As a matter of special interest, you will want to know that the engineer has several devices that help him find a simple mathematical solution. First, he usually can nearly guess the answer so he can often work backwards to help simplify the mathematics, and his knowledge of the answer is always one way of checking the result. Then, too, his understanding of the engineering problem often leads him to expect to need a certain kind of mathematics for the solution. For example, the load coming on the central pier of a two-span continuous bridge can be calculated from a simple algebraic equation. For three spans, the two central pier loads must be computed from the solution of two simultaneous algebraic equations.

Checking

PHYSICS AS ENGINEERING BACKGROUND

Phenomenon of physics

Closely related to mathematics, the second great tool of the engineer is his knowledge of physics. Mathematics is a theoretical subject; physics introduces theory to laboratory experiment. The physicist tries to explain the things he sees in nature, using mathematics and his ability to experiment in his laboratory. Everyone knows the story of Newton's interest in falling objects which led after years of study to his theory of.gravitation. Einstein found that Newton's theory did not explain some movements of the planets, which led him to devise his theory of relativity. Galileo experimented with pendulums and then wrote mathematical equations for their motion. In each of these cases and in most of the physicist's work, we find tied together (a) an unexplained phenomenon of nature, (b) controlled experiments for study of the observed phenomenon, (c) the devising of a mathematical theory that explains the phenomenon. An example is described below.

The blue of the sky

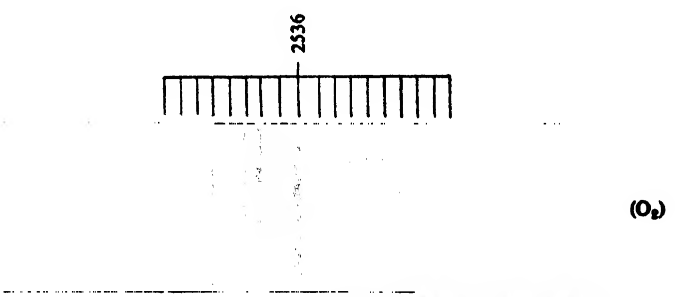
Why are the sky, the sea, and the great glacial ice caps blue? We all now realize that color is not a property of the substance itself but the wave length of the light that enters our eyes after being reflected from the thing we see. Blue light is of shorter wave length than red light. When we look at a colorful painting, our eye is really registering the fact that light of wave lengths 2, 3, 5, 6, and 7 is crowding into it from that picture, and through a

miraculous mechanism these numbers or lengths are transferred into our mind picture of the masterpiece. Viewed in blue light, this many-colored picture appears to be only blue and black.

Early theories

As far back as the start of the sixteenth century, Leonardo da Vinci felt that dust particles in the air caused the blue color of the sky. Some reason had to be found to satisfy the mind of this inquisitive artist-scientist, to whom color produced in colorless air by white sunlight could not be left unexplained. The same theory was again proposed by Tyndall in 1869. He was even able to show in artificial cloud chambers that blue light can be scattered or dispersed when sunlight passes into masses of fine dust particles suspended in air.

It was inevitable, however, that skeptics would point out the increased blueness of the sky as seen from mountain tops above the dust level. It seems simple now to say that it should have been evident that the molecules of the air itself formed a more effective dispersing agent than the most finely divided dust. So it proved to be when Lord Raleigh showed that the blue color and luminosity of the sky were fully explained by molecular scattering of sunlight.



Courtesy Chemical Publishing Company

FIG. 5. RAMAN SPECTRUM OF OXYGEN

Reproduced from *Scattering of Light and the Raman Effect* by S. Bhagavantam, who explains in greater scientific detail the reasons for the color shown by the ocean and the sky.

The blue of the ocean

The simplest explanation of the ocean's color is that it might be a reflection of the sky. Skeptics again arose to show that the ocean was sometimes seen to be bluer when the sky was almost gray. Dissolved coloring matter lost caste as an explanation because the observer was always above the water and could see only by light reflected from near the surface; besides, when sampled, sea water no longer looked blue. There might still be floating particles to

disperse the light, but tests by C. V. Raman, the famous Indian physicist, showed suspended material to be almost nonexistent particularly in the bluest ocean waters. And so Raman concluded that molecular scattering of sunlight explained the blue of the ocean as well as the azure sky. Evidently, the cold blue brilliance of glaciers was another manifestation of the same effect.

A scientific discovery

Thus, the age-long interest in the blue of the sky and the ocean resulted in the discovery of the Raman effect, an important building block in the house of science. It was not long before Raman had shown that every transparent gas, liquid, or solid would disperse light of a different wave length or a color different from the light falling upon it. The discovery of these radiations resulted in full confirmation of the important quantum theory of dispersion. Further research showed that the wave length of this secondary light radiation was a characteristic of the molecular structure of the gas, liquid, or solid passing the light. Now science has developed the Raman spectrum which is a virtual fingerprint of any substance that will pass light. Raman spectra and X-rays are being used regularly by industry as tools for determining the internal structure of liquids and solids of all kinds.

APPLICATIONS OF CHEMISTRY

Organic plastics

In continually greater measure, chemistry is laying the foundations for new industries. Organic chemistry, in particular, seems to be developing rapidly. Synthetic materials stimulated by the war will continue to be produced in greater and greater quantities. Of these, the organic plastics have attracted the greatest popular attention. The thrill we get out of the development of plastics is difficult for the chemist to understand. He had known about the basic kinds of plastics materials for a generation – some were developed nearly a century ago. Plastics materials naturally catch our fancy because of the extraordinarily colorful plastics gadgets that industry is turning out. But, in a larger sense, our interest is an economic one. Through a long depression from 1930 to 1937, we waited hopefully for the new industry that failed to come to our economic rescue. Then about 1938 we became aware that the new plastics industry was already here and *plastics* became a word to conjure with.

Phenol plastics

Everyone should know a few of the scientific facts about these revolutionary materials. The most common plastic (phenolformal-

dehyde or bakelite) comes from these simple chemical changes: Methyl or wood alcohol passes through a heated metal screen, which changes (catalyzes) the alcohol to formaldehyde. Phenol, obtained from coal tar, is then cooked with the formaldehyde, producing the synthetic resin in a half-finished state. The cooking is completed in the hot metal mold into which this resin, mixed with fine sawdust, cotton fiber, or asbestos fiber, is compressed under pressure. The resulting plastic is a cheap hard, amber or dark-colored material useful for many purposes, but brittle rather than shock-resistant. Anyone who has had a phenol plastics radio and a small boy in the same house will understand this reference.

The new technicolor

For a number of years plastics were always dark brown, black, or other dark shades. The first milk-white plastics which appeared in the early 1930's were most attractive as compared with other industrial products. These were the urea-formaldehyde plastics, which were soon available in all the tints of the rainbow. Ammonia and carbon dioxide (soda-fountain gas) combine to produce urea, which reacts with formaldehyde to form a clear resin solution. The filler must be milk-white to take fine colors. Chemically pure wood or cotton cellulose is used. The resin solution and the filler are mixed with color pigment and then dried to form the molding powder. Molded under low heat and pressure, the plastics product is translucent and colorful.

Acrylic plastics

Popular realization of the importance of plastics developed when the crystal-clear acrylic plastics (lucite or plexiglass) came on the market in 1938. So susceptible to color work that the resulting products rivaled jewels in brilliance, the acrylic resins met obvious needs and developed other uses on every hand. They are most valued for light effects, since they conduct light around corners in astounding ways. This is due to their excellent characteristics of internal reflection and transmission which keep the light internally reflected within the plastic itself until the light arrives at a roughened surface or area which lets it escape. The chemical name for this plastics material is methyl methacrylate.

Vinyl plastics

Another important material is vinyl plastic. Vinyl acetate stems from acetylene gas (formerly used to light automobile headlamps) and acetic acid (vinegar). A complex chemical change known as "polymerization" is needed to convert vinyl acetate to a plastic. Ultraviolet light, heat, and pressure speed up this process of linking molecules together to form *polymers*. When completed, these

reactions of several chemicals produce a new material, the elastic plastic. Used for safety glass, the vinyl plastic is an improvement because it stretches and does not break through.

Chemistry and chemical engineering

It is often difficult to distinguish between chemistry and chemical engineering. The subject of plastics production will serve as an example. We know that a plastic can be made from many different raw materials, such as oil, tar, coal, wood, cotton, peanuts, coffee, and cornstalks. The job of the research chemist is to experiment with the raw materials available to find out what kinds of plastics material can be produced and to iron out all difficulties inherent in the chemical reactions involved. It is then the function of the chemical engineer to take over and commercialize the process.

FIG. 6. SANITARY CHEMISTS STUDY WATER POLLUTION

The engineer must not only design a plant to produce the plastics material on a large scale but he must also be certain that the resulting plastic molding powder can be sold profitably on a highly competitive market. The practical engineer brings the dollar sign into the complex chemical reactions involved in the research chemist's process. Of course, some chemists are also able to commercialize their research developments and some chemical engineers direct research, so that the fields are interwoven, but the common division of duties is as indicated.

THE SCOPE OF CHEMICAL ENGINEERING

Unit operations

The main fields of chemical engineering are the design and operation of equipment of chemical plants for sugar refining, paper-making, oil refining, alcohol production, food preservation, rubber synthesis, and other processes in which a chemical change is produced. The operations involved are broken down into the so-called *unit operations* of drying, vaporizing, crushing, grinding, filtering, humidifying, and dehumidifying. The plant process can usually be broken down into certain chemical processes such as alkylation, sulfonation, nitration, hydrogenation, and so forth, known as the "unit processes." When a new product is to be made, the chemical engineer is able to transfer his knowledge of the successful use of each of the unit operations in other plants to the new design. He also has a fund of information about the kinds of machinery that have proved successful in operation under various conditions, although he usually leaves the design of moving machine parts to the mechanical engineer.

Theory

The main theoretical tools of the chemical engineer are mathematics, chemistry, and a knowledge of the transfer of heat and the flow of fluids. Plant processes involve chemical reactions which are usually speeded up by increasing the temperature. Heat must also be added at one step and removed at another stage whenever the process involves vaporization and condensation or drying. Materials are commonly handled either as liquids and vapors, or as solids in suspension since the transportation of dry solids is often troublesome and costly. Hence, every step in the design of the process requires a knowledge of the flow of fluids. More and more, the study of heat flow, fluid flow, and chemical engineering thermodynamics requires a good background of mathematics. It goes without saying that a knowledge of physics is also essential.

Development, design, and operation

Like all other fields of engineering, chemical engineering is broad enough to give people of many interests an outlet and a chance to become successful. The successful chemical engineering student who is also an excellent chemist should consider research and process-development work. After the chemist has produced important new reactions in the laboratory, the development engineer has the job of trying out on a small scale a process which makes use of these new reactions. He must not only understand what the chemist did within his test tubes, but he must be clever at finding and curing the innumerable difficulties arising from this change to "pilot-plant operation." The person with more practical interests will become an operating engineer. In between these two are the plant designer and the engineer who supervises construction or changes in the plant and then directs the adjustments that bring the plant into successful production.

A GREAT BRIDGE COLLAPSES

Tacoma Narrows Bridge

On November 7, 1940, newspaper extras all over the country headlined the collapse of the world's third longest bridge. A few days later, the newsreels showed millions of people the first bridge failure they had ever seen. To the public this was probably the most spectacular newsreel of a decade; to the engineer it has become a scientific record of the greatest significance. Great popular interest was stimulated and the structural engineer for a short time became a public figure criticized and held in awe. Certainly, after seeing the moving pictures of that writhing, tossing monster, equal in over-all length to a half-dozen battleships, people began to realize the tremendous responsibilities involved in the engineer's job of "controlling the forces of nature."

Slender span

The Tacoma Narrows Bridge was opened to traffic on July 1, 1940. It was a slender, graceful, suspension bridge spanning 2800 ft. between the main towers from which 1100-ft. side spans extended to the shores. This entire bridge was constructed for about 6,000,000 dollars, a low figure only a fraction of the cost of several of the great suspension bridges. A low-cost structure was possible because the traffic needs called only for a two-lane bridge with sidewalks so that the two supporting cables suspended over the 425-ft. towers were but 29 ft. apart. The light weight of the bridge could be supported by comparatively small cables, 17⅛ in. in diameter, and the designers concluded that the roadway stiffening trusses, which

by ordinary standards would have been 25 ft. or more in depth, would be out of place on this light structure. They used two 8-ft. girders along the outer edges of the sidewalks as seen in Fig. 7.

"Galloping Gertie"

Even during construction, the Tacoma bridge showed some tendency to sway and to oscillate up and down. Other bridges had acted that way too, but this one did not stop when the floor had been completed. Rumors began to pass around the country about the undulations of "Galloping Gertie," as the bridge had been nicknamed. One man related that he drove across the bridge during a 30-mile wind and saw a car, well ahead of him, disappear in the trough of a wave passing along the floor. At that time this account seemed like the exaggeration of a non-engineer, but the official report of the failure states that oscillating waves as great as 5 ft. had previously been observed, which is about the height of a modern motorcar. The oscillations were seen to change back and forth from six or eight up to thirty-six or thirty-eight per minute.

Five-foot oscillations

Courtesy Professor B. W. Farquaharson

FIG. 7. TACOMA NARROWS BRIDGE IN MOTION BEFORE ITS COLLAPSE

Note the angle between the light posts and the vertical suspenders which is even visible in this diagonal camera shot

Restraints

Engineers were engaged in studying the bridge. They were planning additions to increase its stiffness and reduce the oscillations that by then were recognized to be dangerous. Some center diagonals and hydraulic buffers had been added before the bridge was opened to traffic in July. These restraints proved inadequate and, on October 7, tie-down cables were attached to the girders of the side spans anchoring them to the ground. The side spans then quieted down, but the undulations of the center span were not greatly influenced. Promptly the engineers began to study the effects of stay cables to go from the tower tops down to the center span girders. These ties were never installed because the bridge collapsed before they could be designed.

The day of the failure

We get the picture best by reading the words of eyewitnesses to the twisting, writhing, and final collapse of this ill-fated structure. The following accounts are from several observers, since no one person saw every detail of the collapse.

7:30 A.M., Nov. 7. "I went to mid-span and took the velocity of the wind, reading 38 miles an hour on the anemometer. The bridge bounce was noticeable but of no great magnitude."

9:30 A.M. "The motion kept building up and at about 9:30 A.M., I drove out and parked in the center of the main span and counted the number of cycles the bridge was making per minute and found it to be thirty-five."

10:00 A.M. "A violent change in motion was noted. This change (from vertical oscillation or bounce to a writhing motion as shown in Fig. 7) appeared to take place without any intermediate stages and with such extreme violence that the span appeared to be about to roll completely over." From then on the main oscillation was such that the center of the span stayed reasonably quiet, one end rising and the other end falling.

Approaching collapse

10:00 A.M. "The center span was swaying wildly, it being possible (from the side) to see the entire bottom side as it swung into a semi-vertical position and then the entire roadway. It was at once apparent that instead of the cables in the main span rising and falling together, they were moving (now) in opposite directions, thereby tilting the deck from side to side."

10:00–11:00 A.M. "It was plain at this stage that the lower limit of undulation was almost exactly flattening the girder to a straight (horizontal) line from tower to tower while the upper limit must have been something over 25 ft. above this level." This twisting made the deck slant nearly 45 degrees back and forth. At times when the near section of the deck twisted clockwise, the far section beyond the center of the bridge was seen to be twisting counterclockwise.

11:00 A.M. "I went to the observation house and noticed small chunks of concrete falling. I supposed they were the cracked sections around the light posts. Hearing someone yell, I rushed down to the parking area and observed that the mid-span was broken completely in two, the ends working against each other with a sidewise motion."

11:10 A.M. "Shortly thereafter two or three sections of roadway fell. Then approximately 600 ft. of girders and roadway fell in. I heard a deafening crash and looking up saw clouds of dust and a snarled mass of suspenders and deck whipping. The east approach dropped below my vision and then everything became quiet."

Lessons learned

The Tacoma bridge was by far the most flexible long-span bridge ever constructed. Its collapse was the third major bridge disaster of this century. This bridge was amply strong to resist all loads but its tendency to oscillate under the influence of wind gusts caused its failure. Its shallow stiffening girder and its narrow roadway combined to produce little resistance to twisting action. Consequently, as soon as the bridge fell into this kind of oscillation under the action of the wind, the motion became enormously magnified until the floor system was torn to pieces. Studies of models and mathematical investigations will eventually show us how to build slender bridges that are safe from cumulative vibration, but, in the meantime, we can follow the old rules that have proved safe in the past. This "full-scale experiment" may prove to be the stimulus for an important advance in bridge engineering.

CIVIL ENGINEERING

Structures, highways, hydraulics, sanitation

In the failure of the Tacoma bridge you have seen some of the problems faced by the structural engineer. He also designs and constructs buildings in steel and concrete, dams, docks, hangars, and every type of structure needed by industry. The highway engineer lays out and builds roads, streets, and airports. The hydraulic engineer designs irrigation systems, levees, and other flood-control projects. The sanitary engineer is responsible for water supply and sewage disposal and also for the treatment of industrial wastes. The municipal engineer often acts as a city manager. The foundation expert is much needed for the design of deep piers, footings on troublesome soil, and underwater construction. These are the divisions of civil engineering and, in the main, they are involved with public works. More than any other, the civil engineer works directly in the interest of the public.

"FATIGUE" OF METALS

Engineers sometimes use a word in a technical sense that is descriptive because of its meaning in the everyday world. *Fatigue* when applied to the fracture of engineering materials is such a

Axles even break on trains

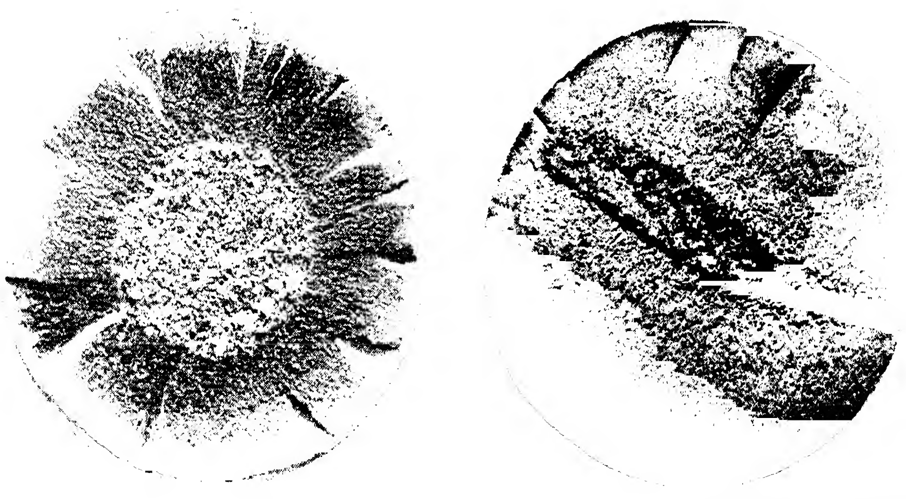
word. You may find something of interest in the following experience with a fatigue fracture. A small group of engineers was riding in a railroad dining car when, without warning, the car began to bounce so severely that the dishes upset and seemed about to leave the table. From the car window, we could see a stream of fire flaring backwards along the train. When the train stopped, we climbed out and found the thick steel axle underneath the rear of the diner broken cleanly through. The apparent fire had been a stream of sparks caused by the wheels and axle pounding against the rails.

Fatigue crack

Under the conductor's flashlight, the surface of fracture of the broken axle showed a characteristic pattern. Around its circumference was a smooth band while the center area appeared very rough and coarsely crystalline by comparison, as illustrated by Fig. 8. The trainmen said that the steel had "crystallized." Actually, this was an example of fatigue failure. After long service, a microscopic surface crack had developed, probably starting at some invisible flaw. At each revolution of the axle, this tiny crack

Progression

opened and closed, its faces being pounded to perfect smoothness as it extended around the axle and gradually worked its way inward. Perhaps months or years after the crack started, it had



Courtesy Professor H. F. Moore

FIG. 8. FATIGUE FRACTURES AS FOUND IN AXLES AND SHAFTS

The fatigue crack in the shaft illustrated at the left gradually extended all around the bar and reduced the area to the small inside circle; then a sharp fracture occurred producing the rough crystalline appearance at the center. In the rotating shaft illustrated at the right the fatigue cracks worked in from two sides.

penetrated deeply enough to cause the axle to fracture. The final rupture occurred suddenly, producing the usual grainlike surface over the center section of the axle characteristic of shock fractures.

Life span of metals

Without thinking about it, you have undoubtedly produced many fatigue fractures yourself. Everyone knows that he cannot break a wire or a strip of metal by bending it once, but that it can be broken readily by bending it back and forth a few times. If you make the angle of bend rather small, you will have to repeat the bending operation several hundred times to break the metal. In fact, several million repetitions may be necessary if you bend the metal part so lightly that it acts as a spring and automatically becomes straight when you release it. Actually, there seems to be a limit of bend below which you can repeat the operation forever without rupturing the metal. This is known as the "endurance limit," which is expressed by engineers as a stress in pounds of load per square inch of material. This definition will be clearer if you think of the load not as a bending force but as an axial load alternately stretching and compressing a length of metal rod. Then, the ultimate stress is the static load that will break the rod divided by the cross-sectional area of the rod; and the endurance limit is about 50 per cent of this ultimate stress. Stresses produced by bending can also be computed, and endurance limits can be set for this case and for other cases.

Progressive cracking

The phenomenon of fatigue has not been explained completely, but a working hypothesis that accounts for most observations is as follows. Steel and other metals are crystalline, which means that they are made up of minutely interlocked crystals. Under heavy load or stress there is some slipping between crystals which may be too severe to disappear entirely when the load is removed. Such tiny internal slippage may be of no importance if it occurs just a few times, but if it is repeated indefinitely, and particularly if the slip is reversed from left to right, a crack will begin to form, resulting eventually in a fatigue fracture. In other cases minute or microscopic cracks or flaws may already be present in the material; such flaws tend to form fatigue cracks after repeated loading.

When fatigue becomes critical

For many years engine designers were able to build successful steam engines without giving any consideration to fatigue. These were the old reciprocating engines that you still see as standbys in power plants. They turned at no more than 300 r.p.m. (revolutions per minute) which is about the engine speed of a freight

locomotive. If any part broke, the designer simply made it heavier since weight was not important. Then the automobile came and speeds of rotation went up to 2000 and 3000 r.p.m.; also, excessive weight became objectionable. In the airplane engine not only is the speed high but every ounce of excess weight must be trimmed away. For power plants, the steam turbine turning at 1800 or 3600 r.p.m took the place of the slow-speed reciprocating engine, while supercharger turbines may rotate above 10,000 r.p.m. Tests show that fatigue failure occurs within 10 to 15 million reversals in direction of the loading or usually not at all. At 10,000 r.p.m. a turbine will make 15 million revolutions in a single day. Evidently, if any reversing stresses (caused by alternation in the direction of the loading) occur in the turbine, they must be below the endurance limit or its life will be very short indeed. However, since the turbine has no reciprocating parts, only vibration from lack of balance can produce such stresses. Some parts in automobile and airplane engines can be kept small and therefore light in weight by designing them for replacement after some fixed number of hours of service. Necessity often dictates such practical procedures.

Automotive and aviation engines

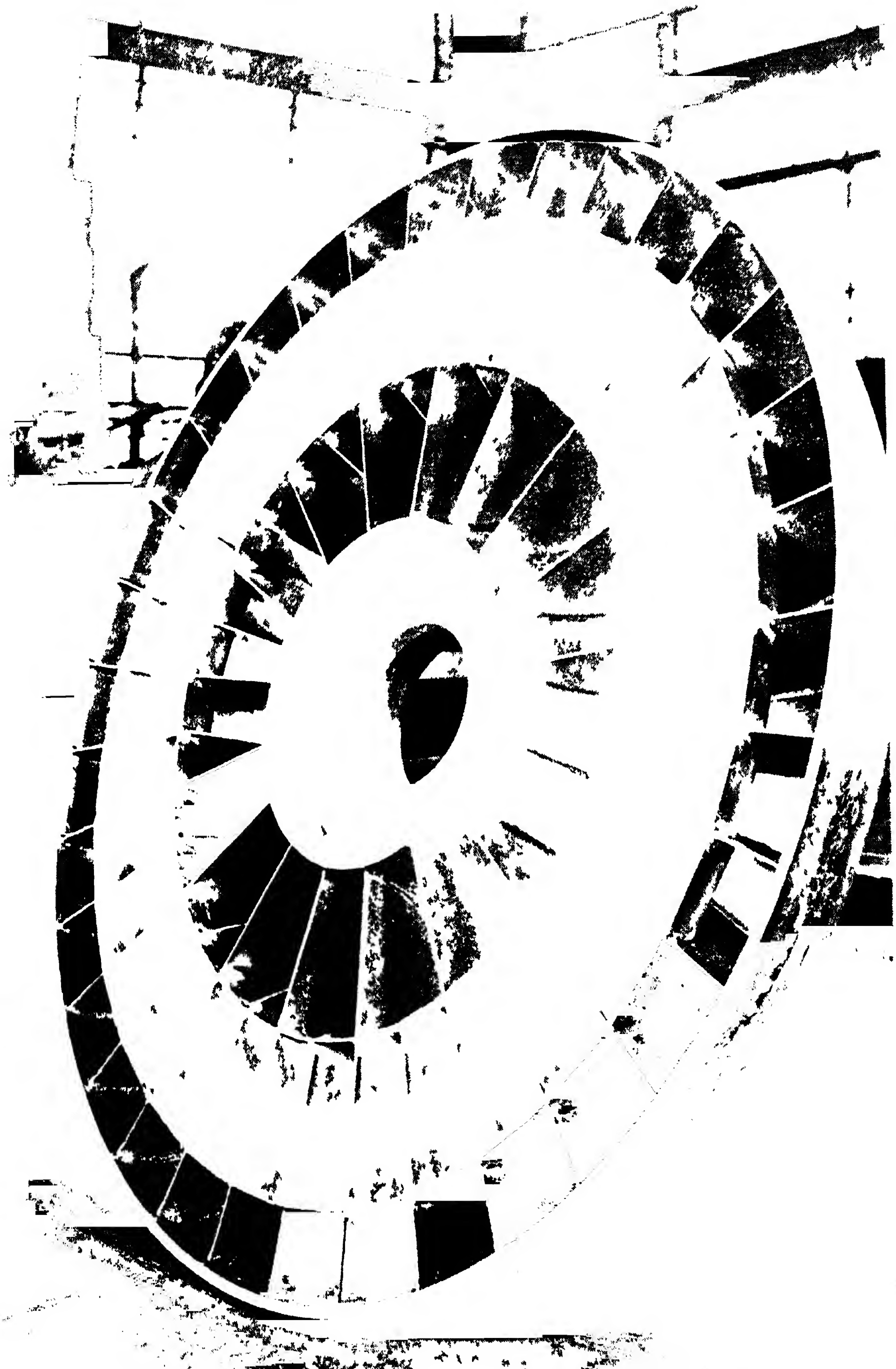
This brief qualitative explanation of fatigue must be supplemented with quantitative data before it can be applied to design. However, it is useful to know something about this phenomenon early in your study of engineering so that you may be looking for its effects.

MECHANICAL AND AERONAUTICAL ENGINEERING

Machines and airplanes

The engineers who have to worry most about fatigue are mechanical and aeronautical engineers because they deal with fast-moving reciprocating machinery. The hydraulic engineer may have to plan the design of a hydraulic turbine and the electrical engineer does design electric motors, but both depend upon the mechanical engineer to design the bearings, the system of lubrication, and the general structure of the machine, based upon the strength of the materials to be used.

Of course, the airplane engine is a natural field of study for the mechanical engineer. The plane itself is a structure rather than a machine since it does not have rapidly moving parts but moves as a whole. Its design is similar to the design of a bridge truss; like the bridge, it carries its own weight and the weight of its cargo. At the same time, the frame of the airplane must resist air pressure



Courtesy Luken Steel Company

FIG. 9 ROTOR OF A HYDRAULIC COUPLING

This huge all welded hydraulic coupling for smoothly transmitting power is

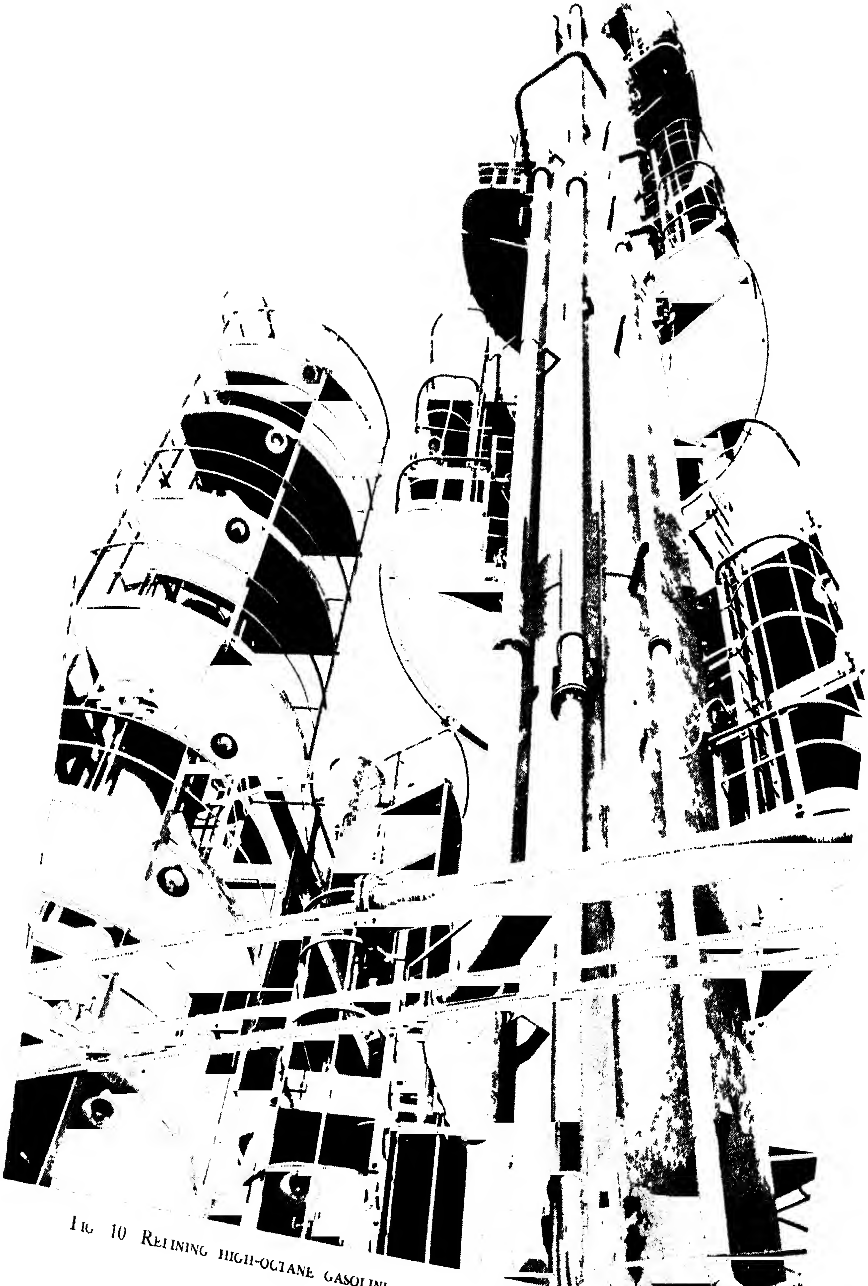


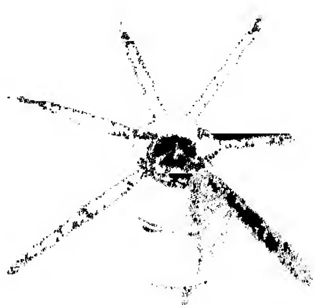
FIG. 10 REFINING HIGH-OCTANE GASOLINE

and aerodynamic forces that produce unusual loads and cause vibration. In this respect the airplane is like the oscillating Tacoma bridge. Unlike the bridge, which oscillated no more than thirty-eight times per minute, the tail or wing of an airplane may vibrate many times per second if there is a slight unbalance and thus become subject to early fatigue fracture. So again we adopt the compromise of designing the plane for a limited life, with the intention of retiring it from service before failure becomes probable.

The scope of mechanical engineering

There are many other fields of mechanical engineering. The steam engineer deals mainly with large power plants. The internal-combustion expert or diesel engineer faces many problems with his engines for automobiles, airplanes, ships, streamlined trains, and small power plants. Air conditioning, which today includes heating, ventilating, humidifying, cleaning, cooling, and purifying air for any building, is an active field for study. Refrigeration engineering is important in the storing, preserving, and transporting of foods. And, of course, we must not forget the production engineer, usually of mechanical-industrial training, whose job it is to keep the machines of industry continually at work turning out goods by mass production. He must be expert in dealing with men, in routing materials through the plant, in developing safety measures, in cost control, and in promoting generally efficient procedures. Years ago he was the unwanted "efficiency expert," but today he is accepted and respected in every progressive industry.

Industrial engineering



From Diesel Engineering Handbook

FIG. 11. OIL SPRAY INJECTED INTO A DIESEL ENGINE CYLINDER

EXPERIMENTS ON MODELS

Usefulness of model studies

When a new structure, machine, or electrical system is proposed, we face the problem of making a design largely from theoretical or mathematical studies. Practical engineering works are often too complex to analyze exactly by mathematical methods; hence, we must depend upon experience with similar engineering works or upon experiment. Since it is impractical to build a 30-story building or a 1000-ft bridge or a 100-mile electrical network for testing, we naturally turn to model tests. When we were discussing chemical engineering, pilot-plant operation was mentioned as one step in the development of a new process. The pilot plant is a model and the information gained in its operation is used to aid the designer of the full-scale plant. When theoretical calculations prove correct in pilot-plant operation, we feel more confident in applying them to the full-scale plant.

Mississippi River model

Dam and waterway construction is a field in which model experiment has been highly successful. The design of a model to represent its full-sized counterpart is no simple matter, however. The reduction of all dimensions to one-tenth or one-hundredth of natural size might seem a reasonable device, and while this geometric similarity is one essential of perfect similarity, it is not the most important factor to be taken into consideration. Similarity of motion must be studied, which means that the corresponding flow paths must be similar and that the velocities at corresponding points bear a definite constant ratio determined by the relative size of the model to the prototype.

At Vicksburg, Mississippi, United States Engineers have produced a model of the Mississippi River which is about a mile long. In this model, which is illustrated in Fig. 12, it was necessary to dispense with geometric similarity to obtain closer dynamic similarity. The influence of straightening reaches of the river, placing a dam on a tributary stream, or providing floodways to relieve the levees in time of flood have been studied effectively. There is reason to believe that these studies have led to construction work which will reduce the damaging effects of the disastrous floods that occur every decade or so along the lower Mississippi River.

Models of structures

Structural models for buildings and bridges are proving increasingly useful. One type is the bakelite plastics model viewed with plane polarized light or light restricted to vibration in a single

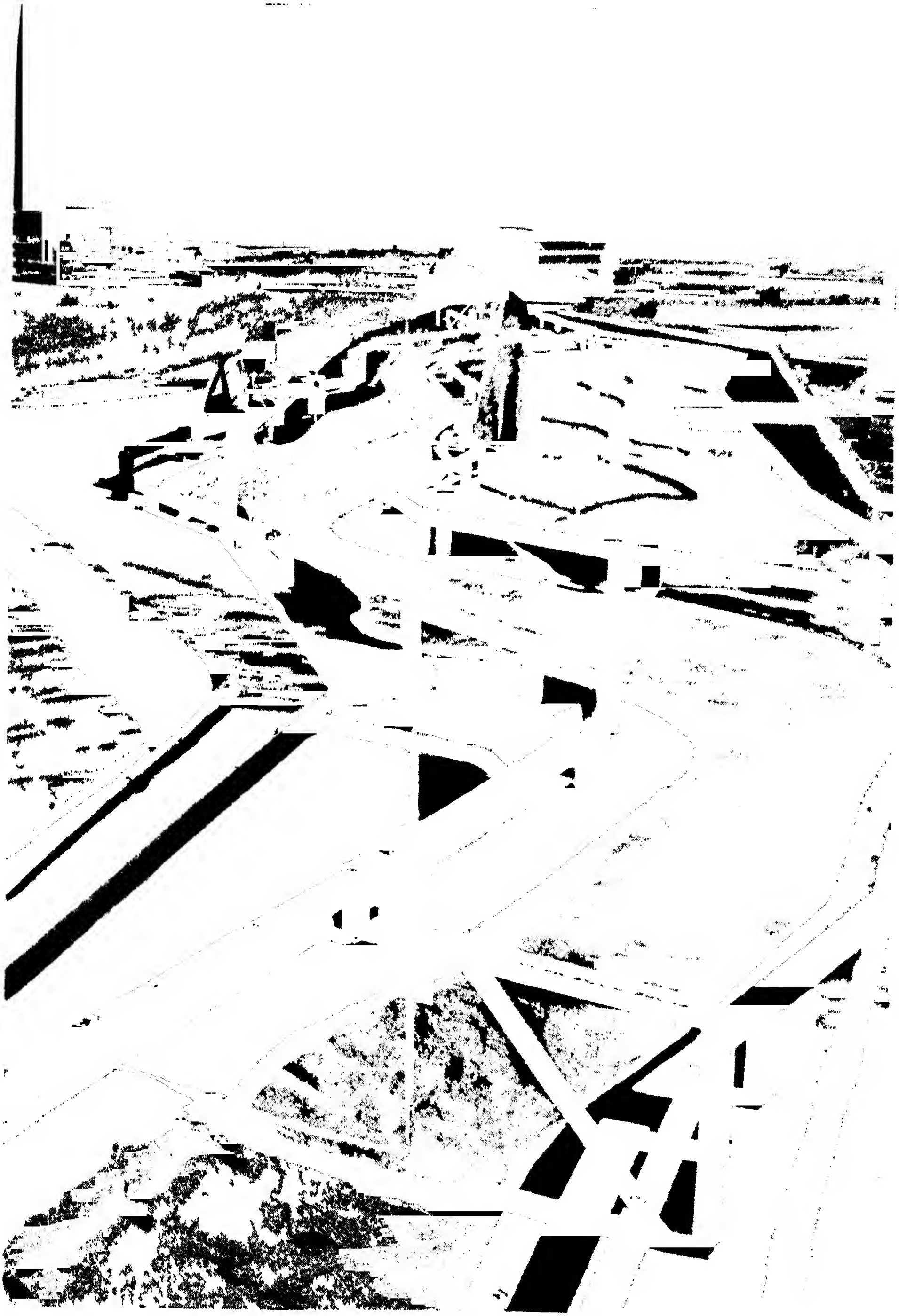
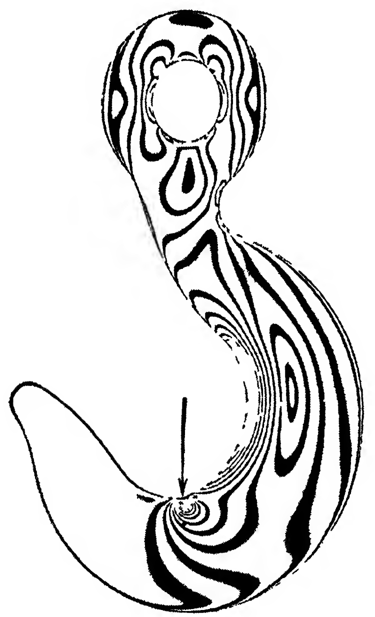


FIG 12 MILE-LONG MODEL OF THE MISSISSIPPI RIVER AT VICKSBURG

direction such as the vertical plane. It is actually possible to *see* the distribution of the stresses in such models because of colored patterns produced by the effect of the polarized light passing through the stressed bakelite model. By using a light of one wave length, we can change the colors into light and dark bands, as shown by Fig. 13, so clearly defined that stresses may actually be calculated from a photograph of the loaded model. However, the engineer then faces the problem of translating these stresses of the small bakelite model into significant stresses for a steel or concrete structure a hundred times as large. This transference is not really practical so another procedure is used; we develop theoretical equations to compute the stresses in the model and check the accuracy of these equations by photoelastic determination of the stresses in the model. This check gives us greater assurance that our equations may be applied without probability of serious error to the structure itself. Photoelastic models have also proved useful in the study of stresses in gears and in other machine and airplane parts.

Photoelastic models



Courtesy M. Hetenyi

FIG. 13 TRANSPARENT MODEL OF A LOADED HOOK VIEWED IN POLARIZED LIGHT

Models of Tacoma bridge

The failure of the Tacoma bridge brought forth numerous models of suspension bridges for study. Many were direct models of Tacoma itself. Most of these models showed a tendency to gallop or oscillate when exposed to an air current crossing the roadway at an angle. Of course, it is no simple matter to design a proper model of such a complex structure as a suspension bridge and particularly a vibrating model. Here, too, a true-scale model would have little meaning. For example, the stiffness of a bent bar varies as the length and breadth but it changes as the cube

of the depth. Nevertheless, these variables can be studied and a simplified model obtained that will simulate the action of the real structure. It seems unlikely that any major suspension bridge will again be built without a model study of its vibrational characteristics.

Models of electrical networks

One of the simplest engineering works to reproduce by a model is an electrical network. Each interconnection between two power systems changes the conditions under which the current flows. Yet the system may be so complex that a mathematical study would be a long, tedious process. Fortunately, a model of almost any electrical network can be arranged on what is commonly called a *calculating board.*

The calculating board

The fundamental relation (Ohm's Law) controlling the flow of current in any system is $E = IZ$, or voltage, E, equals the product of the current flowing, I, and the electrical impedance, Z. Impedance, of course, is any restriction to the flow of current. There are no units here that necessarily depend upon size, and we conclude, therefore, that the circuit may be a hundred miles or a few feet in length without influence upon its electrical characteristics. For convenience, we prefer to make the currents and voltages of the model much smaller than those in the field network but still large enough to measure accurately with available instruments. Evidently, by changing impedances of all parts of the circuit proportionately, any convenient magnitude of voltage or current can be obtained. As a matter of interest, such calculating boards exist at a number of educational institutions and central stations, and they are much used. It is possible in a few hours to plug in a circuit of given resistances (or impedances) on a calculating board to serve as a model of a complex network and to complete the study of the network by measuring the voltages and currents at as many points as desired.

ELECTRICAL AND COMMUNICATION ENGINEERING

Power engineering

The most important engineering advancement of the past half-century has been the development of electric power. The electrical power engineer may be concerned with electric power generation; he may also be interested in electric power transmission or in the design and application of the necessary equipment for transmission, such as transformers, power lines, or lightning protection devices. High-voltage, long-distance transmission is a specialized

field in which there have been important recent developments. Hydropower generation represents another specialized power field. Power-consumption devices, from industrial motors to home vacuum sweepers, provide jobs for many electrical engineers. Perhaps the most important of these special fields is illumination. First the Edison incandescent lamp, then the gas-filled lamp, and finally the fluorescent tube have been successive stages in the development of efficient lighting. The lighting expert finds more need for his services as people become appreciative of the importance of good illumination. For example, the standard of light intensity for proper illumination of a drafting room doubled with the introduction of fluorescent lighting.

Communication engineering

When De Forest began to experiment with radio transmission by use of the vacuum tube (1906), he opened up another field for the electrical engineer that may well become equal in importance to the electric power field. The magic radio of 1920 became the public's darling by 1930 and the widest form of popular entertainment by 1940. Television was nearly ready for commercialization by 1937. The academic interest in extremely high frequencies led to the highly secret Radar which the Army took over for the development of detection devices. Its success in these three fields leads us to conclude that the electronic tube has an important industrial future.

Industrial electronics

Communication or industrial electronic engineers will be needed to devise, install, and operate electronic controls for automatic industrial machinery, for medical heating devices, and for dozens of uses not yet imagined. Evidently, this is a most promising field of study for the man who has mathematical talent and theoretical interests. The development of vacuum-tube devices has passed out of the stage of gadgeteering and it is not likely that major contributions to the field of industrial electronics will be made from now on by the inventor without technical education. In fact, the common belief that "wizards" such as Edison, the Wright brothers, and De Forest were geniuses without scientific education is false. Such men were able and willing to educate themselves in any field in which they happened to be at work.

THE DEVELOPMENT OF RADIO

De Forest

Lee De Forest invented the electron tube in 1906. He was so far ahead of his competitors and of scientists in general that no one

even knew that his *audion* was really an electronic device until five years after he had put it into use for his first love, wireless telegraphy or radio. He broadcast Caruso's voice in 1910 and thus earned his title "the father of broadcasting."

Wireless telegraphy

De Forest's first Chicago experiments were in developing a wireless detector. After considerable conflict with his employers because he neglected his job to work on his own ideas, he finally took a night-school teaching position and experimented on his own. From then on, his work progressed rapidly and in 1901 he picked up wireless signals from a boat in Lake Michigan five miles away.

From that time there was a race on between Marconi and De Forest to see who could most rapidly develop and publicize his own inventions in wireless telegraphy. Marconi won but De Forest produced some extraordinary promotional stunts and actually headed a company that mushroomed into a multimillion dollar corporation before it collapsed. The fact that the public's hopes were dashed and that investments were lost left the overoptimistic inventor open to severe criticism. He had reaped the expected reward of rash commercialization not backed by long painstaking experimental development.

Developing the audion

Broke, De Forest began to hunt for another invention to salvage his lost prestige. It is said that he had noticed a gas light flickering in rhythm with a spark coil and this led him by analogy to study the influence of separate currents upon electric-light filaments. If this story is true, it is a case in which an incorrect analogy led to a great invention. Of course, this effect of one current upon another is the fundamental accomplishment of the electron tube which De Forest developed after hundreds of unsuccessful attempts had merely spurred this erratic genius on. When he succeeded, he knew that he had at hand the one device that could take the weak wireless signal coming through the ether and magnify it to original strength.

Lag in commercialization

De Forest in 1906 had radio at his finger tips, but he had to wait until 1920 for its commercialization because the world was not yet ready for such a revolutionary invention. He spent many of the years from 1920 to 1934 fighting in the courts for his rights as inventor of the electron tube. His patents were finally upheld and he received needed royalties. De Forest then returned to his laboratory, hoping to add another milestone to his list of contributions to electrical technology.

SPECIAL FIELDS OF ENGINEERING

Metallurgy

In twenty years, from 1920 to 1940, the alloys of iron and steel increased a hundredfold in number. Aluminum, dropping greatly in cost, became a competitor of steel for those structures and machines in which weight was a handicap. Magnesium, finally produced from sea water, began to compete with both steel and aluminum for light-weight uses. Following enormously increased war production, aluminum and magnesium may be expected again to drop sharply in cost and to compete on a peacetime basis with steel possibly even for bridge and building construction. The alloys of the light metals may then increase in number for special uses as did the alloys of iron a generation earlier. Rarer metals such as tungsten, vanadium, and molybdenum have become common or at least less rare and have contributed greatly to technology for particular uses or as alloying metals.

Production metallurgy

With all of these developments in the air, it is small wonder that metallurgy has attained the status of a special field of engineering. Sometimes called a metallurgical engineer, the metallurgist has two major opportunities for service. As a production metallurgist, his job is to produce pure or alloyed metals from the raw materials and ores available. The open-hearth furnaces in our steel mills, for example, are under the eye of the production metallurgist. He must control the heat, the contents of the charge, and the time in the furnace to assure a uniform quality of steel ingot.

Physical metallurgy

The other major field of metallurgy is under the control of the physical metallurgist. His work is more scientific in that it is always changing. He is responsible for investigating the usefulness of new alloys. He studies the crystal structure of metals often by X-ray photographs and develops methods of treatment to improve their strength or ductility or resistance to fatigue. He also investigates the causes of corrosion and devises means for reducing the deterioration of metals. In manufacture, the physical metallurgist has the responsibility of selecting special metals for particular products and of devising, along with the mechanical engineer, methods by which these products will be worked during production. Weldability of metals is an important part of his investigations.

Metallurgical engineering was long considered a branch of chemical or mechanical engineering. Certainly the metallurgist

must understand chemistry since physical chemistry along with physics forms the background of most of his work. The scope of metallurgy is such that we must look upon this field as so important that it requires the full-time attention of specialists and not the casual attention of the mechanical or chemical engineer.

Architecture

The architect is not really an engineer although he is a builder. Naturally, the fields of architecture and structural engineering flow into each other without a sharp line of demarcation. Some architects become structural engineers and some structural engineers become architects. The main function of the architect, however, is to design the arrangement of the building and its details to serve the needs of the owner and to harmonize with its surroundings. The structural engineer designs the supporting structure and then one or the other follows the building through the construction period to see that it fulfills all plans and specifications.

Structural design

FIG. 14. A NEW ALLOY OF EXCEPTIONAL HARDNESS AND ELECTRICAL CONDUCTIVITY

Notice the imprint left on the steel bar at the left when the harder alloy at the right is forced down upon it. The alloy is not only harder than mild steel but it has about the electrical conductivity of copper. This is a combination of great importance to industry.

The architect and engineer do not always see eye to eye because the engineer is interested primarily in designing a safe, economical, and simple structure while the architect often complicates the structure to achieve some element of convenience or to maintain style. The modern architect follows closely the ideal of "arrangement to serve the intended function" and is more likely to be in agreement with the engineer than were his classical predecessors.

City planning

The architect must have most of the qualifications of the engineer plus an artistic ability. He has less need for mathematics and is not usually expected to study calculus. He must be imaginative and he should exhibit a special interest in drawing. Some architects have extended their planning function beyond the single building to include residential or business areas, towns, or whole regions; their work is called City Planning or Regional Planning. On the inside of the front cover of this book is a fine example of replanning to eliminate the smoke nuisance of the city. The unsatisfactory appearance, congested areas, and inadequate traffic ways of our cities call for replanning and rebuilding on a large scale. The reconstruction of shattered European cities after the Second World War should bring before us the need for replanning America's metropolitan areas.

Naval architecture

The naval architect is really a marine engineer. He designs and supervises the building of ships of war and ships of commerce or pleasure. His training corresponds to that of the structural engineer since a ship is really a structure similar in many respects to a bridge or a building. The marine engineer is also a specialist in propulsion and in diesel engine operation — the usual motive power for modern vessels. In fact, he needs to be a specialist in heating, ventilation, refrigeration, electrical engineering, and a half-dozen other fields but, since this is manifestly out of the question, he usually acts as supervisor to a group of engineers — each being a specialist in some problem of ship design. Naval architects are trained at only a few colleges since the demand for their services is not great in peacetime.

Engineering specialties

There is a large number of engineering specialties for which courses of study are available in certain institutions. We have already mentioned sanitary engineering which is one phase of public health engineering. Biological engineering extends the scope of these studies to the food industries, where the biological engineer is interested in the handling and packaging of foods and,

especially, in the processing of foods that require fermentation as one step in their preparation.

There are a number of engineering specialties that are directed toward a single industry. The major one has already been mentioned — aeronautical engineering. Others that are considered less critically important today are railroad engineering, automotive engineering, mining engineering, gas engineering, and petroleum-production engineering. Such engineering specialists are much in demand when the industry considered is expanding or rebuilding.

Mining and petroleum engineering

"A roof prism is used in a gun sight's elbow telescope, which enables an anti-aircraft gunner, for example, to look horizontally into the eyepiece and see his target overhead. The elbow telescope inverts the image; the roof prism's function is to turn the image right side up. Roof prisms are thumb-sized, polished crystals whose two top facets are shaped like a peaked roof. In manufacture, a piece of glass is first sawed roughly to shape, then ground to exact proportions by a delicate hand. In the final product, every facet must be absolutely flat, the right angle between the roof facets must be accurate within one 1800th of a degree."

Fig. 15. Roof prism for gun sight

Courtesy Perkin-Elmer Corp.

The roof prism is the most precise article ever manufactured in quantity. The above quotation is from *Time* magazine, October 4, 1943.

When it becomes stabilized, as in the cases of mining and railroad transportation, the opportunities for young engineers become limited. Other specialties of possible long-term interest are safety engineering, fire-protection engineering, and public-service engineering. You should not pick any specialized field for study until you have familiarized yourself with its purpose and its opportunities. For most persons, the best opportunities are likely to exist in the broad fields of civil, electrical, mechanical, and chemical engineering.

TECHNOLOGY AND GOVERNMENT

Engineers in politics

When this was written there were 280,000 engineers, 70,000 chemists, 7000 physicists, and perhaps 5000 trained mathematicians in our country. Obviously the key positions in technology are

held at any given time by a small part of this group, perhaps by no more than 10,000 or at most 25,000 persons. There have been some misgivings expressed by writers who did not understand engineers and scientists that this small group of professional people might attempt at some time to exert an undue influence upon government. Evidently, if such a small group of technologists could be organized and controlled by unscrupulous leaders, the public safety might be jeopardized because the power supply of the country would be under the control of this small minority. Evidently, too, political pressure could be exercised through such a group.

Public office

Worry on this score seems wholly unnecessary because the entire conception of pressure groups conflicts with the ideal of public service deeply ingrained within the engineering profession. Also, engineers and scientists form the least politically minded group imaginable. The only engineer who became a president of the United States, Herbert Hoover, 1929–1933, had never before run for political office. Few engineers or scientists have been elected to Congress and it is a fact that they are not sought for public office. The reason, it appears, is that the technologist is not trained as a public speaker nor does his work offer him many speaking engagements; he is seldom a spellbinder.

Public speaking

Public speaking is the ability of the lawyer and, of course, the law has always been heavily represented in our elective offices. It would be healthy for the country if more engineers and also scientists could be induced to run for elective office. But, from the practical point of view, until the engineer trains himself as a public speaker he is not likely to be chosen as a candidate because he would stand little chance of being elected. A young engineer is well advised to take advantage of each opportunity to improve his ability to speak well. It will influence every contact that he has with others.

HOW TO BECOME AN ENGINEER

College study

Of course, the most direct procedure for becoming an engineer is to spend four academic years of nine months each in an accredited engineering school, although it has always been possible for an industrious person to study during the summers and to complete the regular engineering curriculum in three years. Shorter courses train technicians who may become engineering assistants, but such short courses are not intended for the education of the professional engineer. The cost of college study varies greatly. A maximum

reasonable range appears to be from four hundred to twelve hundred dollars per academic year. By using the summers as work periods and by serving the university during free hours, many young men practically pay for their own college education. Scholarships and loan funds are also available.

Part-time study

Many engineering schools operate cooperative courses of engineering study. The usual plan is for the student engineer to spend from two to four months in class alternating with similar work periods in industry. An engineering degree can usually be earned in five years of such a cooperative program. Incidentally, the income earned in industry may be nearly adequate to cover all educational expenses. The other common part-time program is that of evening study. Evening classes in engineering and science are offered by most of the universities located in the larger cities. Young people holding daytime jobs come regularly to evening school. By using ten or eleven months of the year for study, it is possible for an employed person to complete the requirements for an engineering degree exclusively in evening school in seven or eight years.

Home study and experience

It is also possible to develop your own background of science and engineering by home study and practical experience. This is the hard way. Every textbook author has received letters from persons who have waded through his text without the benefit of an instructor. Ofttimes such persons have a better grasp of the subject than students who have just passed the course. I should like to encourage home study from standard engineering textbooks. They are harder to use than simple home-study texts, but when you have dug your way through to the end, you will have the confidence that comes from knowing that you have covered the same ground as the engineering college student. If you also make regular use of an engineering library, you may even obtain a better knowledge of the subject than some engineering graduates.

Theory, research, and graduate study

American procedures in engineering education follow the belief that at the end of the usual four-year period of college study the engineer should be able to step into industry and perform a useful service; in other words, he should have a practical education. This means that his background of theory has of necessity been neglected in spots and certainly has not been developed as fully as we might wish. Particularly, for those engineers who are to become designers of structures, processes, or machines, and for

those who wish to conduct research, testing, or development work, a further study of engineering theory is essential. Our graduate schools exist for this purpose. They usually grant the master's degree after one year of postgraduate study and the doctor's degree after three years. Although the jobs for which graduate study is a desirable preparation may represent only about a quarter of all positions in engineering, for the person with a scientific turn of mind these are the most attractive opportunities.

Your job in engineering

The engineer deals with men, money, methods, and materials. He plans, constructs, operates, and produces. Wherever capital is invested or government works are constructed, you will find the engineer. Thus he moves into foreign service and returns home when the job is finished. The engineer is behind the man behind the flight because he designed the airplane. But he is also in front of the man on the front line of operation for he is out ahead constructing a new airport. Wherever he is, you may be sure that responsibility follows him. And because of his willingness to accept responsibility and his ability to produce under difficult conditions, the engineer soon finds himself in an administrative position. After twenty years of practice, more than one-half of all engineers have become administrators. It is literally true that in order to remain an engineer, the engineer must often refuse advancement in salary and position. Strange as it may seem, some do.

Chemistry

THE SCIENCE OF MATTER AND MOLECULES

Every engineer and every student of the physical sciences is expected to understand the essentials of chemistry. Perhaps only the chemist, the chemical engineer, the sanitary engineer and the metallurgist come into daily contact with chemistry, but all scientists and engineers should be able to speak the language of chemistry and all must understand its fundamental concepts.

Chemistry in daily life

Chemistry is closely related to the other sciences, to the various fields of engineering, and to the routine of your daily life. The chemical process of digestion makes life pleasant, when it is efficient; the chemistry of the movie film is necessary to your entertainment; and chemistry makes your motorcar possible by contributing alloy steels, storage batteries, lacquer finishes, high-octane gasoline, good rubber tires, and near-perfect lubricants.

Think of surgery without anesthetics and antiseptics; of airships without light, strong alloys, and without gas or gasoline; of clothing without dyes; painting without colored pigments. We might almost as well try to build bridges and office buildings without steel and concrete; to dig tunnels and a Panama Canal without explosives.

THE FIELD OF CHEMISTRY

What is chemistry?

Chemistry treats of the composition and changes in composition of substances. It is also concerned with their properties and their energy relations. We may go further and state that chemistry deals with the detection of substances, their separation from mixtures, and their preparation.

The *detection* of substances is managed so well by chemists that, for example, one part of poisonous mercury vapor in twenty million parts of air may be recognized by the blackening of paper coated with sensitive selenious sulfide. Safe operation of the mercury vapor turbine in power plants is made possible by such a test.

The *separation* of a desired substance from incidental impurities is illustrated well by the winning of gold from the sand or rock

in which it is found and by the removal of the rare gas krypton from air in which it occurs to the extent of one part in one million.

The *preparation* of desired substances is so familiar to the chemist that he makes thousands of useful compounds not found in nature.

Energy transformations

Physics is largely a study of the various forms of energy such as light, heat, electricity, sound, mechanical energy, and so on, while chemistry deals more with matter. However, each science finds it necessary to include both matter and energy in its field. Chemistry, like physics, is fundamental.

Matter and energy are closely related. Matter is difficult to define with exactness. We usually consider matter as anything that has mass or weight and that occupies space.

Chemical energy

The term *energy* is applied to work or to anything that can do work. Therefore light, heat, motion, and electricity are forms of energy. Chemical energy locked up in coal or in beefsteak may be released for our benefit.

The free transformation of one form of energy to another is

FIG. 16 APPARATUS SET-UP AND REAGENT BOTTLES IN A CHEMICAL LABORATORY
This is a laboratory demonstration of the production of high-octane or aviation gasoline.

vital to our existence and welfare. Chemical energy stored up in coal is released as heat by combustion. Chemical energy contained in our food is converted into the motion and heat of our bodies. When the composition of a given portion of matter is changed, energy may be released yet, conversely, light, heat, and electrical energy may be applied to a portion of matter so as to force a change in composition. Heat applied to limestone in the kiln changes it into lime and carbon dioxide.

The *Law of the Conservation of Energy*, which holds that in all ordinary transformations energy is neither created nor destroyed, is merely a generalized statement of a great many observations.

Substances

A *substance* is a particular kind of material with specific properties, such as gold, sugar, common salt, or sulfur; a *body* is merely a definite portion of material such as a bottle, a kettle, a stove, or a statue.

Elements and compounds

Substances that we can decompose into two or more simpler substances are called *compounds*. We recognize limestone as a compound substance because on being heated in the limekiln it breaks down into quicklime and carbon dioxide. There are nearly 500,000 known compounds and millions of new ones can be made when desired.

An *element* is a substance that cannot be decomposed into two or more substances by our present ordinary chemical methods. Until 1808 quicklime, now recognized as calcium oxide, was called an element, but in that year Sir Humphry Davy prepared calcium and showed that it unites with oxygen of pure dry air to form a white solid identical with quicklime. There was then no denying that quicklime could be prepared by the union of two simpler substances, and is, in fact, a compound of calcium and oxygen.

Although 96 elements are known (all above 92 coming from atomic research), many are rare. Oxygen in its compounds makes up one-half the known crust of the earth and silicon one-fourth. In contrast, it takes seventy-four elements to make up only 1 per cent of the earth's crust. The table on page 62 lists all elements and gives their common abbreviations or symbols along with *atomic numbers* and *atomic weights* which will be discussed later.

Properties of substances

To identify different substances we must know, not size and shape, but such characteristics or properties as color, odor, taste, hardness, crystalline form, melting point, boiling point, density, solubility in water or in other solvents, ability to conduct electric-

ity, and index of refraction. We may even ask if the given substance burns in air. Hydrogen sulfide, once smelled, is never forgotten and never separated from the thought of rotten eggs.

In fact, we pay money for specific properties of substances. Copper would not sell normally for 10 cents a pound or more to the amount of over a million tons in an ordinary year if it were not such an excellent conductor of electricity. Obviously, if glass were opaque or if it dissolved readily in water, it would have no market for use as windowpanes.

Changes

Changes in properties seem to be necessary to life itself. Earth, air, and water are transformed by nature's chemical processes into green grass, trees, flowers, and yet each year these beautiful new products decay and return to earth, air, and water.

A *physical change* is a more or less temporary change in certain properties such as color, density, conduction of electricity, and ductility. When the original conditions are restored, the original properties return. Ice may be melted and the resulting liquid water changed into steam without any change in the composition of the water.

A *chemical change* is a very definite permanent change of certain properties with formation of new substances and is always accompanied by a gain or loss of energy. Wherever a chemical change takes place we have a *chemical reaction.*

Lavoisier as early as 1785 believed that in chemical reactions the mass of the system is not changed (*Law of the Conservation of Mass*). Naturally, this belief came from careful weighing of the reacting materials and all the resulting products. In the instance of burning wood, the gaseous products as well as the ashes must be caught and weighed.

Pure substances

Every pure compound has a definite composition by weight. Water from any portion of the earth, if pure, is composed of hydrogen and oxygen in the proportion of 1 part of hydrogen to 7.94 parts of oxygen by weight. If other elements are present, they represent impurities and can readily be separated from the water. Any mixture of alcohol and water might look like water, but it would not be a pure substance. Incidentally it would not boil at 100° C. under 1 atmosphere pressure as does water.

Mixtures and compounds

The difference between pure compounds and mechanical mixtures is evident from the *Law of Definite Composition.* The proportions in any mechanical mixture may be varied considerably with-

out any abrupt change in properties, as in a sand-sugar mixture, but if the proportions of carbon, hydrogen, and oxygen in pure sugar were changed, the substance would no longer be sugar.

In a mixture the properties are the sum of the properties of the constituents, as demonstrated so tastefully in a well-flavored soup. The properties of a compound, however, are nearly independent of the properties of the substances used in making it.

A classic illustration of the difference between a mechanical mixture and a chemical compound is found in an experiment with iron and sulfur. If very fine iron filings are mixed with flowers of sulfur, it may be shown that the sulfur still retains its own characteristic properties, as does the iron, and that each may be removed by simple mechanical means. A magnet attracts the iron as if the sulfur were not present. Carbon disulfide dissolves the sulfur out of the mixture as if the iron were not present and, on filtration, a clear solution (the filtrate) is obtained. This filtrate yields the sulfur if it is allowed to evaporate.

Iron sulfide

The mixture of iron and sulfur may be heated until a hard black solid is obtained which is not attracted by a magnet and from which sulfur is not dissolved by carbon disulfide. Apparently neither free iron nor free sulfur is present — *they have reacted to form a compound.*

The experiment goes as described if the two elements are taken in a very definite proportion. If more iron than is called for by that proportion is used, some free iron is left over and may be removed, leaving the pure product of the reaction. If more sulfur than is called for is used, some sulfur is left over and may be removed. The hard black solid obtained in the above experiment is a *compound* called *iron sulfide.* It has several properties quite different from those of iron or sulfur.

This and similar experiments show us that elements react in very definite proportions to form compounds.

Problems

1. How did the capture and control of fire serve man as a tool?

2. How do you recognize copper, lead, paraffin, salt, vinegar, ammonia, water, alcohol, gold, silver, soap, gasoline?

3. What properties make rubber useful? Gold? Diamonds? Water? Asphalt?

4. Give a few illustrations of physical and chemical changes.

5. What type of change is observed when a rubber band is stretched? When a rubber band is burned?

6. Define a body, a substance, a compound, an element, a chemical reaction.

7. When sand, lime, and soda are fused together to form glass, is the change physical or chemical?

8. How could you separate a mixture of oxygen and ammonia? Ammonia is very soluble in water.

THE ATOMIC THEORY

Dalton's atomic theory

Dalton, in 1803, was convinced that "the ultimate particles of a pure substance, simple or compound, are alike in size and weight."

The smallest particles of these compound substances formed by the union of two or more atoms of different elements we call *molecules*.

In other words, a molecule is the smallest particle of any pure substance with the same composition as any larger portion. If a molecule of sugar were taken apart, these smaller particles would no longer taste like sugar, they would in fact be atoms of carbon, hydrogen, and oxygen. The smallest particles of elements taking part in chemical reactions we call *atoms*. A reaction, then, is merely a regrouping of the atoms in new combinations. Dalton pictured his "ultimate particles" or atoms as definite, concrete grains of matter, indivisible, unaffected by the most violent chemical change.

Conservation of mass

This theory explains why, during a reaction between substances in a closed vessel, there is neither gain nor loss of weight (*Law of Conservation of Mass*). All the atoms of the reacting substances are accounted for in the new substances.

It explains also the *Law of Definite Composition*, for when one element unites with another the combination always takes place between definite numbers of atoms of each kind. For example, if two atoms of hydrogen unite with one atom of oxygen to form a molecule of water, it surely follows that the percentage of oxygen and hydrogen by weight in pure water is always the same.

Molecules are so small that if we empty one liter of water into the ocean, with thorough mixing, and then dip out one liter, we will recapture 2220 of the original molecules of water added to the ocean. A row of 40,000,000 molecules in contact would measure one inch.

Modern atomic theory

Based on study of the X-ray tube and of radium we have come, since 1897, to a more modern conception of the atom, yet Dalton's theory served us well for nearly a century.

The first step was the recognition of cathode rays in the X-ray tube (Fig. 17) as a stream of negatively charged particles (*electrons*)

only $\frac{1}{1837}$ times the mass of the hydrogen atom, the lightest of all atoms. Next came the discovery of the *proton* (a positive charge of electricity with a mass practically that of the hydrogen atom). More recently we discovered another building brick of the atom called the *neutron* (an electrically neutral unit with the mass of the proton).

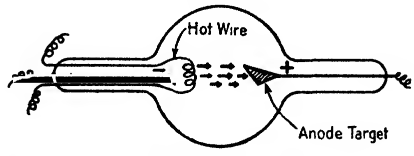
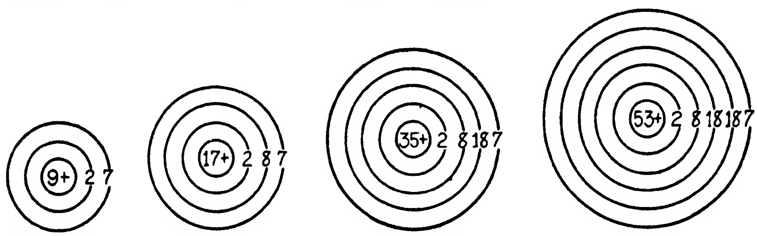


FIG. 17. THE COOLIDGE X-RAY TUBE

Electrons called "cathode rays" are given off by the heated cathode. Their impact upon the anode target produces X-ray radiation.

We now feel certain that the ordinary neutral atom is a miniature solar system with a compact central nucleus made up of neutrons, a number of free protons, and enough planetary electrons to balance electrically the protons in the nucleus (Fig. 18). So small are these three fundamental units, neutrons, electrons, and protons, that the atom is mostly space. So, too, is the solar system. *Miniature solar system*



Fluorine Chlorine Bromine Iodine

FIG. 18. PLANETARY SHELLS OF ELECTRONS AROUND ATOMIC NUCLEI

Positive protons in any atomic nucleus are electrically balanced by an equal number of negative planetary electrons in the electron shells or "energy levels." Neutrons in the nucleus are omitted here for simplicity.

If the nucleus of an atom were magnified to a 2-in. sphere, the nearest electron would be 2000 ft. away. The gold atom, according to Rutherford, has 100,000 times the diameter of its own nucleus. *Atomic nucleus*

In all probability the planetary electrons revolve at great velocities, 150,000 miles per second, around the nucleus, some in circles and some in elliptical orbits (Fig. 19).

The simplest atom is hydrogen, made up of a single proton and a single electron. (The most complex atom is uranium, with 92 protons, 92 electrons, and 146 neutrons.) Since the hydrogen atom weighs 1.008 and the electron mass is only $\frac{1}{1837}$ of the hydrogen

Mass of proton

mass, it is evident that *the proton has a mass of approximately one.* In other words, the mass of a few electrons is negligible, and practically all the weight, or mass, of an element is concentrated in the nucleus, a relatively minute fraction of the volume of the entire atom.

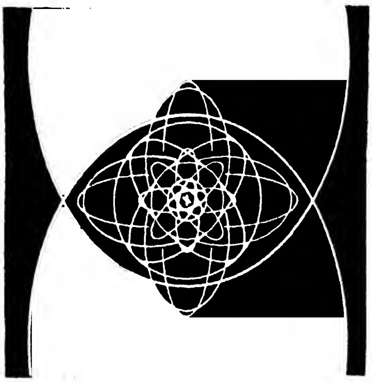


Fig. 19. Complex orbits of electrons in the tantalum atom

It is helpful at this stage to note that with many common atoms the outer shell of electrons may contain from one to eight electrons. In chemical reactions this number is often raised to eight either by a transfer of electrons from another atom (electrovalence) or by a sharing process (covalence).

Exchanging electrons

Although the electrons (negative) are attracted by the nuclear protons (positive), some of the outer planetary electrons are known to escape to other atoms. Some of the most active atoms, like those of the metals potassium and sodium, have a very great tendency to give away outer electrons, while some of the most active nonmetallic atoms like those of chlorine and fluorine exhibit a reluctance to give away electrons, but show an eagerness to take on electrons. Sodium reacts with chlorine, because the chlorine atom

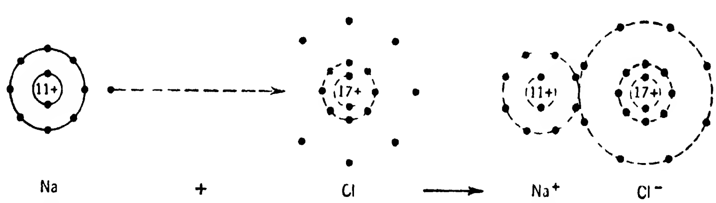


Fig. 20. Reactions between atoms of sodium (Na) and chlorine (Cl) showing electron transfer

eagerly takes on the electron that the sodium atom is so willing to discard. (See Fig. 20.) Chemical reaction takes place and a molecule of sodium chloride, common salt, is formed.

Problems

1. If a neutral atom contained 40 protons as well as many neutrons in the nucleus, how many electrons must be present? Where are those electrons?

2. If the single electron in the hydrogen atom were knocked out, what would be left? What would be its weight?

THE LANGUAGE OF CHEMISTRY

Symbols and formulas

According to our present conception, symbols stand not only for the elements but for definite parts by weight of the elements. For example, H represents a single atom of hydrogen and it may also mean the atomic weight in grams, that is, 1.008 g. The symbol O means either a single atom of oxygen or 16 g. of oxygen. See the table on the following page. On page 86 we explain the derivation of atomic weights.

The formula O_2 for the oxygen gas tells us that there are two oxygen atoms in each molecule with a molecular weight of $2 \times 16 = 32$. To represent the formula of a compound we simply join the symbols of the constituent elements together and attach small subfigures to represent the actual numbers of atoms in each molecule. H_2O as a formula for water tells us that two atoms of hydrogen and one atom of oxygen make up a single molecule of water. The weight of the whole molecule, 18.016, is of course the sum of the weights of all the atoms in it. $(2 \times 1.008 + 16.0 = 18.016.)$ H_2SO_4, the formula for sulfuric acid, tells us that a molecule of the acid contains two atoms of hydrogen, one of sulfur (atomic weight, 32.06), and four of oxygen. The formula weight is 98.076 g.

Chemical equations

With the use of symbols and formulas we can now represent chemical changes (reactions) very simply and clearly. For example, the equation

$$(1) \qquad CuO + H_2 \longrightarrow H_2O + Cu$$

tells us that one molecule of copper oxide reacts with one molecule of hydrogen to form one molecule of water and one atom of copper. The weight of one molecule of CuO is obviously the sum of the weights of all the atoms in it. Cu = 63.57 and O = 16.0. Therefore CuO = 79.57. The weight of $H_2 = 2.016$; of $H_2O = 2.016 + 16$, or 18.016; and of Cu = 63.57.

Metric units

Before introducing chemical problems it may be helpful to define a few metric units.

Metric Units

1 meter = 1000 millimeters (mm.) = 39.37 inches
1 meter = 100 centimeters (cm.)
1 liter (1 l.) = 1000 cubic centimeters (cc.) = 1.06 quarts
1 cubic foot = 28.32 liters
1 kilogram (kg.) = 1000 grams (g.) = 2.2046 pounds
1 cc. of water weighs 1 g. (at 4° C.)

ELEMENTS AND ATOMIC WEIGHTS

	Symbol	*Atomic Number*	*Atomic Weight*		*Symbol*	*Atomic Number*	*Atomic Weight*
Aluminum	Al	13	26.97	Molybdenum	Mo	42	95.95
Antimony	Sb	51	121.76	Neodymium	Nd	60	144.27
Argon	A	18	39.944	Neon	Ne	10	20.183
Arsenic	As	33	74.91	Nickel	Ni	28	58.69
Barium	Ba	56	137.36	Nitrogen	N	7	14.008
Beryllium	Be	4	9.02	Osmium	Os	76	190.2
Bismuth	Bi	83	209.00	Oxygen	O	8	16.0000
Boron	B	5	10.82	Palladium	Pd	46	106.7
Bromine	Br	35	79.916	Phosphorus	P	15	30.98
Cadmium	Cd	48	112.41	Platinum	Pt	78	195.23
Calcium	Ca	20	40.08	Potassium	K	19	39.096
Carbon	C	6	12.010	Praseodymium	Pr	59	140.92
Cerium	Ce	58	140.13	Protactinium	Pa	91	231
Cesium	Cs	55	132.91	Radium	Ra	88	226.05
Chlorine	Cl	17	35.457	Radon	Rn	86	222
Chromium	Cr	24	52.01	Rhenium	Re	75	186.31
Cobalt	Co	27	58.94	Rhodium	Rh	45	102.91
Columbium	Cb	41	92.91	Rubidium	Rb	37	85.48
Copper	Cu	29	63.57	Ruthenium	Ru	44	101.7
Dysprosium	Dy	66	162.46	Samarium	Sm	62	150.43
Erbium	Er	68	167.2	Scandium	Sc	21	45.10
Europium	Eu	63	152.0	Selenium	Se	34	78.96
Fluorine	F	9	19.00	Silicon	Si	14	28.06
Gadolinium	Gd	64	156.9	Silver	Ag	47	107.880
Gallium	Ga	31	69.72	Sodium	Na	11	22.997
Germanium	Ge	32	72.60	Strontium	Sr	38	87.63
Gold	Au	79	197.2	Sulfur	S	16	32.06
Hafnium	Hf	72	178.6	Tantalum	Ta	73	180.88
Helium	He	2	4.003	Tellurium	Te	52	127.61
Holmium	Ho	67	164.94	Terbium	Tb	65	159.2
Hydrogen	H	1	1.0080	Thallium	Tl	81	204.39
Indium	In	49	114.76	Thorium	Th	90	232.12
Iodine	I	53	126.92	Thulium	Tm	69	169.4
Iridium	Ir	77	193.1	Tin	Sn	50	118.70
Iron	Fe	26	55.85	Titanium	Ti	22	47.90
Krypton	Kr	36	83.7	Tungsten	W	74	183.92
Lanthanum	La	57	138.92	Uranium	U	92	238.07
Lead	Pb	82	207.21	Vanadium	V	23	50.95
Lithium	Li	3	6.940	Xenon	Xe	54	131.3
Lutecium	Lu	71	174.99	Ytterbium	Yb	70	173.04
Magnesium	Mg	12	24.32	Yttrium	Y	39	88.92
Manganese	Mn	25	54.93	Zinc	Zn	30	65.38
Mercury	Hg	80	200.61	Zirconium	Zr	40	91.22

It is a very useful practice when working problems to consider these weights as expressed in grams. Of course, the relative values are just the same whether expressed in the minute units of *atomic*

weights, or in *grams*, or *tons*. Expressed in grams we should read the equation thus:

79.57 g. copper oxide + 2.016 g. hydrogen
form 18.016 g. water + 63.57 g. copper.

Balancing equations

These so-called equations name the reacting substances and their products. "Balancing the equations" consists in prefixing such numbers before the molecular formulas as indicate the relative number of molecules concerned.

We admit that unless the student knows from his own (or some other chemist's) observations and tests in the laboratory just what the products are, it will be mere guesswork for him to attempt to balance the equation.

Since facility in balancing equations comes only with considerable practice, it might be well to try another example. Suppose we tell you that aluminum powder can be burned in air or in oxygen gas and that the product is aluminum oxide, Al_2O_3. The oxygen molecule is always represented as O_2.

$$Al + O_2 \longrightarrow Al_2O_3 \text{ (wrong).}$$

Balancing by trial

To have any aluminum oxide at all we must write down at least one molecule, that is, Al_2O_3. This forces us to start with 2 Al on the left and also with three atoms of oxygen. It is hardly consistent to represent 3 O as $1\frac{1}{2}\,O_2$ because *half molecules cannot occur free.* However, as a trial we may write

$$2\,Al + 1\tfrac{1}{2}\,O_2 \longrightarrow Al_2O_3 \text{ (trial).}$$

To obtain whole numbers, each side of the equation should be multiplied by 2, giving us

$$(2) \qquad 4\,Al + 3\,O_2 \longrightarrow 2\,Al_2O_3 \text{ (balanced).}$$

A chemical equation, then, must represent the reacting substances, the products formed, and the relative weights involved. It does not, however, name the conditions — such as temperature — necessary to cause the reaction to take place, nor does it tell how rapidly it occurs nor to what extent the reaction develops.

Percentage composition of compounds

If the correct formula for black iron oxide is Fe_3O_4, it is a simple matter to calculate the percentage of iron, for example. From the formula we learn that the molecular weight is the sum of three times the atomic weight of iron (3×55.85) plus four times the atomic weight of oxygen (4×16), or a total weight of 231.55.

The iron fraction is evidently $\frac{167.55}{231.55} = 0.7236 = 72.36$ per cent of the oxide.

Problems

1. The hard sapphire (except for traces of coloring matter) is aluminum oxide, Al_2O_3. Synthetic sapphires are now made by the thousands for jewel bearings in the control instruments of airplanes and battleships and therefore are of strategic importance. Explain the meaning of the formula, Al_2O_3.

2. Red lead, so common in the priming paint coating on metals, is Pb_3O_4. Explain this formula. What percentage of the weight of Pb_3O_4 is oxygen?

Ans. 9.33%.

3. What per cent of lead is found in pure lead nitrate, $Pb(NO_3)_2$?

Ans. 62.56%.

4. What is meant by the formula, C_2H_6O (alcohol)? Determine the weight of the molecule *Ans.* 46.07.

5. (a) What weight of red iron oxide, Fe_2O_3, may be obtained from 250 g. of oxygen and sufficient iron? (b) What weight of black Fe_3O_4?

Ans. (a) 831.77 g.; (b) 904.5 g.

Note Atomic weights of lead, Pb, nitrogen, N, carbon, C, iron, Fe, and other elements are given on page 62

OXYGEN

Occurrence

Oxygen makes up one-half of all matter that we know. Two-thirds of the human body is combined oxygen. Water contains 89 per cent; ordinary clay, sand, limestone, and granite about 50 per cent. One-fifth of the air is free oxygen, the other four-fifths are nitrogen and other gases. This is the only occurrence of free oxygen except the little that is found dissolved in water. Elsewhere it is found in compounds.

Preparation of oxygen

There are at least four general methods of preparation of oxygen, but its most important commercial source is the atmosphere. Air is chiefly a mixture of oxygen and nitrogen but separation is not easy.

1. Air Liquefaction Process. The separation is accomplished by liquefying air and then allowing it to evaporate. The more volatile nitrogen escapes first, leaving the oxygen, which is usually stored in strong steel cylinders at 100 atmospheres pressure. The gas thus obtained naturally contains a little nitrogen, which does no harm for commercial uses. This modern process is now the only commercial method used on a large scale.

2. Electrolytic Method. The *electrolysis of water* (decomposition by passage of an electric current in the presence of a little added sulfuric acid) releases hydrogen at the cathode and oxygen at the

anode. This process is used commercially to some extent because the product is very pure and because there is considerable demand for the hydrogen.

3. Thermal Decomposition of Oxygen Compounds. The laboratory methods, where convenience is more important than cheapness, in most cases consist in heating oxygen compounds. When potassium chlorate is heated to 400° C. or higher, we obtain $2\ KClO_3 \longrightarrow 2\ KCl + 3\ O_2$. With addition of a mere pinch of black manganese dioxide, MnO_2, the reaction takes place at 200° C., your introduction to a *catalyst* (chemical reaction helper). *Laboratory methods*

4. Sodium Peroxide Method. Sodium peroxide (made by heating sodium in air) reacts with water to form oxygen and sodium hydroxide.

(3) $$2\ Na_2O_2 + 2\ H_2O \longrightarrow 4\ NaOH + O_2.$$

This method is expensive but very convenient since the water may be allowed to drop slowly from a dropping funnel on the peroxide, releasing oxygen as needed.

Oxygen may be liquefied at and below − 118° C., which may then be called its *critical temperature*. At higher temperatures it remains a gas. At the critical temperature, 50 atmospheres pressure are required to liquefy oxygen; hence this is the *critical pressure*. Of course, at temperatures below − 118° C., a lower pressure than 50 atmospheres (735 lb. per sq. in.) would cause liquefaction. *Physical properties*

Oxygen is only moderately active chemically at ordinary temperatures, but at elevated temperatures it unites with nearly every element. *Chemical properties*

If the metal sodium is heated and then placed in a jar of oxygen, it burns brilliantly with a yellow flame. Calcium burns almost as brilliantly, forming ordinary quicklime or calcium oxide.

(4) $$2\ Ca + O_2 \longrightarrow 2\ CaO.$$

An oxide is a compound of oxygen with one other element. Water is an oxide of hydrogen; and rust, Fe_2O_3, is an oxide of iron. When substances react, heat is usually (but not always) liberated. If the reaction becomes so vigorous that light is also produced, it is called *combustion*. As a rule this term is applied to a union of oxygen with other substances, but many other reactions produce light. *Oxides*

Reactions accompanied by the evolution of heat are *exothermic*, and those that require the application of external heat to sustain

them are *endothermic.* In measuring heat we use as a unit the *calorie (cal.), the amount of heat required to raise the temperature of 1 g. of water 1° C. (at 15° C.).* The large calorie (Cal.), one thousand times as great, is sometimes called the *kilogram calorie.*

Heat of combustion

The heat of combustion of any pure substance is the number of calories liberated when the formula weight (molecular weight in grams) is burned. For example, a molecule of carbon monoxide weighs 28.01, hence its formula weight is 28.01 g. When 28.01 g. of this gas burns, it liberates 67,623 calories and becomes carbon dioxide.

When enough hydrogen is burned in air or oxygen to form 18.016 g. of water, 68,310 cal. of heat are evolved and exactly this much energy as heat or electricity is required to decompose 18.016 g. of water into hydrogen and oxygen. Obviously the substances which give off the greatest *heat of formation* require the greatest amount of external energy for their *decomposition.*

Uses of oxygen

Oxygen is a most useful element. Essential to animal life in burning out waste tissue and furnishing heat, beneficent in its purification of sewage and other waste matters that would otherwise be a nuisance and a menace to health, vital in developing heat, light, and power from fuel — it is, in short, the central element. Oxygen, compressed in steel cylinders, is used for the oxy-hydrogen (or oxyacetylene) torch. This is a burner by which hydrogen or acetylene is burned in pure oxygen instead of in air. The extremely hot flame produced cuts through iron like a hot knife through butter (by melting and burning the iron).

A college freshman, impressed by man's great dependence upon an atmosphere containing 20 per cent oxygen, asked, "What did we breathe before Priestley discovered oxygen in 1774?"

Influencing reaction speed

The speed of chemical reactions is greatly influenced by temperature, by the concentration of the reacting substances, by the presence of catalysts, and by the amount of contact surface. A rise in temperature of 10° C. doubles or trebles the speed of most chemical reactions. A rise from 20° C. to 200° C. means a millionfold increase in reaction speed. This is understood when we remember that a rise in temperature means that the molecules move faster, consequently hit each other harder and oftener. A reaction requiring 1 sec. for completion at 200° C. will require 11 days at zero. Admiral Byrd found that he could not use flashlights in the

south polar regions because the intense cold slowed down the necessary chemical action of the dry batteries.

With two substances that cannot be mixed thoroughly, such as a solid and a gas, the amount of surface of the solid is of great importance in combustion. Splinters or shavings burn much faster than a solid log — a matter of "opportunities for contact." Coal dust is now blown into cement kilns with air and burned like a spray of oil.

Dust explosions

Fine, dry dust from flour, starch, grain, sulfur, coal, leather, alfalfa, sugar, soap, cocoa, fertilizer, or feeds may also be *inflammable*. A dust explosion is merely a *fast chemical reaction* with formation of expanding gases.

Catalysis

A *catalyst* is a substance that aids or retards a reaction without itself being permanently changed or consumed. In some cigar lighters the vapors of methyl alcohol mixed with air pass over a thin roughened platinum wire. There is little oxidation without the platinum. Perhaps the oxygen atoms become "excited" by contact with the platinum catalyst with a resulting displacement of some planetary electrons out of their regular orbits. Manganese dioxide aids the release of oxygen from potassium chlorate at 200° C. by reacting first to form another compound that decomposes readily, releasing oxygen and the original manganese dioxide. Since the manganese dioxide is not used up in the reaction, it is a catalyst.

Catalysts are of great importance in industry; for example, in the manufacture of sulfuric acid, nitric acid, ammonia, and aviation gasoline.

Problems

1. Why do we select oxygen as the first element to study?

2. How much phosphoric acid, H_3PO_4, is formed by reaction of 112 g. of phosphorus pentoxide, P_2O_5, with sufficient water? *Ans.* 154.64 g.

3. If oxygen suddenly became very soluble in water, what changes would take place in the world?

GASES, LIQUIDS, AND SOLIDS

Boyle's gas law

Boyle (1660) observed that *the volume occupied by a given sample of any gas kept at a constant temperature varies inversely with the pressure.* Accurate measurements showed that if the pressure was doubled on a given volume of a gas (keeping the temperature constant) the volume was reduced to one-half. With three times the pressure the volume was reduced to one-third. This is illustrated by Fig. 21.

This generalization can be formulated mathematically thus:

$$\frac{P_1}{P_2} = \frac{V_2}{V_1} \quad \text{or} \quad P_1V_1 = P_2V_2$$

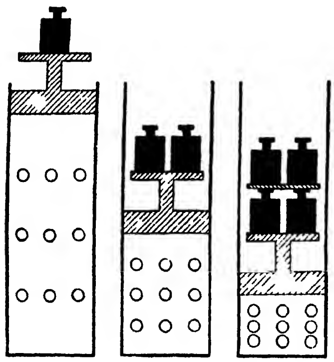


FIG. 21. BOYLE'S LAW ILLUSTRATED The volume occupied by a given number of molecules of a gas varies inversely as the pressure

from which we obtain

(5) $$V_1 = V_2 \times \frac{P_2}{P_1}.$$

Here P_1 and V_1 are the first pressure and volume, and P_2 and V_2 any other pressure with its accompanying volume. Since the product P_1V_1 is equal to any other *pressure-volume product*, such as P_2V_2 for the same weight of the gas, it is sometimes said that the product of the pressure times the volume of a gas is a constant value. It is our custom to refer to the average atmospheric pressure at sea level (enough to hold up a 760-mm. column of mercury in a barometer) as the standard pressure.

Volume calculation

We do not always measure gases at exactly 760 mm. pressure, but we calculate from Boyle's law what the volume would be if the pressure were changed to the standard 760 mm. For example, 400 cc. of air at 740 mm. would occupy what volume at 760 mm.? Using the formula

$$V_1 = V_2 \times \frac{P_2}{P_1},$$

we may let V_1 represent the volume to be calculated and P_1 the standard pressure, 760 mm. To make it clearer,

(6) $$V_1\ (\text{standard}) = V_2\ (\text{observed}) \times \frac{P_2\ (\text{observed})}{P_1\ (\text{standard})}.$$

Substituting known values, we have

$$V_1\ (\text{standard}) = 400 \times \frac{740}{760} = 389\ \text{cc.}$$

High vacuums

High vacuums, or even partial vacuums, minister to our comfort and serve well in many ways. They made possible the original incandescent lamp (although the type has since changed), the X-ray tube, the radio tube, the photoelectric cell of television, and the thermos bottle. By exhausting into a vacuum we have increased the efficiency of steam engines and turbines.

Charles in 1787 observed that on cooling any gas from 0° C. to − 1° C. it contracted $\frac{1}{273}$ of its volume. It was a natural inference that if cooled to − 273° C. its volume would be nothing at all. However, in practice all gases become liquid before they have been cooled to − 273° C., and after they become liquid they no longer obey the gas laws. There is a real convenience, however, in calling this imaginary point of no transitional motion of molecules and no gaseous pressure, *absolute zero*, − 273° C. In honor of the eminent physicist, Lord Kelvin, this temperature is usually termed zero degrees Kelvin and written 0° K. *Charles's law*

In recent years temperatures lower than 0.01° K. have been reached, giving reality to the conception of absolute zero.

Charles's law is merely an observation that *the volume of a given sample of gas varies directly as the absolute temperature (if the pressure is kept constant)*. For example, to double the volume that a gas occupies at 20° C. by heating we should double the absolute temperature. This temperature is 20° + 273° = 293° K.; and 2 × 293° = 586° K. or 313° C. *Example of expansion*

When heated to 313° C. (not 313° K.), the gas would have twice its original volume. Charles's law is, therefore, formulated as follows:

(7) $$\frac{V_1}{V_2} = \frac{T_1}{T_2} \quad \text{or} \quad V_1 = V_2 \times \frac{T_1}{T_2},$$

where T_1 and T_2 refer to absolute temperatures. When we speak of a liter of oxygen weighing 1.429 g., we mean a liter at 760 mm. pressure and 0° C. Since gases expand on heating, there can be no accuracy without reference to a definite standard temperature. Obviously, we do not care to measure gases in rooms at 0° C., but we can apply Charles's law and calculate what the volume would be if the gas were cooled to 0° C. For example, what is the volume of 400 cc. of air measured at + 18° C. when cooled to 0° C.?

(8) $$V\ (\text{standard}) = V\ (\text{observed}) \times \frac{273}{273 + t^\circ},$$

where t° is the Centigrade temperature. Substituting numerical values, we obtain

$$V\ (\text{standard}) = 400 \times \frac{273}{291} = 375.25\ \text{cc.}$$

The observed volume of a gas can be brought to 760 mm. and 0° C. by applying both corrections at once. For example, 400 cc. *Standard conditions*

of dry air at 740 mm. and $-18°$ C. would have what volume at standard temperature and pressure?

$$V \text{ (standard)} = V \text{ (observed as 400)} \times \frac{740}{760} \times \frac{273}{255} = 412.68 \text{ cc.}$$

Partial pressures

Since the total pressure of a mixture of gases (as the air) is the sum of the pressures that would be exerted by each if the others were not present, it is obvious that the barometric pressure of any gas measured must be corrected for that part of the pressure due to water vapor present. A gas collected over water will be saturated at that temperature.

Moisture correction

Suppose the problem is to correct to standard conditions 400 cc. of air collected over water at 740 mm. and 18° C. Now the *observed pressure* of 740 mm. is not all due to the air, for in this case 15.5 mm. of it is due to water vapor.

$$V_1 \text{ (standard)} = V_2 \text{ (observed as 400)} \times \frac{740 - 15.5}{760} \times \frac{273}{291}.$$

ACTION OF GASES

Gases must be composed of flying molecules because such a kinetic theory explains the following observed facts:

Facts	*Explanation*
1. Gases diffuse and mix readily.	Relatively wide spaces between flying molecules permit diffusion.
2. Gases are very compressible.	A given volume of gas is mostly space.
3. Gas pressure extends in all directions.	The molecules fly in all directions and bombard the walls of the containing vessel.
4. Boyle's law holds.	Pressure crowds molecules closer together.
5. Charles's law holds.	Molecules fly faster with rise in temperature.
6. Boyle's law and Charles's law both fail when gases are nearly liquefied.	The gas laws refer to change in the amount of free space between molecules rather than to total volume. At high pressures the molecules themselves occupy a very appreciable fraction of the total volume. Also attraction between molecules when close together aids compression.
7. Liquefaction of gases	Forcing molecules into rather close contact brings in the aid of molecular attraction in restricting freedom of movement.

Facts	*Explanation*
8. Evaporation of liquids	Molecules at the surface of a liquid escape beyond the range of attraction of the liquid.
9. Each gas in a mixture exerts the same pressure it would show in absence of other gases.	Molecules have plenty of room for free movement, even in the presence of other kinds of molecules.

Liquefaction of gases

Liquefaction of gases by pressure is comparatively easy to accomplish with ammonia, chlorine, carbon dioxide, and sulfur dioxide, but both high pressure and great cooling are required to liquefy such gases as oxygen, nitrogen, hydrogen, and helium.

LIQUIDS

Boiling point

The boiling point of a liquid is the temperature at which the upward pressure of its vapor equals the downward pressure or weight of the atmosphere above it. Obviously the boiling point varies with the pressure of the atmosphere (Fig. 22). The bubbling observed in any boiling liquid is due to the expansion of bubbles of vapor forming within the liquid. Water has a relatively high heat of vaporization, requiring 540 cal. to change 1 g. of water at 100° C. into steam at 100° C.

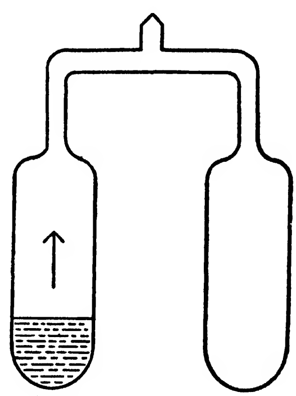


FIG. 22. THE EFFECT OF PRESSURE ON BOILING POINT
When the empty bulb on the right is chilled with dry ice, the ether in the bulb on the left boils (if air had previously been removed from the sealed apparatus). After the ether vapors condense to liquid on the right, the process may be reversed. The movement of gas molecules is decreased by chilling.

The boiling point of any pure liquid at 760 mm. pressure remains constant (100° C. for water, 78.4° C. for alcohol, and so on) until all the liquid has changed into vapor. Not so with mixtures of two or more. For example, alcohol-water may start boiling somewhat above 78.4° C. but the temperature gradually rises, with escape of *vapor of steadily varying composition*, until the *distillate* (condensed vapor) attains a concentration of 95 per cent alcohol in the commercial column still.

Water in the teakettle absorbs heat from the flame while boiling. Liquefied sulfur dioxide in a beaker resting on the table (not on the stove), will boil, absorbing heat from its surroundings, cooling the room as it does in the coils of your refrigerator. The boiling point of

liquefied sulfur dioxide is $-10°$ *C. and that of liquefied ammonia is* $-32.8°$ C.

Distillation

The process of evaporating and recondensing a liquid (by cooling) is termed *distillation.* To expedite matters the liquid is usually boiled, sometimes under less than atmospheric pressure. Vacuum distillation is desirable with liquids or solutions likely to decompose at the normal boiling temperature.

SOLIDS

Freezing point

Liquids, on cooling sufficiently, all lose their fluidity and become solids. They freeze to solids without further lowering of temperature until all the liquid has solidified. When this change of state occurs at a definite temperature, the solid formed is always crystalline.

Water sometimes may be chilled below 0° C. without separation of ice crystals. However, the addition of a minute fragment of ice starts crystallization or freezing. Consequently, we are forced to define the freezing point (same as melting point) as *the temperature at which liquid and solid will remain in contact with each other without change of temperature.*

Heat of fusion

To increase the freedom of movement of the molecules of a solid so that a fluid results, requires increased kinetic energy which involves both mass and velocity. This is introduced by heating the solid. There is a rise in temperature until some liquid is formed and then no further rise until all the solid is melted. To convert 1 g. of ice at 0° C. into water at 0° C., 79.7 cal. of heat are required. This is termed the *heat of fusion of ice.* The corresponding value for sulfur is 9.37 cal. and for aluminum 76.8 cal. Amorphous solids (such as plastics and glass), unlike crystalline solids, have no definite melting points, no definite heat of fusion.

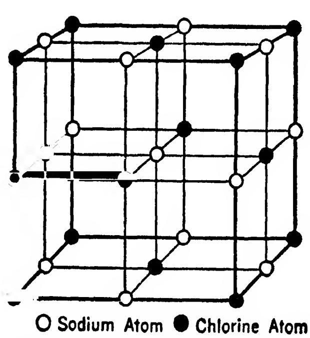


FIG. 23. CUBIC ARRANGEMENT OF ATOMS IN A CRYSTAL OF SODIUM CHLORIDE

Ions in crystals

As Bragg pointed out in 1914, the atoms in crystalline solids are arranged in a geometric pattern or lattice (Fig. 23). The lattice units are not single atoms in all cases. The familiar groups or radicals (such as $—SO_4$ in Na_2SO_4 and $CaSO_4$) are found in

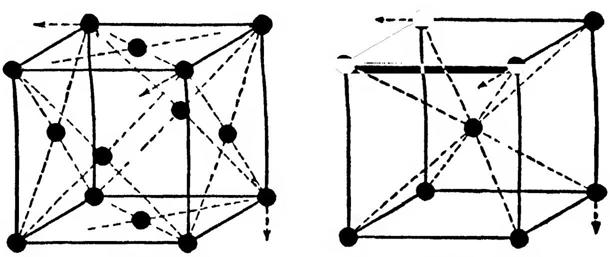
some crystalline salts as lattice units and may carry positive or negative charges. Such charged atoms or groups are termed *ions.*

Planes of cleavage

Since the attraction between atoms in a crystal lattice varies inversely as the square of the distance, it is evident that crystals cleave or split most readily along planes where these distances between atoms are greater than in adjoining parts of the solid. This is notably true with mica and graphite. X-ray photographs show that in solid carbon dioxide, CO_2, each carbon atom is much closer to a pair of oxygen atoms than to the other carbon atoms. In this sense the molecule, CO_2, is a unit in the space lattice.

Lattices

The commonest lattice system is the *face-centered cubic* (Fig. 24) with atoms at the corners of a system of equal closely packed cubes and also at the centers of each cube face.



(a) Face Centered (b) Body Centered

FIG. 24. ARRANGEMENTS OF ATOMS IN CUBIC CRYSTALS

Wyckoff[1] states that X-ray analysis has so far shown that:

1. There is no metal crystallizing in the face-centered system which is not ductile throughout a considerable range of temperature.
2. All "noble" metals (such as gold and silver) are in the face-centered system.
3. All the best conductors of heat and electricity are in the face-centered system.

The ordinary crystal visible to the eye is built up of smaller units or "cells" each having a shape like the larger crystal.

Problems

1. In the tropics drinking water is cooled by storing it in large porous jars. Explain how cooling takes place.

2. What is the effect on the air pressure of the tires of a car driven at high speed with a heavy load? Consider the friction of tire on road.

3. Assume that a gallon of gasoline weighs approximately 3200 g. and that C_8H_{18} represents its composition. (Actually gasoline is a mixture of such hydrocarbons.) How many kilograms of water will be formed, as vapor, during the oxidation of this gallon of gasoline in the engine cylinder of your car? The carbon atoms form carbon dioxide and the hydrogen atoms form water. *Ans.* 4.54 kg.

[1] *Structure of Crystals*, Reinhold Publishing Corporation, New York.

4. Calculate the volume at 0° C. and 760 mm. pressure of 450 cc. of a gas measured at 25° C. and 750 mm. *Ans.* 406.8 cc.

5. Ten liters of nitrogen at 20° C. and 1 atmosphere pressure are heated to 500° C. What is the new volume? *Ans.* 26.38 l.

6. Three liters of oxygen at 22° C. and 740 mm. pressure were collected over water. Calculate the volume, dry, at 0° C. and 760 mm. Allow 19.6 mm. for the pressure of water vapor. *Ans.* 2.63 l.

HYDROGEN

In the combined form hydrogen is found in water, plant and animal tissues, petroleum, asphalt, and natural gas. Among laboratory reagents we learn that it is an essential element of all acids and bases. We will discuss six ways in which it may be prepared.

From metals and acids

1. Hydrogen is a constituent of such common materials as water, acids, and bases. In the laboratory it is convenient to displace hydrogen from dilute acids by the more active metals. The hydrogen is collected by displacement of water.

(9) $Zn + 2\ HCl \longrightarrow H_2 + ZnCl_2$ (zinc chloride).
(10) $Zn + H_2SO_4 \longrightarrow H_2 + ZnSO_4$ (zinc sulfate).
(11) $2\ Al + 3\ H_2SO_4 \longrightarrow 3\ H_2 + Al_2(SO_4)_3$ (aluminum sulfate).
(12) $2\ Al + 6\ HCl \longrightarrow 3\ H_2 + 2\ AlCl_3$ (aluminum chloride).

When a metal displaces hydrogen from an acid, a *salt*, such as a chloride or a sulfate, is formed. The symbol Zn used above represents zinc; Al represents aluminum, while HCl is the formula for hydrochloric acid.

From metals and water

2. The most active metals, such as potassium and sodium, violently displace hydrogen from cold water. Calcium acts at moderate speed. Less active metals react with water only when heated. Magnesium displaces hydrogen from boiling water if a little of a magnesium salt is present to keep the surface of the metal clean. Steam passed over red-hot iron reacts as follows:

(13) $3\ Fe + 4\ H_2O \longrightarrow Fe_3O_4 + 4\ H_2.$

By electrolysis

3. When a current of electricity is passed through water made conducting by the presence of some substance such as sulfuric acid or sodium hydroxide, we are able to collect two volumes of hydrogen at the *cathode*, where the current leaves, to one of oxygen at the *anode*, where the current enters.

From water gas

4. "Water gas," a mixture of CO and H_2, is the principal commercial source of hydrogen in the United States. Water gas is

prepared by the process of passing steam over coke at about 1000° C. yielding a mixture of carbon monoxide and hydrogen.

(14) $$C + H_2O \longrightarrow CO + H_2.$$

Of course, the coke is soon cooled by the steam below the reaction temperature. Steam is then shut off and a blast of air turned on. Part of the coke burns, heating the remainder up to 1000° C., and preparing it for another blast of steam

From hydrocarbons

5. Any compound of hydrogen and carbon, such as petroleum, may be broken down or *cracked* into its elements if sufficiently heated without contact with air.

From coal gas

6. The gas obtained when 1 ton of bituminous coal is turned into coke contains about 27 lb. of free hydrogen. In Germany the separation of this hydrogen has been an economic success, but water gas is a preferred source in the United States.

Properties of hydrogen

Hydrogen, when pure, is a colorless, odorless, tasteless gas. A liter weighs 0.08987 g. at standard conditions, so it is the lightest substance known. Consequently, as indicated by Graham's law, it has the greatest speed of *diffusion* of all gases. The loss by diffusion through balloon fabric is a serious problem. Deuterium (heavy hydrogen) is discussed in the section on water. Hydrogen is rather inactive at ordinary temperatures, but at higher temperatures it unites with most of the nonmetals and many of the metals. Some typical reactions are

(15)	$S + H_2 \longrightarrow H_2S$	(hydrogen sulfide).
(16)	$Cl_2 + H_2 \longrightarrow 2\,HCl$	(hydrochloric acid).
(17)	$O_2 + 2\,H_2 \longrightarrow 2\,H_2O$	(water).
(18)	$N_2 + 3\,H_2 \longrightarrow 2\,NH_3$	(ammonia).

Combustion

The hydrogen flame is intensely hot, as 68,390 calories are released when 2.016 g. (formula weight) of hydrogen burn. Hydrogen, when hot, removes oxygen and even some other elements from many compounds. Such a process is called *reduction*, the exact opposite of *oxidation*. For example, $CuO + H_2 \longrightarrow Cu + H_2O$.

For many years hydrogen has been used in the oxyhydrogen blowtorch to weld metals or cut up large masses of scrap iron. It also has a large and growing use in hardening liquid fats. The catalytic and high-pressure hydrogenation of petroleum (chemical addition of hydrogen) to yield more gasoline now consumes large quantities of hydrogen. Hydrogen is also used in the reduction of

tungsten from its ores, and in the manufacture of ammonia by direct union of nitrogen and hydrogen (Haber process).

ACTIVITY SERIES OF METALS

Oxidation of metals

(Most active)		
1. Lithium, Li	9. Manganese, Mn	16. Lead, Pb
2. Potassium, K	10. Zinc, Zn	17. Hydrogen, H
3. Sodium, Na	11. Chromium, Cr	18. Copper, Cu
4. Barium, Ba	12. Cadmium, Cd	19. Mercury, Hg
5. Strontium, Sr	13. Iron, Fe	20. Silver, Ag
6. Calcium, Ca	14. Nickel, Ni	21. Platinum, Pt
7. Magnesium, Mg	15. Tin, Sn	22. Gold, Au
8. Aluminum, Al		*(Least active)*

Hydrogen is placed in the *activity series* above to show its relative affinity for oxygen. Silver and the metals below it do not unite directly with oxygen, but the higher in the list above silver a metal is, the more vigorously it unites with oxygen. Since the metals below hydrogen in this activity series do not displace hydrogen from the carbonic acid of natural waters and thus turn into salts, they are the only ones found free as minerals. Only the metals above hydrogen displace it from acids.

Reversible reactions

In Fig. 25 you observe that steam passed over very hot iron powder in a tube *oxidizes* the iron to the black oxide, Fe_3O_4, with escape of hydrogen. Yet in the second tube when hot hydrogen is passed over this hot Fe_3O_4 it is *reduced* to metallic iron, with escape of steam. In a closed tube (containing iron and steam — or iron oxide and hydrogen) all four substances will appear and remain in definite proportions as long as the particular temperature is maintained. Both reactions occur, since no products can escape, and the speed of the two opposing reactions soon become equal. We arrive at an equilibrium to be disturbed right or left only by change of temperature or by removing at least one of the products. Many

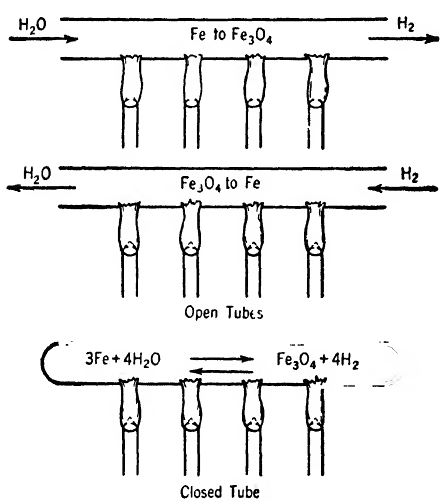


FIG. 25. REACTIONS IN OPEN AND CLOSED TUBES

Equilibrium

reactions are reversible if all products are kept in contact, but others are not.

How many grams of hydrogen are formed when 80 g. of zinc react with sufficient hydrochloric acid to dissolve the metal? *Solving chemical reactions*

1. Write the equation: $Zn + 2\,HCl \longrightarrow H_2 + ZnCl_2$.
2. Select the two substances that are being discussed.
3. *Select four numerical values relating to this pair of substances,* two in grams and two in terms of atomic weight units. Of course x g. is the unknown weight of the product in grams. The 80 g. in the problem and the x g. desired, together with the weight of the molecules (or atoms) compared in the equation, give you these four values. From the balanced equation and the problem we see that one atom of zinc, Zn, and one molecule of hydrogen, H_2, are to be compared. Write the numerical values appropriately above and below Zn and H_2. The weight of Zn is 65.38 and of H_2 is 2.016.

$$\overset{80\text{ g.}}{\underset{65.38}{Zn}} + 2\,HCl \longrightarrow \overset{x\text{ g.}}{\underset{2.016}{H_2}} + ZnCl_2.$$

4. Write these values in a chemical proportion exactly as indicated by step 3. Solve for x. The proportion could logically be written, $80:65.38 = x:2.016$. Expressed as a fraction the proportion becomes

$$\frac{80}{65.38} = \frac{x}{2.016}, \quad \text{or} \quad x = 2.47 \text{ g. of hydrogen.}$$

Problems

1. How many grams of iron will be required to release 88 g. of hydrogen from sulfuric acid? *Ans.* 2437.9 g.

2. Will silver react with hydrochloric acid? Aluminum? Gold?

3. Hydrogen was passed over heated copper oxide. The water formed was collected and found to weigh 56 g. (a) What is the weight of the hydrogen that reacted with the copper oxide? (b) What is the loss in weight of the CuO? *Ans.* (a) 6.26 g.; (b) 49.73 g.

VALENCE

Valence is most easily understood by reasoning from the following experiment.

Experiment

Suppose we weigh out 23 mg. (milligrams) of sodium, 24 mg. of magnesium, and 27 mg. of aluminum and introduce them under inverted tubes of dilute hydrochloric acid. (The sodium should be closed in a gelatin capsule which dissolves slowly.) The hydrogen

gas released pushes the water solution down the tubes as shown in Fig. 26. By experiment (at 0° C. and atmospheric pressure) we observe that

23 mg. sodium release 11 cc. hydrogen which weigh 1 mg.
24 mg. magnesium release 22 cc. hydrogen which weigh 2 mg.
27 mg. aluminum release 33 cc. hydrogen which weigh 3 mg.

Since 23, 24, and 27 are the atomic weights of these metals, we must have weighed out an equal number of atoms of all three. Consequently,

1 atom of sodium displaced 1 atom of hydrogen

$$Na + HCl \longrightarrow NaCl + H.$$

1 atom of magnesium displaced 2 atoms of hydrogen

$$Mg + 2\,HCl \longrightarrow MgCl_2 + 2\,H.$$

1 atom of aluminum displaced 3 atoms of hydrogen

$$Al + 3\,HCl \longrightarrow AlCl_3 + 3\,H.$$

Definition of valence

Valence is a number which indicates how many atoms of hydrogen or chlorine one atom of a given element will hold or displace. The valence of aluminum is 3 because one atom of it displaces three atoms of hydrogen, or holds in a compound three monovalent atoms of chlorine. Why is the valence of magnesium 2? Prove your point by reference to magnesium compounds.

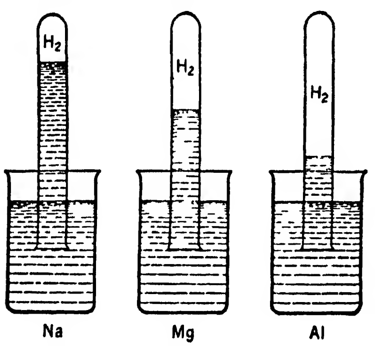


FIG. 26. RELATIVE VOLUMES OF HYDROGEN DISPLACED BY MILLIGRAM-ATOMIC WEIGHTS OF SODIUM, MAGNESIUM, AND ALUMINUM

Atoms with a valence of 1 are monovalent; with a valence of 2, divalent; and with a valence of 3, trivalent. In water, H_2O, the oxygen is *divalent* because one atom of it holds two atoms of *monovalent* hydrogen. From analysis and the use of atomic and molecular weights we know that Al_2O_3 is a correct formula. Why is aluminum *trivalent* in this case?

Radicals

We have already noticed in equations, such as

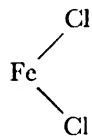
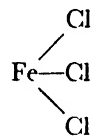
$$2\,Al + 3\,H_2SO_4 \longrightarrow Al_2(SO_4)_3 + 3\,H_2, \tag{19}$$

that one sulfur atom and four oxygen atoms seem to hold together as a group or *radical* during reactions, passing from one compound

to another unchanged, much as a family of Arabs might move from one tribe to another without the loss of any of its members. We call the SO_4 group of atoms the *sulfate radical*. It is incapable of existing free without its negative charge, and its compounds are usually called *sulfates*. For example, H_2SO_4 is known as "sulfuric acid," but it could be called *hydrogen sulfate*. Sodium sulfate is Na_2SO_4, magnesium sulfate is $MgSO_4$, and aluminum sulfate is $Al_2(SO_4)_3$. In nitric acid, HNO_3, and the nitrates derived from it such as $NaNO_3$ and $Cu(NO_3)_2$, we find the NO_3 group of atoms holding together pretty well through all sorts of reactions; so we call it the *nitrate radical*.

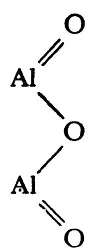
Structural formulas

It is convenient, although wholly diagrammatic, to represent the valence units between elements by straight lines or "bonds of valence." The two structural formulas

 and 

Valence of iron

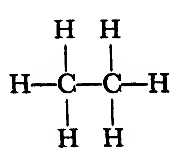
represent ferr*ous* chloride and ferr*ic* chloride, in which iron is seen to have valences of 2 and 3. The *-ic* compounds of a metal are those in which the metal has a higher valence than in the *-ous* compounds. The formula



merely represents Al_2O_3, showing the usual valences of 3 for aluminum and 2 for oxygen.

Valence of carbon

In ethane, C_2H_6, the valence of carbon might seem to be 3, yet, if the structural formula is written, it is seen that each carbon atom is holding the other and that the correct valence is 4.



Electrovalence

When sodium unites with chlorine to form molecules of sodium chloride ($NaCl$), the sodium really loses an electron to the chlorine atom (page 60). This leaves an excess of *one positive charge* on the sodium atom — and, in the compound formed, the sodium atom has a positive valence of 1. When the neutral chlorine atom gains an electron, it possesses an excess of *one negative charge* — and in the compound it has a negative valence of 1. (These charged atoms are called *ions*.) The bond between atoms set up by electron transfer is called an *electrovalent bond*. It is limited to acids, bases, and salts.

In any compound the algebraic sum of the positive and negative valences is zero. Thus in H_2O the total positive valence of the two hydrogen atoms is 2 and the negative valence of the oxygen atom is also 2.

Covalence

In many reactions electrons are not transferred but are shared in pairs by two atoms. Each atom contributes one electron to make a pair by which the two atoms are held together with a valence of 1. If the valence between these atoms is 2, it is necessary that two pairs be shared. (See Fig. 27.)

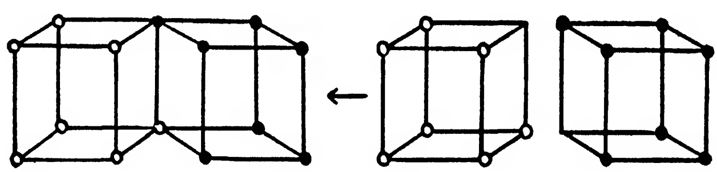


FIG. 27. A MOLECULE OF FLUORINE FORMED FROM TWO ATOMS BY THE SHARING OF ONE PAIR OF ELECTRONS

Oxidation and reduction

Although the original conception of *oxidation* means the addition of oxygen, and *reduction* means the loss of oxygen, we advance now to a broader view. Increase in valence of the central positive element in a compound has come to be considered as oxidation and decrease in valence as reduction. For example, the process

$$2\,FeCl_2 + Cl_2 \longrightarrow 2\,FeCl_3, \tag{20}$$

involving an increase in the valence of iron from 2 to 3, is recognized as oxidation of ferrous chloride to ferric chloride. The free chlorine bringing this about is called the *oxidizing agent*. Note that oxygen is not even involved.

Valence changes

In the reaction above, one iron atom in each molecule of ferrous chloride has lost one electron (to a neutral chlorine atom) and so has been oxidized.

(21)
$$\overset{2+\ 1-}{2\ FeCl_2} + \overset{0}{Cl_2} \longrightarrow \overset{3+\ 1-}{2\ FeCl_3}.$$
$$\underbrace{\quad\quad\quad}_{2(-)}$$

Oxidation is the loss of electrons by the atom or substance oxidized. Reduction is the gain of electrons by any substance reduced.

Problem

Assuming that chlorine is monovalent and that oxygen is divalent, state the valence of each of the metals in the following compounds: MgO, $AgCl$, $AlCl_3$, FeO, Fe_2O_3, SnO_2, Cr_2O_3, $NiCl_2$, $AuCl_3$.

WATER

Water, in a sense, is the central liquid of our life. It makes up two-thirds of the human body and 50 to 75 per cent of all plant tissue. Even in engineering, water is important: for marine transportation, water power, steam power, cooling systems, and many other uses.

Properties of water

One cubic centimeter of water weighs one gram at 4° C.; so by definition its *density* is 1 g. per cc., a standard of comparison for all liquids and solids. Lead, for example, has a density of 11.34 g. per cc., mercury, of 13.6 g. per cc. At 0° C. ice is lighter than water and floats.

The boiling point of water is 100° C. when the air pressure is 760 mm. as measured by the mercury barometer. The *specific heat of water* is 1, which means that one calorie of heat is required to raise the temperature of one gram of water one degree C. This relatively high value makes a hot-water heating system effective.

Reactions

Chemically, water is reactive although sometimes extra heat is needed.

(22) $$Cl_2 + H_2O \rightleftharpoons HCl + HClO.$$
(23) $$Ca + 2\ H_2O \longrightarrow Ca(OH)_2 + H_2.$$

Even carbon, if red hot, reacts to form the excellent fuel called *water gas.*

(24) $$C + H_2O \longrightarrow H_2 + CO.$$

Quicklime, CaO, and ammonia, NH_3, react with water to form bases.

(25) $$CaO + H_2O \longrightarrow Ca(OH)_2.$$
(26) $$NH_3 + H_2O \rightleftharpoons NH_4OH.$$

Calcium carbide, CaC_2, with water yields acetylene, C_2H_2.

Heavy water

Heavy water (deuterium oxide), D_2O instead of H_2O, occurs as one molecule out of every five thousand of ordinary water.

Deuterium is the name for hydrogen of atomic weight 2.016 whereas common hydrogen weighs 1.008. These two forms are explained by Fig. 28.

Hydrates

Certain substances crystallize from their water solutions on slow evaporation of the liquid, carrying definite amounts of the water with them. This *water of crystallization*, as it is sometimes called, is not held in a wet-sponge fashion, for no amount of pressure will squeeze it out. Moreover a sponge can hold any amount of water within wide limits, but not so with these *hydrates*. Hydrates must be definite chemical compounds because atomic weights are essential in expressing their composition. Some substances, it is true, can crystallize with two or three definite amounts of water, but the shape of the crystals, solubility, and some other properties are then different in each case.

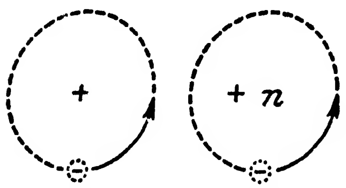


FIG. 28. ATOMS OF HYDROGEN AND DEUTERIUM

The hydrogen atom (left) has one planetary electron revolving around one nuclear proton; the deuterium atom (right) has, in addition, one nuclear neutron designated by the letter *n*.

Loss of water

These hydrates feel dry to the touch but easily lose their combined water on heating. For example, blue vitriol is a hydrate with five molecules of water of crystallization, $CuSO_4 \cdot 5\,H_2O$. When heated above 200°, all the water is lost and a white powder is obtained, $CuSO_4$. Some hydrates such as washing soda ("soda ash"), $Na_2CO_3 \cdot 10\,H_2O$, lose water in ordinary air and crumble to a powder, but others do not.

Drying agents

Gases are dried by passing them through granules of anhydrous calcium chloride, $CaCl_2$, over fragments of sodium hydroxide, NaOH, or quicklime, CaO (or a mixture of both), through ultraporous "silica gel," SiO_2, or aluminum oxide, Al_2O_3, or by freezing out the moisture. Some substances take up water chemically, while others condense it in submicroscopic capillaries (as with silica gel).

CARBON, ITS OXIDES AND CARBIDES

The importance of carbon and its compounds is evident as we learn that they make up our fuels, whether coal, wood, oil, or gas; our lubricants; our foods; our clothing; and that they make possible the change from iron to steel. Drugs, dyes, explosives, auto finish, pencils, picture film, printer's ink — these and a host of

other necessities are based upon carbon and its compounds. *Organic chemistry is the study of these compounds of carbon.*

The element occurs in nature in two crystalline forms, the diamond and graphite, and is readily prepared from its compounds in such amorphous forms as coke, lampblack, charcoal, bone black, and carbon black.

Diamonds The diamond, a pure form of crystalline carbon, is the hardest substance known and hence scratches all other substances. Faulty specimens are used for the cutting edge of rock drills and in the form of dust to polish good diamonds or to make cutting wheels for the machine shop.

Graphite Graphite, another form of carbon, crystallizes in flat plates which slip over each other with pressure. This explains its use in the common lead pencil and as a lubricant (suspended in oil or water). Carbon and graphite electrodes make possible the modern electric furnace and electrolytic cell. Iron takes up 1 or 2 per cent of carbon from the coke used in the blast furnace, but much of this dissolved carbon crystallizes out on cooling as graphite.

Coke, charcoal When bituminous coal is heated in retorts without contact with air, many volatile products escape. These include ammonia, illuminating gas, and coal tar, from the last of which benzene, toluene, phenol (carbolic acid), creosote, naphthalene, and many derivatives of great value are secured. Coke is left behind in the retort. It is used in making pig iron from iron ore, for melting metals in foundries, as a domestic fuel, and in the manufacturing of some carbon compounds. Just as coke is the nonvolatile product of coal distillation, so is charcoal the residue remaining when wood is distilled in the absence of air.

Carbon black When a smoky gas flame is chilled, the incandescent carbon of the yellow flame deposits as a form of soot called *carbon black*. An inferior carbon called *lampblack* is prepared by chilling the smoky flame from tars and oils.

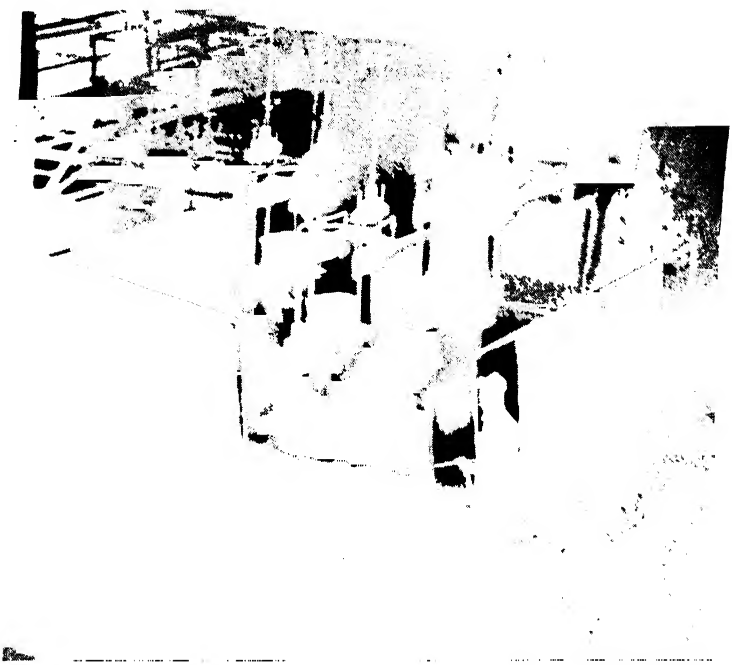
Carbon black doubles the tensile strength of rubber for tires and quadruples its resistance to abrasion. In a set of tires for a small car there are 12.5 lb. of carbon black. Over 330,000,000,000 cu. ft. of natural gas are burned annually in the United States to make about 500,000,000 lb. of carbon black.

Carbides A carbide is a simple compound of carbon with one other element. A carbide of iron, Fe_3C, is found in steel and has great influence on the properties of the steel. Calcium carbide, CaC_2, which

reacts with water to form acetylene, C_2H_2, is the product of electric-furnace heating of lime and coke (Fig. 29).

$$(27) \qquad CaO + 3\,C \longrightarrow CaC_2 + CO.$$
$$(28) \qquad CaC_2 + 2\,H_2O \longrightarrow C_2H_2 + Ca(OH)_2.$$

Preparing calcium carbide



Courtesy Monsanto Chemical Company

FIG. 29 AN ELECTRIC FURNACE

The heat of the arc between large carbon electrodes is necessary for some chemical reactions. Resistance heating is also used.

A carbide of silicon, sold by one firm as carborundum, SiC, one of the most useful *abrasives* known, is likewise prepared in an electric furnace but from a mixture of coke and sand.

$$(29) \qquad SiO_2 + 3\,C \longrightarrow SiC + 2\,CO$$

Tungsten carbide and tantalum carbide are also used because of their extreme hardness. Boron carbide is hard enough for use as the nozzle of sandblast apparatus.

***Carbon dioxide,* CO_2**

The combustion of coke or carbon compounds such as coal, wood, and oils with plenty of oxygen present yields carbon dioxide gas. Any other elements present would probably burn to their oxides.

In the lime kiln carbon dioxide is driven off by heating limestone (calcium carbonate) to a red heat, quicklime being left behind.

(30) $$CaCO_3 \longrightarrow CaO + CO_2.$$

In the laboratory a convenient method is the treatment of any carbonate with a dilute acid.

(31) $$CaCO_3 + 2\ HCl \longrightarrow CaCl_2 + H_2CO_3.$$
(32) $$H_2CO_3 \rightleftharpoons H_2O + CO_2.$$

Carbonic acid, H_2CO_3, a very weak acid, is unstable as shown above. The *reversibility* of the last reaction reminds us that carbon dioxide reacts with water to form soluble carbonic acid. Under pressure, like all gases, its solubility may be greatly increased. When this pressure is released at the soda fountain for your pleasure, your nose is tickled by carbon dioxide escaping from the drink. "Dry ice" is frozen carbon dioxide.

Bicarbonates

Complete neutralization of carbonic acid, H_2CO_3, by bases such as sodium hydroxide, NaOH, or calcium hydroxide, $Ca(OH)_2$, yields ordinary calcium carbonate, $CaCO_3$, or sodium carbonate, Na_2CO_3.

(33) $$Ca(OH)_2 + H_2CO_3 \longrightarrow CaCO_3 + 2\ H_2O.$$

However, an excess of carbonic acid may react with calcium carbonate (as in limestone caves) to yield the soluble calcium bicarbonate.

(34) $$CaCO_3 + H_2CO_3 \longrightarrow 2\ Ca(HCO_3)_2.$$

Hardness of water is due, in part, to this soluble bicarbonate.

Sodium bicarbonate, $NaHCO_3$, is mixed with solid acidic salts to form baking powder. When heated dry it breaks down into the ordinary carbonate and carbon dioxide.

(35) $$2\ NaHCO_3 \longrightarrow Na_2CO_3 + CO_2 + H_2O.$$

Carbon monoxide

Carbon monoxide, a colorless, poisonous, almost odorless gas, is formed by burning carbon, or its compounds, with insufficient air.

(36) $$2\ C + O_2 \longrightarrow 2\ CO.$$

It is also formed when carbon dioxide is passed over red-hot carbon.

(37) $$CO_2 + C \longrightarrow 2\ CO.$$

It is a valuable fuel gas as shown by the great use of "water gas," a mixture of $H_2 + CO$.

A concentration of 0.02 per cent of carbon monoxide, CO, may cause headache in a few hours; *0.06 per cent will produce unconsciousness* in two hours. Automobile exhaust gases contain about 7 per cent CO; hence, the danger of warming up the engine in a closed garage.

Problems

1. Balance the equations below following the procedure from p. 63.

$$CH_4 + O_2 \longrightarrow ?$$
$$C_2H_5OH + O_2 \longrightarrow ?$$

The first equation represents the burning of natural gas and the second the burning of alcohol.

2. How many grams of carbon dioxide result upon heating adequately 7 kg. of pure marble (calcium carbonate, $CaCO_3$)? *Ans.* 3.078 kg.

MOLECULAR AND ATOMIC WEIGHTS

Avogadro's theory (1811), since confirmed, states that *equal volumes of all gases, under the same conditions of temperature and pressure, contain the same number of molecules.*

Exact molecular weights

From the above it is evident that if we weigh equal volumes of gases under like conditions, we get weights that bear the same ratios as the weights of single molecules. In other words, we can thus obtain *the relative molecular weights* of all gases. (Solids must be handled differently.)

For good reasons the weight of the oxygen molecule was finally set at 32 and all others compared to it. For example, if a liter of a new gas weighs ten times as much as a liter of oxygen, its molecule must weigh ten times 32 or 320 in the system of relative weights. In the next section, methods of determining the molecular weights of solids will be given.

Atomic weights

In 1858 Cannizzaro laid the foundations for the determination of atomic weights. He suggested, in line with Avogadro's law, that:

1. The molecular weights of all the volatile compounds of the element studied should be determined.

2. These compounds should all be analyzed and the part of each molecular weight due to the particular element determined.

3. The smallest weight of the element found in any of the molecular weights of its compounds should be selected as the atomic weight.

By this logic the atomic weight of chlorine, for example, would be found from a study of many chlorine compounds to be 35.46. Similarly, H = 1.008. Since the molecular weight of oxygen was taken as 32 and the weight of a single atom found to be 16, the formula of oxvgen gas must be O_2.

Formula for water

Let us derive the formula of water. Analysis would show that about eight-ninths of any quantity of water taken, or of one drop, or of the smallest particle with all the properties of water (the molecule), is oxygen. We apply this approximate fraction to the molecular weight, $\frac{8}{9} \times 18.016 = 16$. Since the weight of one atom of oxygen is 16 (as just shown), there must be only one atom of oxygen in the molecule of water. The rest of the molecular weight is $18.016 - 16 = 2.016$, which is, of course, the weight of two atoms of hydrogen, and the formula must be H_2O. Of course, *molecular weights are expressed in atomic weight units.*

The molecules of most, but not all, elementary gases are diatomic. For example, a molecule of chlorine is Cl_2, and of nitrogen, N_2. Yet mercury vapor is monatomic and its molecular formula is Hg.

The molar volume

One liter of oxygen at atmospheric pressure and 0° C. weighs 1.429 g. Hence the number of liters occupied by 32 g. of oxygen gas $= \frac{32}{1.429} = 22.4$. Now, by Avogadro's law, this is the volume occupied by the molecular weight in grams of any gas (molar volume). If we weigh a convenient volume of any gas, correct the volume to 0° C. and 760 mm. pressure, we can calculate the weight of 22.4 l. This weight is also the relative molecular weight. Note that the gram molecular volume, 22.4 l., is approximately 1 cu. ft.

Another useful trick follows. Suppose you want to know the weight of a liter of methane gas. The formula is CH_4. The combined weights of one atom of carbon and four atoms of hydrogen give the molecular weight, 16.042. Of course 22.4 l. weigh 16.042 g. (the gram-molecular weight). Therefore, 1 l. weighs $\frac{16.042}{22.4} = 0.7162$ g.

Problems

1. Does the molar volume of water (weighing 18 g.) refer to liquid or gaseous water? The molar volume is nearly 1 cu. ft. or exactly 22.4 l.

2. If 1 l. of nitrogen at 0° C. and 760 mm. weighs 1.251 g., what must be the molecular weight? *Ans.* 28.02

3. Find the molecular weight of a gas, of which 500 cc. at 18° and 752 mm. weigh 2.5 g. *Ans.* 120.65

4. What will be the volume occupied by 12 g. of carbon tetrachloride, CCl_4 (molecular weight 153.84), in the vapor state at 0° and 760 mm.? *Ans.* 1.747 l.

SOLUTIONS

True solutions A solid and a liquid may be mixed so intimately that separate particles of the solid are no longer seen. With certain reservations, to be given later, we may call this mixture a *solution.* Salt water is a good illustration. If the mixture is less intimate, so that minute particles of the solid may be seen with the microscope, we may call it a *suspension.* Two liquids may also mix so completely as to deserve the term solution. If less thoroughly mixed, minute droplets of one liquid suspended in the other may be seen, under proper conditions. Such a mixture is called an *emulsion.* Milk is an example.

True solutions are nonsettling and are homogeneous to the eye. Suspensions or emulsions settle, sooner or later, and are not homogeneous under severe optical tests. In a true solution the particles of a dissolved substance are single molecules or, at most, groups of a few molecules. Consequently, a solution is a homogeneous molecular mixture of substances. It should be noted that the proportions of the components of a solution (sugar and water, for example) may be varied within limits. This distinguishes a solution from a compound.

Solid solution Some alloys are mixtures of metals or intermetallic compounds so intimate as to deserve the name *solid solutions.*

Calcium carbonate (limestone) does not actually dissolve in hydrochloric acid but the calcium chloride formed does dissolve.

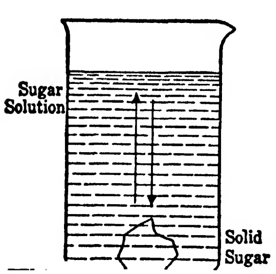


FIG. 30. SOLUTION EQUILIBRIUM

In a saturated solution of sugar a sugar crystal dissolves at the same rate that sugar in solution deposits on the crystal.

The liquid in which a solid or gas has been dispersed to form a solution is called the *solvent* and the substance dissolved is the *solute.* By concentration we mean the amount of solute dissolved by a definite quantity of the solvent, often referred to as so many grams per 100 cc. of solvent.

A *saturated solution* is one which will remain unchanged in concentration if placed in contact with some of the undissolved solute. It is said to be in *equilibrium* with excess of solute. That is, when we stir sugar, for example, into water, it continues to dissolve until the rate at which the sugar molecules in solution deposit on the lumps of solid sugar just equals the rate at which the solid throws molecules into the solution. (See Fig. 30.)

Solubility measurements

There are many degrees of solubility. Limestone ($CaCO_3$), called *calcium carbonate* in its pure form, dissolves to the extent of 0.0013 g. in 100 cc. of water, while 143 g. of potassium hydroxide dissolve in the same quantity of water. By a "10 per cent solution" we mean one containing 10 g. of solute and 90 g. of water (or other solvent).

The temperature effect on solubility must be considered in any of these quantitative statements. Nearly all solids are more soluble

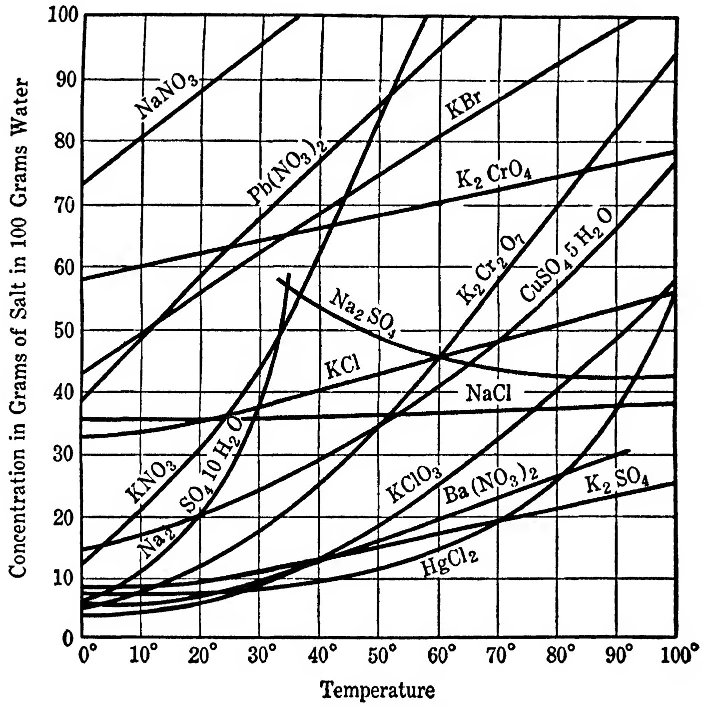


FIG. 31. SOLUBILITY CURVES — TEMPERATURE CENTIGRADE

with rise in temperature, but there are a few exceptions, notably some calcium salts of organic acids. (Gases, however, are less soluble with rise in temperature.) Potassium nitrate is seven times as soluble at 70° C. as at 8° C. There can be no accuracy, then, in a solubility statement unless the exact temperature of the measurement is given. This is usually 20° C.

Temperature effects

The solubilities at various temperatures may be plotted in the form of curves, using the ordinates to represent the number of grams of solid dissolved by 100 cc. of water. The abscissas represent temperatures. With the curves in Fig. 31 the solubility at any

temperature may be read at a glance. What is the solubility at 18° C. and 90° C. of lead nitrate? Of sodium chloride at 77° C.?

Solubility of gases in liquids

In 1803 Henry observed that at any definite temperature the solubility by weight of a gas in a liquid is proportional to the pressure of the gas, except for the very soluble gases which, like ammonia, react with water. With rise in temperature all gases are less soluble in liquids.

A *molar solution* contains a molecular weight in grams (one mole) of solute in a liter of solution. Since the molecular weight of sulfuric acid is 98, a molar solution of this acid contains 98 g. per liter.

A *normal solution* of an acid, base, or salt contains one gram-equivalent weight of the solute in one liter of solution.

Equivalent weights

The *gram-equivalent weight* of an acid contains 1.008 g. of acidic hydrogen (which requires 36.468 g. of HCl, for example) while a gram-equivalent weight of any base is the weight containing 17.008 g. of basic hydroxyl (− OH) as is the case with 40.008 g. of NaOH. The acid and base are exactly equivalent to each other as shown in the equation below. The base exactly neutralizes the acid. By "acidic" hydrogen we mean only those hydrogen atoms replaceable by a metal.

$$\underset{40.008\text{ g.}}{Na}\boxed{OH + H}\underset{36.468\text{ g.}}{Cl} \longrightarrow \underset{18.016\text{ g.}}{H_2O} + \underset{58.46\text{ g.}}{NaCl}.$$

With H_2SO_4, the formula weight (98.076 g.) would contain 2.016 g. of acidic hydrogen. Hence, 49.038 g. of this acid would be the acid equivalent of 36.468 g. of HCl and either the H_2SO_4 or the HCl would neutralize 40.008 g. of NaOH.

Normal solutions

In the case of a salt such as "blue vitriol" or copper sulfate, $CuSO_4$, we note that one copper atom has taken the place of the two hydrogen atoms in H_2SO_4 and so we divide the weight of the molecule by 2, as we did with the acid, to obtain the equivalent weight. Since "blue vitriol" is a hydrate, $CuSO_4 \cdot 5\,H_2O$, we include the weight of the $5\,H_2O$ (90.04) in the formula weight which is then divided by two, $\frac{249.71}{2} = 124.85$ g., to arrive at the equivalent weight. Consequently, one liter of solution containing 124.85 grams of $CuSO_4 \cdot 5\,H_2O$ is a *normal solution*. The symbol for normal is N. Twice normal is $2\,N$ and one-tenth normal is $0.1\,N$ or $N/10$, with other decimal values representing exact concentrations.

Freezing-point lowering

A most interesting point is brought out if solutions of gram-molecular weights of such substances as alcohol, glycerine, sugar, and acetone, in 1000 g. of water, are frozen. They all freeze at about $-1.86°$ C. Hence, the *molar depression* of the freezing point of water is 1.86° C. We can usually obtain the molecular weight of a soluble substance by noting how many grams of it must be dissolved in 1000 g. of water to lower the freezing point to $-1.86°$ C. Acids, bases, and salts are excepted.

In a similar way the boiling point rise caused by one formula weight in grams of solute per 1000 g. of solvent is 0.52° for water (with the exception of acids, bases, and salts as explained later).

Problems

1. How could you make 1 l. of a normal solution of hydrobromic acid (HBr)? Of potassium hydroxide? Of barium hydroxide, $Ba(OH)_2$?

2. (a) How many grams of $Na_2SO_4 \cdot 10\ H_2O$ are there in 800 cc. of $N/5$ solution? (b) Of Na_2SO_4 in 800 cc. of $N/5$ solution? *Ans.* (a) 25.78 g.; (b) 11.36 g.

3. Prove that a solution is a mixture rather than a compound.

CHLORINE, BROMINE, IODINE, AND FLUORINE

Occurrence of the halogens

These extremely active elements (with fluorine), called the *halogens*, resemble each other so much that they are grouped as a family. In fact, the first three sometimes occur together in salt deposits.

Chlorine occurs in nature chiefly in common salt (sodium chloride), NaCl. The ocean contains 3 per cent of salt, but salt beds, often hundreds of feet below the ground surface, are the chief commercial source. *Bromine* is obtained from the sodium bromide occurring as an impurity in some salt deposits and from sea water. *Iodine* is obtained from an impurity in Chile saltpeter and from a seaweed. *Fluorine* is found in fluospar (calcium fluoride), CaF_2, and in cryolite, $AlF_3 \cdot 3\ NaF$.

Preparation of chlorine

The most convenient laboratory methods for the preparation of chlorine depend upon the oxidation of hydrogen chloride into water and free chlorine:

$$(38) \qquad \boxed{O_2 + 4\ H}\,Cl \longrightarrow 2\ H_2O + 2\ Cl_2.$$

Oxygen alone does this only with extreme slowness unless aided by a catalyst at a favorable temperature, but good oxidizing agents such as potassium permanganate, manganese dioxide, sodium dichromate, and lead dioxide will oxidize it rapidly without a catalyst. A concentrated solution of hydrogen chloride in water

(called *hydrochloric acid*), when warmed with the oxidizing agent, gives this reaction.

(39) $$MnO_2 + 4\ HCl \longrightarrow 2\ H_2O + MnCl_2 + Cl_2.$$

Preparation of bromine

Bromine may be prepared similarly by oxidation of hydrogen bromide, HBr, and iodine by oxidation of hydrogen iodide, HI. Most of our bromine is displaced from sodium bromide by the more active chlorine.

(40) $$Cl_2 + 2\ NaBr \longrightarrow 2\ NaCl + Br_2.$$

Chlorine by electrolysis

The leading commercial method of making chlorine, especially in the United States, is the electrolysis of water solutions of sodium chloride. Chlorine is evolved at the anode, which must be made of carbon, usually in the form of graphite, because the gas is so active it would react with metal anodes. At the cathode, usually of iron, sodium is released, but of course it reacts at once with water to form sodium hydroxide, NaOH, and hydrogen. The sodium hydroxide is a valuable by-product. Since chlorine in the presence of water reacts with sodium hydroxide, it is necessary to keep the solutions around the two electrodes from mixing. The current of electricity enters at the anode and leaves at the cathode.

Molten sodium chloride conducts electricity and can thus be decomposed into chlorine and sodium. With no water present the sodium accumulates. This is one of the commercial sources of metallic sodium.

Properties of chlorine

Chlorine, a very poisonous, greenish-yellow gas, can be liquefied at 18° C. by 16.5 atmospheres pressure. It is shipped under pressure in tank cars containing 15 tons of the liquid. Dry chlorine does not attack steel unless it is very hot.

Chlorine unites with most elements to form chlorides, although extra heat may be required.

(41) $$H_2 + Cl_2 \longrightarrow 2\ HCl.$$

It will even pull the hydrogen out of some compounds to form hydrogen chloride.

(42) $$CH_4\ \text{(methane)} + 2\ Cl_2 \longrightarrow C + 4\ HCl.$$

Chlorine not only dissolves in cold water (6.3 g. per liter) but reacts with it:

(43) $$Cl_2 + H_2O \rightleftharpoons HCl + HClO.$$

This is a reversible reaction, but the hypochlorous acid, HClO, slowly decomposes into hydrochloric acid and oxygen.

Two-thirds of all the chlorine produced in America is used in bleaching paper pulp, the remainder in bleaching textiles, in water purification, in laundry bleaching, and so on. Until recent years chlorine was used in combination with lime as "bleaching powder." *Uses of chlorine*

Most city water is purified by the aid of chlorine, and the soldiers' drinking water at the front may be made safe by a little *hypochlorite*, NaClO. Chlorine is needed in the preparation of one synthetic rubber (neoprene) and in making glycol (antifreeze), degreasers, and certain solvents. Chlorine is essential to most of the poison gases used in modern warfare. Rolling clouds of chlorine were used in the earlier gas attacks in World War I.

Bromine, although less active, is similar in its properties to chlorine. However, it is a heavy, red-black liquid at room temperatures. Iodine is a violet-black solid at room temperatures, easily volatilized by heating without melting (*sublimation*).

Bromine is used in making some aniline dyes, in making potassium bromide (a valuable nerve sedative), and in the form of silver bromide it is the sensitive material on photographic plates. In 1924 *tetraethyl lead* came into extensive use as an "antiknock" addition to gasoline. About 2 cc. of ethylene bromide, $C_2H_4Br_2$, per gallon of gasoline are required to furnish enough bromine to change the lead set free in the explosion into $PbBr_2$ (lead bromide). *Bromine*

Iodine is rather expensive, finding application as silver iodide in photography, iodides in medicine, and in the manufacture of some dyes. *Iodine*

Fluorine as the free element is not yet very important, but some of its compounds have great value. *Hydrofluoric acid* and its acid salts are used in etching glass; calcium fluoride, fluorspar, is used as a slag former in the steel industry; antimony trifluoride and hydrogen fluoride are important catalysts in certain organic chemistry reactions; and "Freon," CF_2Cl_2, is a safe refrigerating substance (substituting for ammonia and sulfur dioxide). *Fluorine*

THE HALOGEN ACIDS

The most used laboratory and commercial method of preparation of hydrogen chloride, HCl, is the heating of any suitable chloride (sodium chloride is the cheapest) with concentrated sul-

Hydrochloric acid

furic acid. The gas is dissolved in water for convenient use as *hydrochloric acid.*

(44) $$NaCl + H_2SO_4 \longrightarrow NaHSO_4 + HCl.$$

Hydrogen fluoride, HF, is prepared in similar fashion from calcium *fluoride (fluorspar).*

(45) $$CaF_2 + H_2SO_4 \longrightarrow CaSO_4 + 2\,HF.$$

This method gives a badly contaminated product if it is used to release HBr and HI from metallic bromides and iodides. These two acids are so unstable that they are oxidized, to some extent, by the concentrated sulfuric acid which, when hot, can give up some of its oxygen to the hydrogen of the halogen acids, forming water and free bromine or iodine with by-products of SO_2 or H_2S. Phosphoric acid (not an oxidizing agent) is sometimes substituted for the sulfuric acid.

Uses of hydrogen chloride

Hydrogen chloride is a colorless gas of sharp, choking odor, extremely soluble in water. The concentrated "chemically pure" acid of commerce contains 37 per cent hydrogen chloride and 63 per cent water. It is one of the strongest acids.

Metals more active than copper displace hydrogen from it. (See p. 76.)

(46) $$Fe + 2\,HCl \longrightarrow FeCl_2 + H_2$$

Oxides of the metals react with it to form water and a chloride of the metal:

(47) $$2\,Ca\boxed{O + 4\,H}Cl \longrightarrow 2\,H_2O + 2\,CaCl_2.$$

Preparation of salts

Hydroxides of the metals (bases) react with hydrochloric acid to form water and metallic chlorides. This suggests general methods of preparing sulfates, nitrates, and other salts.

(48) $$Na\boxed{OH + H}Cl \longrightarrow H_2O + NaCl.$$

(49) $$Ca\boxed{(OH)_2 + 2\,H}Cl \longrightarrow 2\,H_2O + CaCl_2.$$

The base and acid can be mixed in such proportions that the solution is no longer sour (acidic) or soapy (basic), nor does it change the color of litmus, a dye (p. 96). We say the acid has neutralized the base and vice versa.

Since hydrochloric acid attacks all metallic oxides, hydroxides, and carbonates, it is useful in preparing metallic chlorides. It also is useful in "pickling" steel wire before galvanizing with melted

zinc. The black furnace scale, Fe_3O_4, formed when hot iron or steel remain in the air, must be removed to permit adhesion of zinc. Sheet steel is usually pickled in diluted sulfuric acid (cheaper).

Hydrofluoric acid

Hydrofluoric acid, HF, is the only acid that rapidly attacks glass or silica; so it is useful in etching glass apparatus (thermometers, and so on), polishing silicate rock, and removing sand from straw (for hats). Recently anhydrous (water-free) hydrogen fluoride has found favor as a catalyst in the petroleum refinery. This acid is so corrosive and toxic that it is very dangerous stuff to handle.

Other chlorine compounds

Chlorine reacts with water to form, in an equilibrium, *hypochlorous acid*, HClO, a strong oxidizing agent.

(50) $$H_2O + Cl_2 \rightleftharpoons HCl + HClO.$$

Sodium hypochlorite, NaClO, is a useful bleaching agent and bactericide. Its calcium salt occurs in "bleaching powder." The chlorates, such as $NaClO_3$, readily give up oxygen on heating so are used in making matches, fireworks, and some explosives.

Problems

Balance the following:

$Ca + Cl_2 \longrightarrow ?$

$CaCO_3 + 2\,HCl \longrightarrow ?$

$CuO + 2\,HBr \longrightarrow ?$

$BaCO_3 + 2\,HCl \longrightarrow ?$

$KOH + HCl \longrightarrow ?$

BASES OR ALKALIES

Sodium hydroxide

Electrolysis of melted, dry sodium chloride yields the very active metal sodium at the cathode and chlorine at the anode. If "salt" water is electrolyzed, sodium hydroxide, NaOH, is the cathode product. Much sodium hydroxide is also prepared by treating boiling soda solutions with slaked lime in some excess; thus we obtain

(51) $$Na_2CO_3 + Ca(OH)_2 \longrightarrow CaCO_3 + 2\,NaOH.$$

The precipitated calcium carbonate (same as limestone or marble) is filtered off, calcined to form more lime, and the filtrate is concentrated. In commerce, sodium hydroxide is sold in white sticks or lumps. It is very corrosive on the hands or in the lungs.

Bases, compounds of metals with —OH groups such as NaOH, $Ca(OH)_2$, $Ba(OH)_2$, are useful for neutralizing acids. Water and a salt are the products.

(52) $$Na\boxed{OH + H}Cl \longrightarrow NaCl + H_2O.$$

The hydroxides of sodium and potassium are the strongest, or most reactive, common bases. Ammonium hydroxide, NH_4OH, ranks as a weak base. *Another weak one is slaked lime, $Ca(OH)_2$, but it is so cheap that it has great use. Although only mildly soluble in water, it is effective as "milk of lime," a suspension of excess lime in water.* As fast as that in solution reacts, the rest dissolves and reacts. Ferric hydroxide, $Fe(OH)_3$, is a typical insoluble base.

IONS AND ELECTROLYSIS

Acids, bases, and salts

All acids contain hydrogen, replaceable by active metals, although there are thousands of hydrogen compounds (sugar, for example) that do not contain acidic hydrogen and do not taste sour. *Litmus* paper, an "indicator," is turned red by acids.

Bases contain hydroxyl groups, —OH, linked to a metal. They neutralize acids and are recognized by the change of red litmus to blue. A salt may be formed by reaction between an acid and a base or by displacement of hydrogen from an acid by certain metals or by other methods. Copper sulfate, $CuSO_4$, is a salt.

Evidence for the ionic theory

Arrhenius in 1887 sought an explanation of the fact that acids, bases, and salts in water solution conduct electricity and are decomposed in the process, while no other substances dissolved in water act in this way. He was also concerned with the abnormally large (and quite unexpected) lowering of the freezing point of water by acids, bases, and salts as a unique class.

For example, a molecular weight in grams of glycerine in 1000 g. of water froze at $-1.86°$. So it proved with thousands of such substances but not with a salt like sodium chloride. A solution of 58.43 g. (the molecular weight in grams of sodium chloride) in 1000 g. of water froze at nearly twice 1.86° below zero. Since it was known that freezing point lowering depended upon number of dissolved particles, young Arrhenius was forced to conclude that *each molecule of salt split into two in water solution.*

$$NaCl \longrightarrow Na + Cl.$$

A fallacy?

The reader should at once recognize the necessity of accounting for more particles than indicated by a gram-molecular weight and he may see that this explanation accounts for them, but he does not see how ordinary metallic sodium can exist in contact with water. For that matter he is skeptical about the presence of ordinary chlorine in salt water. Chlorine water is greenish yellow, poisonous,

and a powerful oxidizing agent, and he knows by experience that salt water has none of these properties. He remembers that metallic sodium reacts violently with water, but there is nothing exciting to be seen when salt is dissolved in water.

Ions

The reply by Arrhenius to his critics was that this was not common sodium metal but electrically charged sodium atoms (*ions*) and that the chlorine was not ordinary chlorine but charged chlorine atoms (ions) with entirely different properties Solutions of copper salts are blue due to charged copper atoms (ions) while common copper is red and insoluble.

Rewritten to meet this explanation the equation above now becomes

(53) $$NaCl \longrightarrow Na^+ + Cl^-$$

Positive and negative ions

because in electrolysis it is the positive sodium particle or ion that migrates to the negative pole, while the negative chloride ion is attracted to the positive pole. Other acids, bases, and salts ionize similarly although some, like Na_2SO_4, yield three ions per molecule.

Since bathers in the ocean never seem to be electrically shocked, there must be as many positive charges developed from each molecule as negative charges.

Hence

(54) $$CuSO_4 \longrightarrow Cu^{++} + SO_4^{--},$$
(55) $$AlCl_3 \longrightarrow Al^{+++} + 3\ Cl^-.$$

The number of charges on an ion is the same as the valence.

A modernized ionic theory holds that in the crystal lattice of solid acids, bases, and salts, the charged ions already exist and are simply pried apart by water. Certainly we know that fused salts conduct electricity and are decomposed by it.

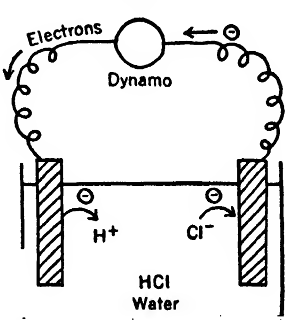


FIG. 32. ELECTROLYSIS OF HYDROCHLORIC ACID

Electrolysis

Suppose in the cell of Fig. 32 we use a solution of hydrochloric acid and pass a direct current through the circuit. The electric current is merely the passage of a stream of electrons along a conductor. The electrode where the current enters (or electrons leave) is called the *anode* and the one by which it leaves, or electrons enter, is called the *cathode*. Ions attracted to the anode are anions while those attracted to the

cathode are cations. The passage of the current merely keeps the anode positively charged and the cathode negatively charged Hydrogen escapes at the cathode and chlorine at the anode. In other words, hydrochloric acid is *electrolyzed.*

Ion travel

Hydrogen particles from the entire solution travel to the cathode and are there released. Why, unless they are electrically attracted? The cathode is negatively charged and consequently attracts positively charged bodies. The conclusion is that the hydrogen particle is simply an atom carrying a positive charge of electricity which in this form is soluble in water. On contact with the negative electrode it receives an electron from the circuit, its charge is neutralized, and the plain hydrogen atoms then unite in pairs to form ordinary hydrogen gas molecules, which escape.

The chlorine particles move from all parts of the solution to the anode, or positive electrode. This attraction must be due to the presence of a negative charge on the chlorine particle or atom. On contact with the electrode the charge on the chlorine atom is neutralized; that is, it gives its extra electron to the pole while common chlorine escapes and may be detected by the usual tests. Both the hydrogen and chlorine are primary products of electrolysis.

Electrolysis of sulfuric acid

When sulfuric acid is electrolyzed, its hydrogen is liberated at the negative electrode, as with all acids, but the discharged sulfate radical is not found as a molecular substance at the positive electrode; instead, oxygen escapes. Evidently the sulfate radical is attracted by the positive electrode because of the negative charge carried by the radical. After discharge the simple radical, unable to exist in molecular form, reacts with water.

$$2\,SO_4 + 2\,H_2O \longrightarrow 2\,H_2SO_4 + O_2. \tag{56}$$

Of course this sulfuric acid of the equation, a secondary product of electrolysis, is itself electrolyzed, and so on indefinitely.

Degrees of ionization

Under similar conditions (surface and temperature) normal acetic acid releases 1 cc. of hydrogen in contact with zinc in the same time that the zinc discharges 65 cc. of hydrogen from normal sulfuric acid and 100 cc. from normal hydrochloric acid. The only possible explanation is that there are 100 times as many hydrogen ions present at any one time in the hydrochloric acid solution as in the acetic acid solution. Yet equal normality would seem to guarantee equal concentrations of hydrogen ions.

Ionization is considered to be an equilibrium process:

$$\text{Acetic acid} \quad \text{or} \quad H \cdot \text{Acetate} \rightleftharpoons H^+ + \text{Acetate}^- \text{ (ions)}.$$

Now we might say that only a small fraction of the acetic acid molecules are ionized at any instant. On the other hand, in the reaction

$$H \cdot Cl \rightleftharpoons H^+ + Cl^-, \tag{57}$$

nearly all the molecules are ionized at a given instant. We refer, therefore, to acetic acid as a "weak acid" and to hydrochloric as one of the "strongest acids." Sulfuric acid is intermediate. Similarly, ammonium hydroxide is a weak base, sodium hydroxide a strong base. Carbonic acid, H_2CO_3, is even weaker than acetic acid. Salts with few exceptions are well ionized.

Equilibrium disturbed

It is a curious fact that a liter of normal acetic acid will neutralize just as much base as a liter of normal hydrochloric acid despite their relative "strengths." Refer to the equilibrium equations given above. If neutralization of H^+ ions from acetic acid by the OH^- ions of the base (to form H_2O) prevented the reaction from going to the left (because there would no longer be any hydrogen ions) soon all the acetic acid molecules would have ionized and the *equilibrium would have been disturbed to the right.*

In the light of this section, we may now state that acids yield hydrogen ions in aqueous solution and that bases yield hydroxyl ions.

SULFUR AND SOME OF ITS COMPOUNDS

Occurrence of sulfur

In ordinary years the Louisiana-Texas sulfur field yields nearly 3,000,000 tons of remarkably pure sulfur. At depths of 900 ft. it is melted by hot water and steam, led down through pipes with air to make it lighter. Through one of the three or four concentric pipes the frothy, aerated sulfur rises.

Combined sulfur is very common. Iron pyrites, FeS_2, once called *fool's gold*, is found in most of our states and in nearly all countries. In fact some of our most important metals occur combined with sulfur, as lead sulfide or galena, PbS; zinc blende, ZnS; and chalcopyrite, $CuFeS_2$. The sulfates also occur in great quantities. The most common are calcium sulfate or gypsum, $CaSO_4 \cdot 2\,H_2O$; barite, $BaSO_4$; celestite, $SrSO_4$; and Epsom salt, $MgSO_4 \cdot 7\,H_2O$.

Properties of sulfur

Sulfur is a pale-yellow solid, tasteless, and nearly odorless, insoluble in water but *soluble in carbon disulfide*, CS_2. When sulfur is

*melted it first turns into a thin, pale-yellow liquid, easily poured, then shows a rapid increase in viscosity at 159.5° C. At 230° C. the liquid becomes black and very viscous but at higher tempera-*tures the viscosity again decreases.

Sulfur is very active when heated, uniting with all metals except gold and platinum. It has a high electric insulating power, is water resistant, is inactive toward most acids, and has a fair degree of compressive strength.

Great quantities of sulfur are burned in the manufacture of sulfuric acid and nearly 40,000 tons are used yearly in vulcanizing rubber. Free sulfur is a necessary part of black gunpowder (with carbon and saltpeter). Some insecticides and fungicides require sulfur.

Hydrogen sulfide, H_2S

Hydrogen sulfide is found in some natural waters, such as the so-called sulfur springs, in volcanic gases, and in Mexican oil wells. Hydrogen unites with hot sulfur vapor to form hydrogen sulfide. The most convenient laboratory method of preparation is the addition of some nonoxidizing acid to a suitable sulfide, preferably ferrous sulfide, FeS, because the latter is very cheap.

$$FeS + 2\,HCl \longrightarrow FeCl_2 + H_2S. \tag{58}$$

Since hydrogen sulfide in aqueous solution is an acid (although weak), the sulfides of metals are its salts. Many occur in nature and many are formed by the addition of soluble sulfides to salt solutions. *Any sulfate can be reduced to the sulfide by heating with carbon.*

Properties of H_2S

The odor of H_2S is very disagreeable. Air containing one part of this gas in one thousand *is rapidly fatal*, nearly as dangerous as prussic acid but for its warning odor. Since this gas is evolved in some manufacturing processes, it is an industrial poison.

A solution of hydrogen sulfide open to the air soon has a film of sulfur on the surface, showing how easily it is oxidized. A rapid oxidation is secured by burning the gas in air.

$$2\,H_2S + O_2 \longrightarrow 2\,H_2O + 2\,S, \tag{59}$$
$$S + O_2 \longrightarrow SO_2. \tag{60}$$

Sulfur dioxide, SO_2

Sulfur burns in air with a pale blue flame, forming sulfur dioxide. In the United States this is a great commercial source. It is also formed as a by-product when sulfide ores are roasted in air. As a result, many smelters have attached sulfuric acid plants to utilize the waste sulfur dioxide.

The cheapest sulfide ore is iron pyrites, FeS_2, which burns like coal.

(61) $$4\ FeS_2 + 11\ O_2 \longrightarrow 2\ Fe_2O_3 + 8\ SO_2.$$

The sulfur dioxide obtained in this way from roasting sulfide ores is, of course, diluted with several volumes of nitrogen from the air.

Properties and uses

Sulfur dioxide is a colorless gas with a peculiar choking odor. It is 2.2 times as heavy as air and is easily liquefied — under 3 atmospheres pressure at 20° C. In fact, an ice-salt mixture will liquefy it at atmospheric pressure.

Sulfur dioxide is used to some extent as a refrigerant (because it is volatilized and liquefied at convenient temperatures). (See Fig. 33.) Huge quantities are necessary for the manufacture of sulfuric acid and for calcium sulfite used in papermaking. A great deal of SO_2 is used in *bleaching* and some in fumigation, although formaldehyde has almost displaced it from that field.

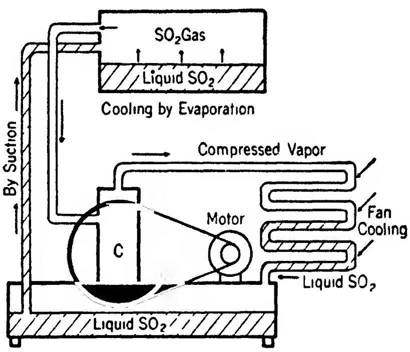


Fig. 33. Refrigeration with SO_2. (After Deming)

Sulfuric acid, H_2SO_4

The sulfuric acid industry is now so enormous that the United States makes over 8,000,000 tons annually. There are two rival processes of manufacture, the lead-chamber process for 62–70 per cent acid and the contact process for concentrated acid of high purity. The less concentrated acid is used in making superphosphate fertilizer and in removing from sheet steel the black oxide (Fe_3O_4) formed in hot rolling.

Lead-chamber process

The essential reactions of the lead-chamber process involve the oxidation of sulfur dioxide to the trioxide in the presence of enough water to form sulfuric acid. A mixture of air and the dioxide reacts very slowly, so oxides of nitrogen are introduced to increase the speed of reaction.

(62) $$SO_2 + NO_2 + H_2O \longrightarrow H_2SO_4 + NO.$$

If the nitric oxide, NO, were wasted, the lead-chamber process would be too costly. But luckily this nitric oxide has a property — seemingly unimportant to the student but of thrilling interest to the men owning lead-chamber plants — of rapidly uniting with

oxygen from the air to form the useful nitrogen dioxide, able to *oxidize SO_2 into SO_3.*

(63) $2\,NO + O_2 \longrightarrow 2\,NO_2$. *and then* $SO_2 + NO_2 \longrightarrow SO_3 + NO$.

In fact nitric oxide is a *catalyst,* making the oxygen of air unite with sulfur dioxide in a roundabout way and itself being regenerated time after time.

Contact process

In the contact process a mixture of sulfur dioxide and air is led through a series of iron tubes containing a porous supporting material such as asbestos or magnesium sulfate holding the catalyst, finely divided platinum at 400° C., or vanadium pentoxide at 500° C.

(64) $$SO_2 + O \rightleftharpoons SO_3 + 22{,}600 \text{ cal.}$$

The sulfur trioxide is caught in 98 per cent acid, reacting with the water present. Steady addition of water or dilute acid maintains this concentration.

$$H_2O + SO_3 \longrightarrow H_2SO_4.$$

NORMAL ANNUAL USE OF SULFURIC ACID BY UNITED STATES INDUSTRIES

Fertilizers	2,100,000 tons	Paints, pigments	500,000 tons
Petroleum refining	1,210,000 tons	Explosives	190,000 tons
Chemicals	975,000 tons	Rayon and cellulose	400,000 tons
Dyes, coal products	740,000 tons	Textiles	116,000 tons
Steel pickling	980,000 tons	Miscellaneous	400,000 tons
Metallurgical uses	570,000 tons	Total	8,181,000 tons

THE PERIODIC TABLE

The Russian, Mendeleeff, in 1869, tabulated the elements in periods of eight in order of increasing atomic weights and showed that the properties of the elements are periodic functions of their atomic weights. See the table on the following page. So fundamental was this great conception that he was able to leave gaps in the table for elements yet to be discovered. Later his prophecy was fulfilled and the new elements fitted into the system. There were a few obvious flaws such as the wrong order of tellurium and iodine but these were corrected by Moseley in 1914 when he arranged the elements in periods in increasing order of their *atomic numbers.*

Atomic numbers

Strangely enough, these atomic numbers are all whole numbers. The atomic numbers are approximately half the atomic weights,

PERIODIC TABLE SHOWING ATOMIC NUMBERS IN RED

PERIOD	GROUP 0	GROUP I	GROUP II	GROUP III	GROUP IV	GROUP V	GROUP VI	GROUP VII	GROUP VIII		
0		H 1.0080									
I	He 4.003	Li 6.940	Be 9.02	B 10.82	C 12.01	N 14.008	O 16.0000	F 19.00			
II	Ne 20.183	Na 22.997	Mg 24.32	Al 26.97	Si 28.06	P 30.98	S 32.06	Cl 35.457			
III	A 39.944	K 39.096	Ca 40.08	Sc 45.10	Ti 47.90	V 50.95	Cr 52.01	Mn 54.93	Fe 55.85	Co 58.94	Ni 58.69
		Cu 63.57	Zn 65.38	Ga 69.72	Ge 72.60	As 74.91	Se 78.96	Br 79.916			
IV	Kr 83.7	Rb 85.48	Sr 87.63	Y 88.92	Zr 91.22	Cb 92.91	Mo 95.95	Ma ?	Ru 101.7	Rh 102.91	Pd 106.7
		Ag 107.880	Cd 112.41	In 114.76	Sn 118.70	Sb 121.76	Te 127.61	I 126.92			
V	Xe 131.3	Cs 132.91	Ba 137.36	Rare Earths	Hf 178.6	Ta 180.88	W 183.92	Re 186.31	Os 190.2	Ir 193.1	Pt 195.23
		Au 197.2	Hg 200.61	Tl 204.39	Pb 207.21	Bi 209.00	Po 210.0				
VI	Rn 222		Ra 226.05	Ac ?	Th 232.12	Pa 231	U 238.07		Elements — are discussed on next page		

The *atomic number* appears before the symbol of each element in red and the *atomic weight* below. With atomic numbers 81–84 and 88–92 are to be assigned a number of other elements, for the most part radioactive, which have the same chemical properties as the element in the table but somewhat different atomic weights.

and represent, in reality, *the number of positive charges of electricity* (protons) *in the nucleus* of the atom of each element. In addition to the free protons there are (except in the case of hydrogen) some neutrons. Electrons are present in the outer shells of the atom. These protons in the nucleus are electrically balanced in the neutral atom by an equal number of planetary electrons; hence the atomic number is measured either by the number of free protons in the nucleus or by the number of planetary electrons.

Use of the table

In Group I at least one of the valences of each element is unity (Na^1, K^1, Ag^1). In Group II the valence is 2, and so on as we proceed to the right. To the right of Group IV, although valence toward oxygen increases, it decreases in hydrogen compounds. For example, N_2O_5 and NH_3. As you see, correct formulas may be deduced from this useful table. Furthermore, the elements on the left are decided base formers (NaOH), those in the middle are weakly basic or weakly acidic — $Al(OH)_3$ acts both ways — while those on the right are strongly acid-forming (HNO_3 and HCl). Proceeding down any column we find that the elements become more metallic (compare Si and Pb). Group 0 elements with *no combining power*, and therefore *no valence*, were discovered after Mendeleeff's time.

RADIOACTIVITY

The Curies

Because of Becquerel's work with the penetrating radiations from uranium compounds, 1896, the Curies (Madame Marie and her husband, Professor Pierre Curie of the Sorbonne) decided to examine some of the residues from pitchblende, a uranium mineral, after all the uranium had been extracted. They found that these residues gave off similar but more powerful radiations than the uranium salt itself.

Finally the Curies isolated polonium, a powerful source of radiations that could go through objects opaque to visible light and record their effects on a photographic plate or discharge an electroscope. As a climax radium was discovered in the residues, in 1898, with an activity 1,000,000 times that of uranium.

Radium rays

Soon after the discovery of radium, scientists found that it gives off three types of radiations. The *alpha rays* are positively charged atoms or nuclei of helium moving 15,000 miles per second (in a vacuum). They do not travel 15,000 miles per second in air which quickly reduces their velocity to zero. In fact, they are stopped

completely before they have gone 8 cm. through the surrounding air. After colliding with several thousand air molecules, their initial velocity is reduced to nothing.

Electron streams

The *beta rays* are identical with cathode rays (see page 58), but move faster and are 100 times as penetrating. These cathode radiations and beta rays are known to be merely *streams of electrons.* When thrown off by radium they move nearly 186,000 miles per second (the velocity of light). Since they have only $\frac{1}{7400}$ the mass of alpha particles, they are deflected much more by a magnetic field. As negative particles they are deflected in the opposite direction from alpha particles. In spite of their tremendous initial velocity, they are completely absorbed after smashing through 1 cm. of water or 1 mm. of lead or 10 mm. of aluminum.

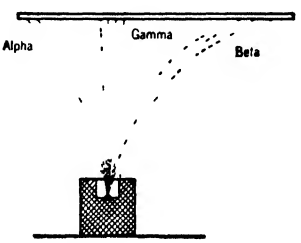


Fig. 34. Three types of radium radiation in a magnetic field

The *gamma rays* resemble X-rays but are far more penetrating. They are electromagnetic disturbances, not streams of small particles, and hence are not deflected in a magnetic field (Fig. 34). Although alpha particles are stopped by 0.1 mm. thickness of aluminum foil, gamma rays penetrate 1 ft. of steel. Since gamma rays affect photographic film, they are used to detect flaws in steel parts.

Radioactivity

The half-life of radium is 1590 years. Half the radium remaining at the end of that period decays in another 1590 years, and so on. This decay is due to the emission of electrons and alpha particles. There are other radioactive elements such as polonium, actinium, mesothorium, and protactinium. The cheapest is mesothorium for half of it is lost in 6.7 years. That is why luminous paint for watch faces is often prepared from a mixture of one to ten parts of mesothorium (worth about 15 cents) with one hundred thousand parts of sensitive, phosphorescent zinc sulfide.

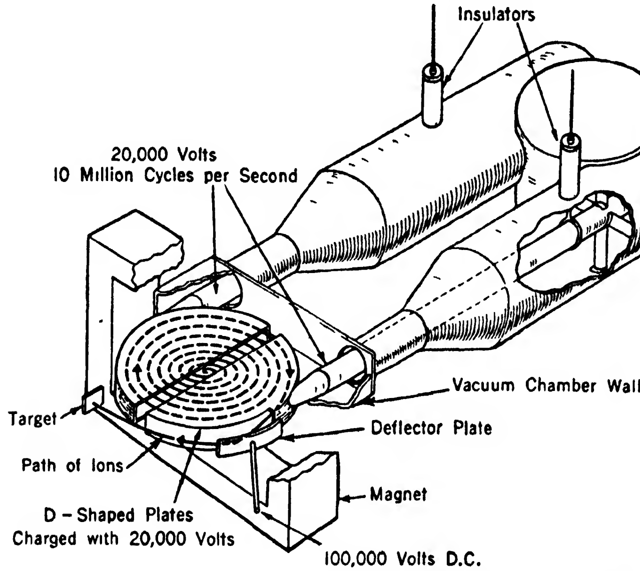
Importance of radium

Radium competes with X-rays in the treatment of cancer and in photography. The world's stock exceeds 800 g. and the price, once $135,000 per gram, has fallen to $25,000 because of discovery of larger ore deposits.

One of the most important results of the research on radioactivity has been the discovery of the inner nature of the atom. In recent years this has led to "atom smashing" with such high-

velocity "bullets" as *alpha rays* from polonium, *protons* (hydrogen nuclei), *deuterons* (ionized deuterium), and *neutrons.* These high velocities are given by powerful X-ray tubes and by the Lawrence cyclotron (Fig. 35). The impact of such bullets energizes the atomic nucleus which throws off various types of particles leaving a different element. Even after the bombardment ceases, this artificial radioactivity may continue for hours or weeks.

Artificial radioactivity



Courtesy Life Magazine

FIG. 35. CYCLOTRON PRODUCES ARTIFICIAL RADIOACTIVITY

Gaseous particles, ionized by a hot filament, are attracted by the two evacuated hollow electrodes, shaped like halves of a round pill box. The magnetic field deflects the ions and, with 20 million reversals of current per second, a succession of kicks is given them, increasing their speed until they go off at a tangent and strike the target where they may split atoms or produce artificial radioactivity.

In 1939 and 1940 the atomic fission (splitting) of the uranium isotope of mass 235 was recognized both in Europe and in America. Neutrons from radium bombardment of beryllium were slowed down by passage through such hydrogen compounds as water or paraffin so that they were effectively captured by the U–235 nucleus followed by its splitting into barium and krypton or into other elements not differing greatly in mass. This fission was accompanied by a tremendous release in energy, nearly 200,000,000 electron volts per atom of U–235.

Uranium fission

There is an important chain of reactions in fission in the sense that secondary neutrons are released to cause other fissions and,

Chain reaction

theoretically, to continue until all the U–235 in a given sample is exhausted. However, the presence of U–238 and other impurities quickly breaks this *chain reaction*. Since U–235 is only one part in 140 of common uranium, the critical problem is separation to obtain pure U–235. The atomic bomb contained purified U–235 or the new element plutonium, No. 94, prepared by suitable bombardment of common U–238. Plutonium also can undergo chain fission. For further details read the note facing page 103.

THE ATMOSPHERE

Air, a mixture About 99 per cent of air is composed of two gases, oxygen and nitrogen, and the proportions of these vary but slightly. Several other substances are present, some in proportions that vary considerably. The approximate composition of dry air by volume is

Nitrogen	78.00 per cent
Oxygen	21.00 per cent
Argon	0.93 per cent
Carbon dioxide	0.03 per cent
	99.96

In addition to the four named above there are found water vapor, helium, neon, krypton, xenon, dust, and sometimes ozone, hydrogen sulfide, and oxides of nitrogen in traces.

Not a compound The evidence that these various constituents are present merely as a mixture, and not as one compound, is convincing.

1. Each constituent of the air retains its own properties irrespective of the presence of the others. The oxygen is just as soluble in water as if the nitrogen were missing.

2. The properties of air are the mean of the properties of its components, which is not true of a compound. The density, for example, is what might be calculated from such a mixture.

3. When liquid air evaporates, the more volatile nitrogen tends to escape faster than the oxygen. In pure compounds, water for example, the distillate has exactly the same composition as the original liquid, unless decomposition occurs. Therefore, air can be separated into its constituents by liquefaction and fractional distillation. Compounds cannot.

4. Nitrogen and oxygen can be mixed in the same proportion as found in air, without change in temperature or volume, producing a mixture like air. Evidently, no reaction occurs.

5. The proportions by weight cannot be represented by a chemical formula because the constituent gases are not found in exact multiples of atomic weights.

Air conditioning

When air is saturated with water vapor at, say 26.6° C. or 80° F., we feel depressed and are incapable of much exertion. If there is only 50 per cent as much water vapor present as could be held at saturation, we feel brisk and alert. Hence the statement, "The most comfortable relative humidity is about 50 to 60 per cent."

Body cooling by evaporation of perspiration gives us comfort, but at high relative humidity this evaporation is slight. Even a fan stirring the air, blowing wet air away from the skin, is helpful. Air-conditioning apparatus for cooling, washing out dust, and maintaining a comfortable relative humidity makes life more pleasant on trains, in offices, homes, hotels. In factories of certain types conditioning of the air is essential.

Argon and helium

Nearly 1 per cent of the air is argon, a very "lazy" gas, valuable for its *inertness* in argon-filled electric-light bulbs. Only a trace of helium is present in air. We separate it from natural gas in the Amarillo district of Texas. Since its lifting power is about 90 per cent that of hydrogen, it is desirable as a filler for balloons and airships. Tracer and incendiary bullets cannot ignite inert helium.

NITROGEN AND AMMONIA

Nitrogen, making up four-fifths of the air, is cheap unless it must be entirely free from oxygen. Then air must be liquefied and the nitrogen separated by careful *fractional distillation*, at considerable cost.

This gas is lazy at room temperatures, but at high temperatures (electric arc) it may be forced to unite with oxygen, forming oxides of nitrogen. Persuaded by suitable catalysts and high pressure and temperature, it may react with hydrogen to form ammonia, a reaction of tremendous importance in war and in peace.

Ammonia

Ammonia gas is conveniently prepared in the laboratory by warming a dry ammonium salt with lime or with any other base.

(64) $$NH_4Cl + NaOH \longrightarrow NaCl + NH_4OH,$$

(65) $$NH_4OH \longrightarrow H_2O + NH_3.$$

The ammonium hydroxide, NH_4OH, breaks down on heating into the gas and water. In fact, warming a concentrated solution of ammonium hydroxide is often found a most convenient method of securing the gas.

Soft coal contains 1 per cent, or more, of combined nitrogen. About one-fifth of this is recovered on distillation as ammonia. The mixed gases of distillation are passed through wash water which retains the ammonia and some impurities. Later, lime is added to this ammonia liquor and steam is passed through. The ammonia is thus driven off and may be neutralized with sulfuric acid to form ammonium sulfate (about 20 lb. from 1 ton of coal). In an average year the United States produces 900,000 tons of ammonium sulfate from coal.

Coke manufacture

Up to thirty years ago coke was made in beehive coke ovens, allowing the ammonia and other gases to waste into the air. Now over 90 per cent of all American coke is made in by-product ovens with recovery of the ammonia, benzene, tars, coal gas, and other products. However, by far the larger part of all commercial ammonia is prepared by the catalyzed high-pressure, high-temperature union of nitrogen and hydrogen (Haber process).

$$N_2 + 3\,H_2 \longrightarrow 2\,NH_3. \tag{66}$$

Reactions of ammonia

Ammonia will "steal" oxygen from metallic oxides.

$$3\,CuO + 2\,NH_3 \longrightarrow 3\,Cu + 3\,H_2O + N_2. \tag{67}$$

Ammonia reacts directly with some hot metals to form nitrides such as Mg_3N_2. It is actually used to harden the surface of steel by formation of an iron nitride.

Acids react with the gas to form ammonium salts, but only in the presence of at least a trace of water.

$$NH_3 + HCl \longrightarrow NH_4Cl. \tag{68}$$

One of the most important reactions of ammonia is with water to form ammonium hydroxide.

$$NH_3 + H_2O \rightleftharpoons NH_4OH. \tag{69}$$

Uses of ammonia

Ammonia is the raw material for the Ostwald catalytic process of making nitric acid. It is combined with acids to make ammonium salts used in enormous quantities for fertilizers. Its salts are used in soldering, galvanizing, in dry batteries, and the gas itself is the chief refrigerant in commercial use.

Nitrogen fixation

In 1913 the German chemist Haber developed the extremely difficult process of *forcing a reaction* between nitrogen and hydrogen.

$$N_2 + 3\,H_2 \rightleftharpoons 2\,NH_3. \tag{70}$$

Haber discovered that at 200 atmospheres pressure and 500° C. the formation of ammonia proceeds satisfactorily if osmium or uranium is present as a catalyst. Later, Claude, in France, improved the process by increasing the pressure to 1000 atmospheres or more. Working at such high temperatures and pressures is difficult, yet success was obtained. Chrome steel free from carbon must be used for the apparatus, as otherwise the hydrogen removes the carbon from ordinary steel, leaving "rotten steel." The catalyst now used is largely iron with some substances such as the oxides of aluminum and potassium present to promote the catalytic action.

World production

World capacity for fixed nitrogen of various types was 4,000,000 tons annually in peacetime. The demand for explosives (derived from ammonia) makes this total very inadequate for war. In peace most of the ammonia so formed is converted into ammonium sulfate fertilizer. In war much of the ammonia is catalytically oxidized by air to yield nitric acid, the basis of most high explosives.

OXIDES OF NITROGEN. NITRIC ACID

There are five oxides of nitrogen, including laughing gas or nitrous oxide (N_2O) the anesthetic, but the engineer is mainly interested in two, nitric oxide, NO, and nitrogen dioxide, NO_2.

Nitric oxide, NO

The first reaction in the catalytic (over platinum gauze at 700° C.) oxidation of ammonia by air is:

(71) $$4\,NH_3 + 5\,O_2 \longrightarrow 4\,NO + 6\,H_2O.$$

Nitric oxide (colorless gas, insoluble in water) is also made by the action of dilute nitric acid on copper and some other metals (in absence of air).

Nitrogen dioxide, NO_2

This very poisonous, red-brown gas is made quickly by direct union of nitric oxide with the oxygen of the air, below 620° C.

(72) $$2\,NO + O_2 \longrightarrow 2\,NO_2.$$

When this NO_2 meets warm water it reacts at once.

(73) $$H_2O + 3\,NO_2 \longrightarrow 2\,HNO_3 + NO.$$

Nitric acid, HNO_3

This useful acid (dangerous to skin and clothing) was once made by action of concentrated sulfuric acid on any nitrate.

(74) $$NaNO_3 + H_2SO_4 \longrightarrow NaHSO_4 + HNO_3.$$

Then on warming, the volatile nitric acid was distilled and caught in condensers. Today it is all made by the catalytic oxida-

tion of ammonia and passage of the NO_2 (from $NO + O_2$) into water, as described above.

The commercial acid, density 1.14, boiling at 120.5° C., is 68 per cent HNO_3, the rest water. As stated before, it attacks many metals — even copper which is below hydrogen in the activity series. In this case this strong oxidizing acid probably first forms copper oxide, CuO, and then reacts with it (as with any metallic oxide).

$$(75) \qquad CuO + 2\,HNO_3 \longrightarrow Cu(NO_3)_2 + H_2O.$$

Explosives

Cellulose (of cotton or wood) is attacked by nitric acid with formation of cellulose nitrates, some of which are high explosives. With glycerine a similar product called *nitroglycerine* is produced. Toluene yields the famous TNT (trinitrotoluene), and even starch is nitrated to form a powerful explosive. Concentrated nitric acid reacts with organic compounds in two ways, as is well illustrated by its reaction with glycerine.

$$(76) \qquad \text{Glycerine, } C_3H_5\boxed{(OH)_3 + 3\,H}NO_3 \longrightarrow 3\,H_2O + C_3H_5(NO_3)_3.$$

$$(77) \qquad \text{Benzene, } C_6H_5\boxed{H + HO}NO_2 \longrightarrow H_2O + C_6H_5NO_2.$$

Dynamite is essentially a solid mixture of nitroglycerine with wood dust, but the mixture usually includes nitrates of sodium or ammonium. Gelatin dynamite contains both nitroglycerine and "nitrocotton."

Problems

1. How many grams of nitric acid can be made by catalytic oxidation from 1500 g. of NH_3? *Ans.* 5549.8 g.

2. To make 10 kg. of nitroglycerine how many kilograms of glycerine are required? *Ans.* 4.05 kg.

THE PHOSPHORUS GROUP

Phosphorus, in the white or yellow form, is extremely active, igniting in warm air at about 35° C., so it is always kept under water. Phosphorus burns are very painful and slow to heal. *White phosphorus is very poisonous,* a fatal dose being only 0.15 g. Red phosphorus, obtained by heating the white variety to 230°–300° C. in absence of air is not poisonous and does not ignite below 240° C.

Phosphorus

Phosphorus is obtained from phosphate rock either in a blast furnace or in an electric furnace.

$$(78) \qquad 2\,Ca_3(PO_4)_2 + 6\,SiO_2 + 10\,C \longrightarrow 6\,CaSiO_3 + 10\,CO + P_4.$$

The phosphorus volatilizes (no air present), leaving the calcium silicate behind. The vapors condense to a liquid and finally solidify on cooling. With admission of air the element burns to the pentoxide, P_2O_5, a white powder which reacts with water to form the most useful of the three phosphoric acids, H_3PO_4. Its salts have wide industrial uses. "Superphosphate," for fertilizers, is a water-soluble calcium hydrogen phosphate, made by treating the insoluble phosphate rock with the proper amount of sulfuric acid.

Arsenic

Arsenic occurs as a sulfide in various ores. In fact most of it is obtained as a by-product in roasting copper sulfide ores. The arsenic burns to the trioxide, As_2O_3, which is removed as a white dust from flue gases by the Cottrell electric precipitator. The element is black and brittle. It has a small use in industry as a hardener for some alloys, notably lead shot. The trioxide is for the greater part converted into lead arsenate, calcium arsenate, and Paris green, as weapons in the terrific battle with insects.

Antimony

Antimony occurs as the sulfide, Sb_2S_3, stibnite. Formerly nearly all of it came from China but now Mexico supplies us. It is a silver-white, brittle element, melting at 680° C.

Its chief value is in making alloys, such as type metal, lead-antimony grids for storage batteries, bearing metals, and others. Its property of expanding on cooling makes it essential to type metal, for sharp edges result when the metal expands against the type forms.

Bismuth

Bismuth is found both free and in the sulfide. It is now obtained chiefly from anode slimes in the electrolytic refining of copper, lead, and tin. It is strictly metallic, brittle, and crystalline. In color it is silver-white with a reddish tint. It melts at 271° C. Bismuth may be electroplated, forming a film capable of taking a high polish. There is little use for the element alone, but in very low-melting alloys it is rather important.

ORGANIC CHEMISTRY — HYDROCARBONS

Homologous series

There are several hundred *hydrocarbons* (compounds of hydrogen and carbon), most of them found in petroleum, natural gas, asphalt, and coal-tar products. These are divided into a few series, with some very definite relationships between the members. Rubber is a hydrocarbon, $(C_5H_8)x$, in which possibly 2000 units of C_5H_8 are *chained together*. Natural and synthetic rubbers are discussed on pages 115 and 116.

HYDROCARBON SERIES

Methane Series C_nH_{2n+2}		*Ethylene Series* C_nH_{2n}		*Acetylene Series* C_nH_{2n-2}		*Benzene Series* C_nH_{2n-6}	
Methane	CH_4	Ethene or		Ethine or			
Ethane	C_2H_6	Ethylene	C_2H_4	Acetylene	C_2H_2		
Propane	C_3H_8	Propylene	C_3H_6	Propine	C_3H_4		
Butane	C_4H_{10}	Butylene	C_4H_8	Butine	C_4H_6		
Pentane	C_5H_{12}	Pentylene	C_5H_{10}	Pentine	C_5H_8		
Hexane	C_6H_{14}	Hexylene	C_6H_{12}	Hexine	C_6H_{10}	Benzene	C_6H_6
Heptane	C_7H_{16}	Heptylene	C_7H_{14}	Heptine	C_7H_{12}	Toluene	C_7H_8
Octane, etc.	C_8H_{18}	Octylene	C_8H_{16}	Octine	C_8H_{14}	Xylene	C_8H_{10}

The CH_2 *variation*

In each hydrocarbon series any one member differs from its neighbor by one carbon and two hydrogen atoms, that is, by a CH_2 unit. For example, in the methane series (so called from the name of its simplest member) we know that a hydrocarbon with five carbon atoms must have twelve hydrogen atoms in the molecule (C_5H_{12}) because in the series formula (C_nH_{n+2}), the value of n is 5 in this case, and $2n + 2 = 12$.

The methane and benzene series are by far the most important in the field of organic chemistry, if their derivatives are considered. The first four members of the methane series are gases at ordinary temperatures; from C_5H_{12} to $C_{15}H_{32}$ they are liquids; beyond $C_{16}H_{34}$ they are solids.

The existence of so many compounds of two elements depends wholly on the *tetravalence of carbon* and the ability of carbon atoms to hold other carbon atoms in *long chains* or in *rings*. For example, we may write

$$\begin{array}{ccccc} & H & \\ & | & \\ H— & C & —H \\ & | & \\ & H & \end{array} \quad \text{and} \quad \begin{array}{cccccc} & H & H & H & H & \\ & | & | & | & | & \\ H— & C— & C— & C— & C & —H. \\ & | & | & | & | & \\ & H & H & H & H & \end{array}$$

From the hydrocarbon table we conclude that these structural formulas represent methane and butane.

Methane

Methane, CH_4, occurs in natural gas up to 90 per cent or more. It is the pressure of methane that forces petroleum up from deep wells.

Ethylene

Ethylene, C_2H_4, is the simplest member of a series of unsaturated hydrocarbons. Note the conversion of ethane, C_2H_6, by "cracking" with heat into ethylene and free hydrogen.

$$(79)\qquad \begin{array}{c}\mathrm{H}\ \ \mathrm{H}\\ |\ \ \ |\\ \mathrm{H{-}C{-}C{-}H}\\ |\ \ \ |\\ \mathrm{H}\ \ \mathrm{H}\end{array}\ \text{plus heat and pressure} \longrightarrow \begin{array}{c}\mathrm{H}\ \ \mathrm{H}\\ |\ \ \ |\\ \mathrm{C{=}C}\\ |\ \ \ |\\ \mathrm{H}\ \ \mathrm{H}\end{array}\ + H_2.$$

The symbol, =, does not mean "equals" but indicates that the two connected carbon atoms are unsaturated, "hungry" for addition of other elements, bromine, for example. This makes possible ethylene bromide, $C_2H_4Br_2$, used in high-octane gasoline to convert the lead of tetraethyl lead into lead bromide, less harmful than lead oxide to ignition points. Ethylene glycol, $C_2H_4(OH)_2$, a similar grouping, is the leading "antifreeze" liquid.

Thiokol

A rubberlike product called Thiokol is made commercially by converting ethylene into ethylene dichloride which is then treated with a sodium sulfide complex. *Polymerization* (joining of like molecules to form larger molecules) follows.

Acetylene, C_2H_2, still more unsaturated than ethylene, is made by reaction of water with calcium carbide.

$$(80)\qquad CaC_2 + 2\,H_2O \longrightarrow Ca(OH)_2 + C_2H_2.$$

Acetylene

Acetylene, fed with oxygen in the acetylene torch, produces an intensely hot flame. With this flame a 6-in. steel shaft can be cut through in less than a minute. As soon as the steel melts, it burns in air and is blown away as oxide, in sparks.

Great quantities of acetic acid are made from acetylene by catalytic hydration and controlled oxidation in the second stage. The new synthetic rubber, Neoprene, is derived from acetylene, which itself is based upon limestone and coke in the manufacture of calcium carbide.

Petroleum hydrocarbon mixtures

Normally the United States produces two-thirds of the world's 2,000,000,000 barrels of petroleum. Since mere distillation of crude oil yields only about 25 per cent of "straight run" gasoline, it is evident that the process of *cracking* (not yet universal) which more than doubles this yield has saved for the future nearly one-half the crude oil otherwise needed for our 30,000,000 automobiles. Even with every device in use for increasing gasoline yield, we are now beginning to face a serious shortage. The solution will be production of light oils from coal by *hydrogenation* as will be explained below.

Although kerosene was once the chief fraction separated (by distillation), gasoline now leads. Such other fractions as lubricants,

fuel oil, gas oil, paraffin wax, coke, and petroleum asphalt are also very important.

Gasoline

Gasoline is a mixture of hydrocarbons ranging from C_6H_{14} up to about $C_{12}H_{26}$ while kerosene hydrocarbons may range from $C_{12}H_{26}$ to $C_{16}H_{34}$; and lubricants range as high as $C_{35}H_{72}$. It is a remarkable fact that gasoline yields six times the power of an equal amount of nitroglycerine — but more slowly. This is because nitroglycerine contains its oxygen for combustion while gasoline uses air.

Cracking

Cracking is already familiar in thermal decomposition, remember $H_2S \longrightarrow H_2 + S$ and $C_2H_6 \longrightarrow C_2H_4 + H_2$. The apparatus used for cracking petroleum is indicated in Fig. 36. In the high-temperature cracking chamber, larger molecules are split, or cracked, into smaller and more volatile molecules. *Vapor-phase cracking*, operating under somewhat higher temperatures, produces large quantities of unsaturated hydrocarbons and "ring" compounds that give a high antiknock value to the gasoline formed. High temperatures, great pressures, and catalysts are now employed in "cracking."

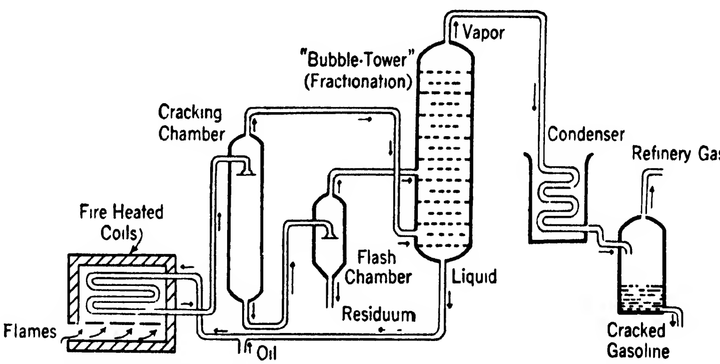


FIG. 36. PETROLEUM CRACKING PRODUCES GASOLINE FROM HEAVIER OILS

Hydrogenation

The success of hydrogenation of coal in Germany in recent years led to *catalytic hydrogenation* of petroleum in this country. The gasoline resulting is remarkably low in sulfur compounds and rich in benzene ring molecules of excellent antiknock value. In Germany and in England, where there is no local petroleum, lignite and bituminous coal are being hydrogenated to produce huge quantities of motor fuel.

Chemicals from petroleum

A large number of compounds are now being prepared, not merely separated, from petroleum by chemical reactions. The pentanes, C_5H_{12}, may be chlorinated, converted into amyl alcohol and finally into amyl acetate, a solvent used in nitrocellulose lac-

quers. Even common ethyl alcohol (intoxicating) can be prepared from petroleum fractions. Another petroleum chemical, isopropyl alcohol (C_3H_7OH), is a formidable rival to ethyl alcohol. One of the butyl alcohols, similarly prepared in large quantity, is converted into butyl acetate, an important solvent for the lacquer industry. Benzene, toluene (for TNT), glycerine, and butadiene (for synthetic rubber) are also triumphs of the petroleum chemist.

100 octane

Pure isooctane (C_8H_{18}) is given an octane number of 100 (high antiknock value) while pure heptane (C_7H_{16}) shows no antiknock value whatever. Any fuel that matched a mixture of 70 per cent isooctane and 30 per cent heptane would be said to have an octane number of 70. (See Fig. 37.) The usual octane numbers of gasolines for automobiles range from 60 to 78. Aviation gas has a 100-octane rating, obtained by use of tetraethyl lead, isooctane, tryptane, neohexane, and benzene.

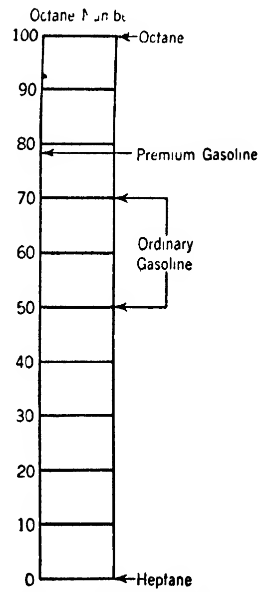


FIG. 37. OCTANE NUMBERS

"Liquid gas"

Propane, (C_3H_8), and butane (or isobutane), (C_4H_{10}), have recently become important by-products of oil refining, whereas formerly they were burned to produce steam. Now these gases are liquefied by pressure in steel cylinders or even in tank cars and are sold as a convenient economical source of gaseous fuel for rural homes.

Rubber, $(C_5H_8)x$

Natural rubber occurs in the latex of many trees and plants, the most important being *Hevea brasiliensis*, grown in plantations in the Dutch-British East Indies and growing wild in Brazil. We have experimented with the guayule shrubs in California, the Russian dandelion, and a vine from the West Indies. In normal years we use 600,000 tons of rubber in the United States.

The milky latex, or sap from the tapped tree, is coagulated with acid, the raw rubber washed, mixed with carbon (carbon black) and zinc oxide to give wearing qualities to tires, and with sulfur and reaction accelerators to make vulcanization possible.

Synthetic rubber

Synthetic rubbers are really not substitutes, for some of them have superior oil resistance as well as better air and light resistance than natural rubbers.

Natural rubber is a polymer (long-chain molecule) of isoprene, $CH_2{=}C(CH_3){-}CH{=}CH_2$ which nature has joined in elastic chains. Vulcanization involves the reaction with sulfur at these double bonds, the sulfur atom serving to tie parallel chains together giving increased strength. The double bond sign (=) indicates unsaturation of carbon, that is, its ability to add sulfur or other atoms.

Monomers

This topic naturally follows a study of petroleum refining, because a number of these synthetic rubbers use as the building brick or unit (*the monomer*) butadiene ($CH_2{=}CH{-}CH{=}CH_2$) which may be prepared from petroleum derivatives or from common alcohol.

SYNTHETIC RUBBERS

Commercial Name	*Monomer (Unit)*
Natural rubber	$CH_2{=}C(CH_3){-}CH{=}CH_2$, isoprene
Neoprene	$CH_2{=}CCl{-}CH{=}CH_2$, chloroprene
Buna S	$CH_2{=}CH{-}CH{=}CH_2$, butadiene, and styrene
Butyl rubber	Petroleum units. Not yet disclosed
Ameripol	Butadiene + copolymer secret as yet
Koroseal	$CH_2{=}CHCl$, vinyl chloride, plasticized with equal weight of tricresyl phosphate
Styrene	$CH_2{=}CH(C_6H_5)$

By 1945 we were making nearly 1,000,000 tons of synthetic rubber yearly, most of it Buna S, although neoprene, butyl, and other rubbers were included. In a competitive market after the war the relative proportions of natural and synthetic rubber sold will be determined by relative manufacturing costs, quality, and policies regarding foreign trade.

Benzene, C_6H_6

When bituminous coal is distilled (for the coke residue) many volatile products escape and are captured as by-products. Ammonia, coal gas, benzene, and toluene are notable products plus coal tar, a mixture that in turn is distilled to yield many valuable products. Our 130,000,000 gallons of benzene (benzol, as the public calls it) are consumed as an antiknock addition to our gasoline and as raw materials for manufacture of dyes and medicines.

Saturation

Chemically, benzene must be classed as an unsaturated hydrocarbon, since it is capable of adding on six hydrogen atoms or six bromine atoms per molecule. It reacts with nitric acid to form nitrobenzene, $C_6H_5NO_2$, and this in turn is reduced by active hydrogen to *aniline*, $C_6H_5NH_2$, the mother substance of a host of

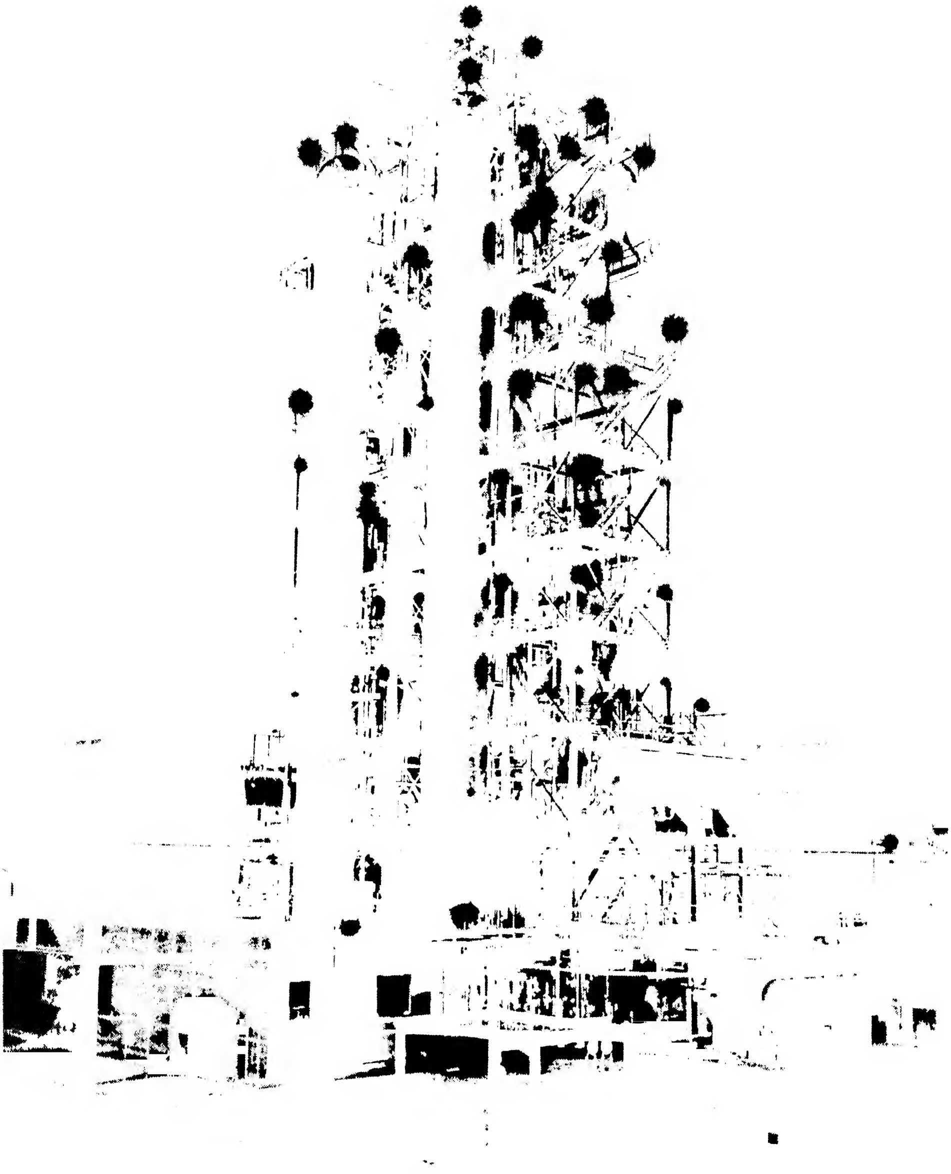


FIG. 38. STYRENE PLANT FOR PRODUCING BUNA-S RUBBER

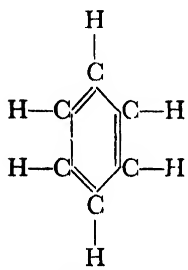
beautiful "aniline dyes." It can also be attacked by hot concentrated sulfuric acid and other reagents. In fact, over one-half of all carbon compounds are derivatives of benzene.

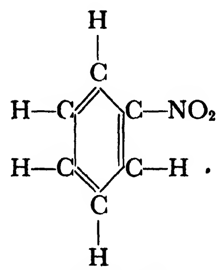
$$C_6H_6 + HNO_3 \longrightarrow C_6H_5NO_2 + H_2O \tag{81}$$

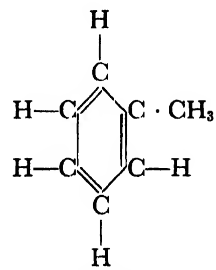
$$C_6H_5NO_2 + 3\ H_2 \longrightarrow C_6H_5NH_2 + 2\ H_2O \tag{82}$$

Ring molecules

A very interesting line of evidence shows us that the six atoms of benzene are tied together in a ring, very different from the chains of carbons in the methane series. A comparison of a few structural formulas of benzene and its derivatives will be worth while.


Benzene


Nitrobenzene

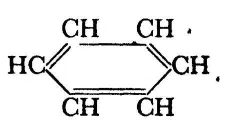

Toluene

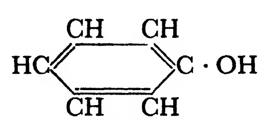
Toluene, $C_6H_5 \cdot CH_3$

Toluene, a colorless liquid boiling at 110° C, is closely associated with benzene in its manufacture from bituminous coal and resembles benzene in many properties. Its special interest to the public is because it is the source of the world's greatest explosive, TNT (trinitrotoluene). Our supply of toluene from coal distillation is quite inadequate for a major war, so it is fortunate that we have learned how to prepare it in large amounts from petroleum.

Phenol, $C_6H_5 \cdot OH$

Phenol is important since it reacts with formaldehyde to yield the important plastic, bakelite. We use 80,000,000 lb. yearly in the United States.


Benzene


Phenol

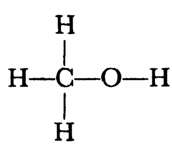
Picric acid, one of the chief high explosives, is produced by nitration of phenol. *Creosote* is a crude mixture of phenol-like compounds separated from coal tar. In a normal year we use 200,000,000 gallons to preserve railroad ties and other timbers.

Coal gas and water gas ($CO + H_2$, from hot coke and steam) *Fuel gases* have already been mentioned as has natural gas. *Producer gas* is one of the cheapest to manufacture. In a bed of coal five or six feet deep in a furnace, the lower layers may be red-hot while the upper layers are not burning at all, possibly only suffering distillation of their volatile matter. If no flame plays over the surface, the escaping gas has good fuel value. Carbon dioxide, CO_2, formed in the lower layers is reduced by the excess of hot carbon above to the valuable carbon monoxide, CO. All the nitrogen of the air used is found in the producer gas, making up nearly one-half its volume; hence the heating value is low — but the cost is also low. The best economy is to use this gas while it is still hot, thus saving all its heat.

A "semiwater gas" is made by blowing some steam with the air through the deep fuel bed in the producer-gas process. Reaction with the steam adds hydrogen from the water and more carbon monoxide.

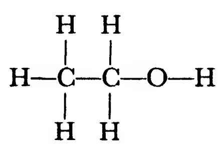
ALCOHOLS, ACIDS, FATS, AND SOAPS

Alcohols in general contain OH groups linked directly to carbon atoms as in methyl alcohol, CH_3OH, common ethyl alcohol, C_2H_5OH, and so on.



Methyl

and



Ethyl

In commercial practice methyl alcohol is made by the retort *Methyl alcohol,* CH_3OH distillation of wood, hence its common name, *wood alcohol.* In 1924 there was developed a catalytic process for preparing methyl alcohol by the direct union of carbon monoxide and hydrogen under considerable pressure in the presence of zinc oxide and copper, or other catalysts.

(83) $$2\,H_2 + CO \longrightarrow CH_3 \cdot OH$$

A very pure product is prepared in commercial quantities at a cost lower than by wood distillation. The wood distillation industry would cease to exist were it not for the superior charcoal produced by it.

Methyl alcohol, CH_3OH, a colorless liquid boiling at 67.4° C., mixes with water in all proportions, is intoxicating, and extremely poisonous. It is much used as a solvent in varnishes, and as the starting point in the manufacture of formaldehyde.

Ethyl alcohol, C_2H_5OH

Ethyl alcohol, commonly known as "grain alcohol," a colorless, inflammable liquid boiling at 78° C., mixes with water in all proportions. It is intoxicating but not poisonous. It is one of the best solvents in the laboratory and the starting point in the manufacture of many useful compounds. Internal-combustion motors can use it, with certain adjustments, and when our coal and oil deposits are exhausted the world will probably turn to cheap alcohol as a power fuel. Luckily our coal supplies are good for hundreds of years.

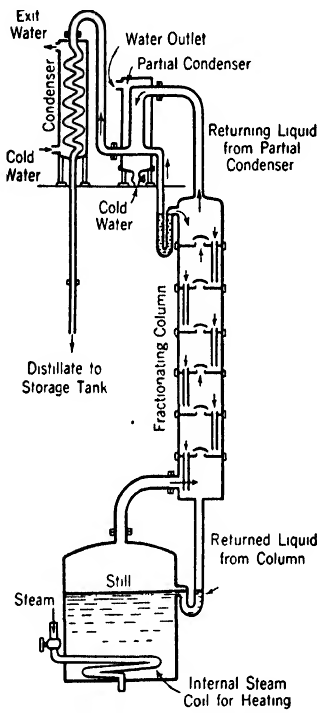


FIG. 39. ALCOHOL COLUMN STILL
The more readily condensed water vapor returns to the still while the more volatile alcohol vapor reaches the condenser.

Alcoholic fermentation of sugars supplies us with a great part of the ethyl alcohol used. The fermented liquid must be concentrated by a column still (Fig. 39) to obtain the 95 per cent alcohol of industry. Glucose is the only common fermentable sugar; hence, cane sugar, malt sugar, starch, and cellulose must all be converted into glucose before they can yield alcohol. The reactions involved are given later (p. 124).

Absolute alcohol, when listed for taxation by our government, is termed *200° proof* and the ordinary 95 per cent alcohol is 190° proof. The standard is 100° proof or 50 per cent alcohol by volume.

Glycerine

Glycerine (glycerol, to be exact) is an alcohol, although we are inclined to forget that fact, since it is nonintoxicating. Its structure shows the presence of alcoholic hydroxyl groups, three of them in each molecule. Glycerol is a colorless, syrupy liquid of sweetish taste. Since it is *hygroscopic* (attracts moisture from the air), it is often put in mixtures because of this property. Both glycerol and

glycol, $C_2H_4(OH)_2$, are much used in antifreeze solutions for automobiles. When fats are broken down in soapmaking, glycerol is a by-product. It is now possible to make great quantities of glycerol from certain products of the petroleum refinery.

Aldehydes, $R \cdot CHO$

The general formula for all aldehydes may be written $R \cdot CHO$, in which R represents any radical such as methyl, CH_3, or ethyl, C_2H_5. Wood alcohol can be oxidized moderately by air and copper oxide, to yield formaldehyde, $H \cdot CHO$. Other aldehydes are made in similar fashion. They are all reducing agents.

(84) $$CH_3OH + (O) \longrightarrow H \cdot CHO + H_2O.$$

Formaldehyde, $H \cdot CHO$

Formaldehyde is used as a 40 per cent solution called *formalin.* It is a valuable germicide and antiseptic. A large amount, 50,000,000 lb. annually, is used to react with phenol (carbolic acid) in the preparation of bakelite, perhaps the most common plastic.

The simplest organic acid is formic acid, $H \cdot COOH$, but a far more important one is acetic acid. Fruit acids (citric acid and so on) and such fatty acids as stearic acid are well known. The general formula for any organic acid is $R \cdot COOH$.

Acetic acid, $CH_3 \cdot COOH$

Acetic acid, $CH_3 \cdot COOH$, found in vinegar and made from acetylene, is used to the extent of 200,000,000 lb. yearly in this country. Much of it is converted into acetic anhydride, $(CH_3CO)_2O$, for the manufacture of cellulose acetate (one form of rayon).

Fats and oils

When an alcohol, $R \cdot OH$, reacts with an organic acid, $R' \cdot COOH$, water and an "ester," $R' \cdot COOR$, are formed. The esters derived from glycerine (as the alcohol) and the higher fatty acids, such as palmitic and stearic (and oleic from a related series), are called fats. This means that fats are a particular kind of ester. We do not have to synthesize the fats — plants and animals have done that for us — but if necessary it could be done.

(85) $$\underset{\text{Glycerine}}{C_3H_5(OH)_3} + 3\ \underset{\text{Oleic acid}}{C_{17}H_{33}\cdot COOH} \longrightarrow \underset{\text{Glyceryl oleate, a fat}}{(C_{17}H_{33}\cdot COO)_3C_3H_5} + 3\ H_2O.$$

There are very few simple fats in animal and plant tissue. Olive oil is largely glyceryl oleate (*olein*) but lard is a mixture of 40 per cent glyceryl palmitate (*palmitin*) and stearate (*stearin*) with 60 per cent olein. Beef suet contains the same fats but is harder because it contains only 25 per cent olein. Olive oil and cottonseed oil are liquids because they contain fully 75 per cent olein.

FATS USED IN THE UNITED STATES, 1939

Cottonseed oil	2,630,000,000 lb.	Fish oils	222,000,000 lb.
Tallow, inedible	878,000,000	China wood oil (tung oil)	91,000,000
Coconut oil	800,000,000	Corn oil	220,000,000
Soybean oil	710,000,000	Castor oil	41,000,000
Linseed oil	345,000,000	Peanut oil	132,000,000
Palm oil	383,000,000	Babassu	24,000,000

In 1943 the United States production of soybean oil was three times the 1939 amount.

The only suitable drying oils to rival linseed oil at present as a paint vehicle are tung oil, soya oil, fish oils, and perilla oil. Such paint oils do not really dry, but are *oxidized* to form tough solids.

Esters

Esters (compounds of alcohols with organic acids) are hydrolyzed by water, especially hot water. The equilibrium reaction below indicates this.

$$(86)\quad \underset{\text{Ethyl acetate}}{CH_3 \cdot COOC_2H_5} + H_2O \rightleftharpoons \underset{\text{Ethyl alcohol}}{C_2H_5OH} + \underset{\text{Acetic acid}}{CH_3 \cdot COOH}.$$

This *hydrolysis* takes place faster if a base is present. Of course, a salt of the fatty acid must be formed when the particular ester is a fat. For this purpose a base such as KOH is used.

$$(87)\quad \underset{\text{Glyceryl oleate}}{C_3H_5(C_{17}H_{33}{\cdot}COO)_3} + 3\,KOH \longrightarrow \underset{\text{Glycerine}}{C_3H_5(OH)_3} + 3\,\underset{\text{Potassium oleate, a soap}}{C_{17}H_{33}{\cdot}COOK}.$$

This reaction is called *saponification* of the ester, and the salt of the fatty acid is a soap.

Soap

Soapmaking on a commercial scale is usually a matter of boiling cheap mixed fats with caustic soda, NaOH, if the firm, "hard" soaps are desired, or with caustic potash, KOH, if "soft" soaps are wanted.

THE CARBOHYDRATES

This important class of compounds, which includes starch, the sugars, and cellulose, contains carbon and just twice as many hydrogen atoms as oxygen atoms in each molecule. This means that the two latter elements are found in the proportion to form water. Hence the name *carbohydrates.*

Cellulose, $(C_6H_{10}O_5)x$

Cellulose, the world's greatest crop, makes up half the weight of dry wood. Pure linen and cotton fibers, as well as the stringy parts of many vegetables and grasses, represent cellulose.

One of the reasons our forest resources of cellulose are rapidly being exhausted is the fact that every American citizen uses over 200 lb. of paper annually. Spruce, hemlock, and balsam are among the woods favored for papermaking. Interlacing fibers give paper its strength.

Starch, $(C_6H_{10}O_5)x$

Starch is stored in roots like potatoes, in grains, in chestnuts, and in the trunk of the sago palm. Rice contains 75 per cent starch, corn 50 per cent, and potatoes 20 per cent. It is a white substance of very high molecular weight $(C_6H_{10}O_5)x$, insoluble in cold water, but swelling in hot water to a pasty translucent mass, useful as an adhesive. Having alcoholic hydroxyl groups, it can, of course, be nitrated. *Nitrostarch* is a very safe explosive and is used to the extent of thousands of tons in filling hand grenades. In the laundry, starch is indispensable. It is also one of the chief sources of ethyl alcohol. When treated with water and a trace of acid as a catalyst, it is hydrolyzed into fermentable glucose.

FIG. 40. THE SUGAR MOLECULE

The formula for sugar is $C_{12}H_{22}O_{11}$. In the model the black balls represent carbon atoms, the grey balls represent oxygen atoms, and the white balls are the hydrogen atoms. Notice the arrangement.

Sugars

Sugars of the formula $C_{12}H_{22}O_{11}$ (such as cane, malt, and milk sugars) can be broken down into simpler sugars of the formula $C_6H_{12}O_6$ (like glucose and levulose). (See Fig. 40.)

Sucrose, $C_{12}H_{22}O_{11}$

Cane sugar, beet sugar, and maple sugar are all of the same composition when purified. The cane is crushed to extract the juice, which is then treated with lime to coagulate impurities that would otherwise ferment. The juice (12 to 18 per cent sugar) is next treated with carbon dioxide to precipitate the lime as calcium carbonate, or with phosphoric acid to precipitate calcium phosphate. The solution is boiled down in vacuum pans at as low a temperature as 50° C. because if boiled at atmospheric pressure and above 100° C., the sugar would be decomposed. Concentra-

tions of 65 per cent are possible. When the sugar crystallizes it is whirled free from molasses in *centrifugals* and shipped to refineries for further purification. When we can get it we use 7,000,000 tons yearly.

Glucose (dextrose)

Glucose, $C_6H_{12}O_6$, is made commercially from starch which is heated with water and a little hydrochloric acid as a catalyst.

$$(C_6H_{10}O_5)_x + xH_2O \longrightarrow xC_6H_{12}O_6. \tag{88}$$

Glucose is used in candies, soda fountain syrups, baked goods, and even as a filler for cheap soaps and leather. As an intermediate compound in the manufacture of alcohol (as well as whisky and beer) from the starch of cereals, glucose is of vast importance.

Enzymes and fermentation

The yeast plant, a very lowly but useful organism made up of microscopic cells, secretes two chemicals known as *invertase* (or sucrase) and *zymase*. These are true catalysts. Sucrase splits sugar (in the presence of water) into glucose and fructose, both fermentable, although fructose is less so than glucose. Zymase finishes the work begun by the sucrase and converts the simple sugars into alcohol and carbon dioxide.

$$C_6H_{12}O_6 \longrightarrow 2\ CO_2 + 2\ C_2H_5OH. \tag{89}$$

Nature's catalysts

It is an interesting fact that a germinating seed (barley, for example) develops enzymes as it sprouts. These catalysts, *amylase* and *maltase*, convert the insoluble starch of the seed into soluble glucose capable of being carried by the sap of the growing plant. The brewer borrows this clever device from the sprouting grain in order to make alcoholic drinks from starchy cereals. He allows barley to sprout — of course amylase and maltase are developed in the process — then heats the sprouts to kill all life. The catalysts, mere chemicals, are uninjured. This "malt" is then mixed with the warm, moist, crushed grain and the starch is hydrolyzed to maltose (malt sugar), $C_{12}H_{22}O_{11}$. Yeast is then added and the malt sugar is further hydrolyzed to form glucose. The zymase completes the change by fermenting this glucose into alcohol. By distillation a 45 per cent alcoholic solution well known as "whisky" is secured.

Starch (by amylase) $\longrightarrow$ maltose,
Maltose (by maltase) $\longrightarrow$ glucose,
Glucose (by zymase) $\longrightarrow$ alcohol.

PLASTICS AND SYNTHETIC FIBERS

We have already mentioned bakelite (made from formaldehyde and phenol) and celluloid (lower nitrated cellulose mixed with some alcohol and camphor). These were the first plastics, but now we have many, readily molded into solid forms. Some soften with heat and can be shaped readily. A waterproof plastics liquid is now used as a bond between the thin layers of plywood. With heat and pressure it sets to a powerful bonding solid used in the bodies of some aircraft, in light boats, etc. Among other plastics we should mention the following: *Plastics*

The protein (casein) of milk as well as the protein of soy beans reacts with formaldehyde, $H \cdot CHO$, to form *protein plastics*.

Urea, $CO(NH_2)_2$, which is now manufactured in this country as a possible fertilizer, a stabilizer in high explosives, and a reactant for plastics manufacture, condenses with formaldehyde to yield a new plastic called *beetle* or *plaskon*. It takes colors well, is not easily broken, and so finds use as tableware. The *urea-formaldehyde resins* or plastics are much used now in treating textiles for anticreasing, waterproofing, and fireproofing.

Vinyl resins, prepared indirectly from acetylene or ethylene, take transparent colors extremely well, are odorless and tasteless, and make useful varnishes or lacquers. Vinyl chloride, $HC_2{=}CHCl$, is polymerized to produce polyvinyl chloride fabrics. *Lacquers*

The rather new *methacrylate resin*, "Plexiglas" or "Lucite," is transparent (even to ultraviolet light), flexible, and is easily scratched (and repolished). Heavy sheets serve as windows for some airplanes. The monomer or unit for polymerization is $CH_2{=}C(CH_3){-}COOCH_3$.

Rayon is a term used to describe various fibers resembling silk in appearance (although chemically different). In every case the material is dissolved to yield a pasty liquid that may be spun (through the many holes of a "spinneret") and coagulated in a suitable bath. (See Fig. 41.) *Rayon*

Viscose is made by the reaction of sodium hydroxide, NaOH, and carbon disulfide, CS_2, with cellulose. The liquid is spun and then decomposed into the original cellulose, but with the silken luster desired. Viscose leads all the rayons. Formed into tough transparent sheets known as *cellophane* it has great sale as a wrapping material. In waterproof form it is popular in food packaging.

Nylon Nylon resembles silk and wool chemically for it is a nitrogen compound. Basically, it is derived from coal, air, and water. It is popular in hosiery and may be used in parachutes. Massive threads of *nylon* are substituted for catgut in tennis rackets.

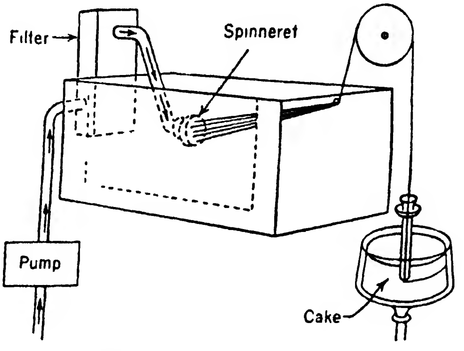


FIG. 41. RAYON SPINNING

Rayon solution is forced through many small holes in a platinum disk (spinneret), coagulated as thread in an acid bath, and spun into lustrous thread. (Colgate and Company)

SILICA AND GLASS

Silicon and Silica, SiO_2 Silicon does not occur free, yet it is the commonest element after oxygen. Over one-fourth the earth's crust is combined silicon. Its oxide (SiO_2) is found everywhere as quartz or in sand and sandstone. Clay and all common rocks except limestone and dolomite are silicates. When a mixture of iron oxide, Fe_2O_3, and silicon dioxide, SiO_2, is reduced with carbon, ferrosilicon is produced. It is much used in steel manufacture to remove air bubbles.

Silicon steel, containing only a few per cent of silicon, has for the past twenty years been used for transformer cores and in that time has saved the electrical industry hundreds of millions of dollars. With the previous iron cores there was a considerable heat loss when the magnetic state of the transformer was changed. This and other losses were greatly reduced by silicon steel. Under low magnetizing forces it is far more magnetic than the best Swedish iron and this quality does not deteriorate.

Quartz Silica, SiO_2, called *quartz* as a mineral, is very common. Its melting point is not sharp due to the high viscosity of the melted material, but it is above 1700° C. Clear *fused quartz* (see Fig. 42), melted under alternate vacuum and pressure to get rid of air

Courtesy General Electric Company

Fig 42 Laboratory glassware of fused quartz

bubbles, is very useful. Its high transmission of ultraviolet light is an important property. Since its coefficient of expansion is only 6 per cent of that of ordinary glass, it can stand sudden changes of temperature without cracking. This property makes it suitable for clock pendulums, tuning forks, and thermometers, as well as for huge lenses and reflectors in astronomical instruments.

Silicic acid, H_2SiO_3

Since SiO_2 is acidic ($H_2SiO_3 \longrightarrow H_2O + SiO_2$) it reacts with bases but not with acids (except hydrofluoric acid). With fused sodium carbonate it displaces carbon dioxide, as would any ordinary acid. The volatility of one of the products, carbon dioxide, prevents reversal of the reaction.

(90) $$Na_2CO_3 + SiO_2 \longrightarrow Na_2SiO_3 + CO_2.$$

The sodium silicates (water glass) formed in this and other ways have enormous use as cheap adhesives in making heavy fiberboard for boxes.

Glass

Glass is a mixture of silicates, usually of sodium and calcium, although many variations are possible. Common glass might be considered as $Na_2SiO_3 \cdot CaSiO_3$, but in practice it contains more silica. The dot used in the formula above indicates close association or addition of molecules.

Glass is made by melting sand, SiO_2, lime, CaO, and soda, Na_2CO_3, in a gas-heated furnace. If iron is present as an impurity, the glass is green, due to ferrous silicate. Since ferric silicate, $Fe_2(SiO_3)_3$, has a very weak yellow color, it is an advantage to oxidize the green ferrous salt with manganese dioxide. Selenium is now replacing manganese dioxide as a color corrector. Soft or low-melting glass is of the lime-soda type. Hard glass, of higher melting point, is prepared by substituting potassium carbonate for the sodium carbonate.

Window panes

Window glass is blown from a ball of hot glass into a cylinder, cut open while still hot, and allowed to flatten in the annealing oven. Bottles are made by blowing the glass in a mold. Where window-glass cylinders were formerly blown by men with strong lungs, they are now blown by machines. A hollow ring is lowered into a pool of molten glass, a bubble started, and then the ring is raised while air is blown into the bubble. As a result a beautiful glowing cylinder 40 ft. high and 2 ft. across is suspended in the dark factory like a pillar of light. This cylinder is lowered on to supports, cut open, and allowed to flatten.

Plate glass is cast on a wide iron table, rolled flat, and when cold it is ground to remove unevenness and polished with rouge. In some factories the plate-glass "melt" is allowed to flow on a revolving drum and is then passed under a roller, cut into sheets, and annealed by slow cooling. *Plate glass*

The automobile industry has created a demand for plate glass exceeding even its use for show windows. The possibility of accident and injury from flying glass has led to the use of nonshattering glass windshields and windows. Two layers of sheet glass and a middle layer of transparent cellulose acetate or vinylite plastic, with a suitable adhesive, are powerfully compressed. Under impact the flexible center layer holds broken fragments. A similar laminated glass (five layers) is in use for bulletproof windows in bank offices. Such a glass 1 in. thick turns aside machine-gun fire at 45 ft. *Shatterproof glass*

Tempered glass, a recent product, is important because it is five times stronger than plate glass. It is also more flexible, better able to stand sudden temperature changes, and although it crumbles it does not splinter when broken. Sudden cooling (by air) places the surface under compression and gives this glass its toughness.

Resistance glass of the pyrex type contains some boron trioxide (B_2O_3) in place of part of the silica (SiO_2). It resists the attack of chemical reagents and is essential to the laboratory. Since it expands very little with heating, it is not likely to crack with sudden temperature changes. This property, with its unusual strength, has made pyrex glass popular for baking dishes. *Pyrex*

Glass textiles have recently come into use wholly because the filaments are now drawn much smaller than ever before with resultant great increase in strength. A single filament has a tensile strength of 1,500,000 lb. per square inch of cross-section, but the filaments are so tiny that 60-strand thread still has a strength of only 17,000 lb. Two pounds of the finest filament would encircle the earth. This fine drawing that eliminates all internal flaws explains this tensile strength of ten times that of the best steel. A large market immediately developed for fibers woven into electric insulating tape, and another market for fluffy blanket heat insulation of houses, ships, and locomotives, and still another for dust removal from conditioned air. (See Fig. 43.) *Glass textiles*

Porcelain enamels are glasses varied somewhat in composition. They are made white by suspended tin oxide or calcium phosphate

or fluorspar. They may contain much Al_2O_3, and B_2O_3, and Sb_2O_3 as well as SiO_2. Bath tubs, sinks, etc. may be made of iron or steel coated with enamel.

Boron compounds

Boron carbide, B_4C, a new product of the electric furnace, is next to the diamond in hardness. Boric acid, reduced at high temperature by pure coke, releases boron which then unites with carbon. Its extreme resistance to abrasion makes it suitable for sandblast nozzles, wire-drawing dies, and jeweled bearings. In powdered form it can be compressed in molds, electrically fused, and molded to desired shapes.

Boric acid, H_3BO_3 (a very weak acid and mild antiseptic), is deposited from certain hot springs in Italy in flaky white crystals.

Borax, $Na_2B_4O_7 \cdot 10\,H_2O$, occurs in nature or can be readily prepared from another mineral, calcium borate. It has great use in making pyrex glass, as a flux in welding or soldering, in soaps, pottery, enamels, and as snow on indoor ski slides

Problems

1 What difference would it make to you if there were no glass? List many known uses of glass and give the preferred type or chemical composition

Fig 43 Uses of glass fiber

Modern glass fibers in the form of cloth (resistant to insects and fire), tape (electrical insulation), fluffy blankets (heat insulation) and thread

2. *Name three of the hardest substances used for cutting and grinding; give their formulas.*

3. How many kilograms of Na_2CO_3 are required to react completely, on fusion, with 15 kg. of pure quartz? *Ans.* 26.45 kg.

PRIMARY CELLS AND STORAGE BATTERIES

Flow of current

When two strips of different metals are placed in a solution of an acid, base, or salt and connected outside by a wire, a current of electricity flows, while the more active of the two metals dissolves. If zinc dipping in a solution of zinc sulfate and copper dipping in a solution of copper sulfate (separated by a porous clay cup) are used, the current of electrons flows along the wire from zinc to copper and through the solution to the zinc.

Primary cell

This device (Fig. 44) is a primary cell (as distinguished from a secondary or storage cell) and converts chemical energy into electrical energy. The zinc throws atoms into solution as positive ions. This means that some zinc atoms give up electrons to the rest of the zinc plate, making it negative. These electrons flow along the wire (current of negative electricity) to the copper plate, where they are readily taken up by the adjoining positive copper ions in solution. This action neutralizes the charge on the copper ions and causes a deposition of metallic copper on the plate. The zinc plate is finally dissolved or the copper ions all removed. The excess positive Zn^{++} ions attract negative SO_4^{--} ions from the other compartment, so all portions of the solutions are kept electrically neutral. Many types of such cells are possible. The one described is the Daniell cell.

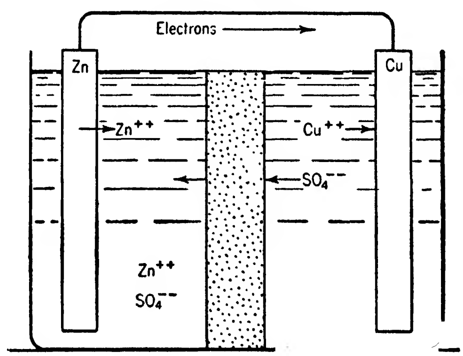


FIG. 44. A PRIMARY CELL

Voltage

The force that drives the electrons along a metallic circuit is termed *electromotive force* (e.m.f.) or *voltage* and its unit of measurement is the volt. The difference in this pressure or potential between the two poles of a Daniell cell is 1.10 volts. Other cells, such as the Clark and Weston, are better standard cells. If a cell is made of two metals close together in the electromotive series

(same as the activity series, p. 76), the voltage is very low. The farther apart they are in the series the higher the voltage.

Storage batteries

A charged lead storage cell consists of a lead plate surfaced with lead dioxide as the positive pole and a second lead plate surfaced with spongy lead as the negative pole — both immersed in 20 per cent sulfuric acid. (See Fig. 45.)

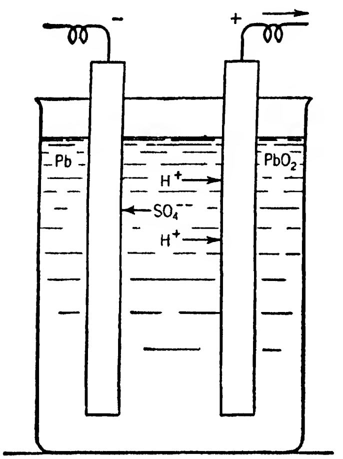


FIG. 45. A CHARGED LEAD STORAGE CELL WITH H_2SO_4 ELECTROLYTE

When the two poles are connected by a metallic circuit, some of the lead atoms on the surface of the lead plate give up electrons to the rest of the plate (making the plate negative) and go into solution as Pb^{++} ions. They do not go far, however, for they meet SO_4^{--} ions in the solution and are redeposited on the plate as insoluble $PbSO_4$.

The two negative electrons given to the lead plate, when Pb^{++} ions are formed, travel along the metallic conductor to the brown lead dioxide plate, PbO_2. There they unite with H^+ ions of the solution which on discharge promptly reduce the acidic PbO_2 to PbO, a basic oxide that readily reacts with sulfuric acid to form $PbSO_4$.

DISCHARGE REACTIONS

$$(91) \quad \left\{ \begin{array}{r} PbO_2 + 2\,H^+ + 2\,\epsilon \longrightarrow PbO + H_2O \\ PbO + H_2SO_4 \longrightarrow PbSO_4 + H_2O \\ Pb + SO_4^{--} \longrightarrow PbSO_4 + 2\,\epsilon. \end{array} \right.$$

Charging

On passing a current through the exhausted cell (as in electroplating), sulfuric acid is electrolyzed, its H^+ ions discharged on the $PbSO_4$ surface of the lead electrode reducing the lead sulfate again to the original spongy lead and forming more H_2SO_4. At the other electrode (brown PbO_2 now surfaced with $PbSO_4$) the current discharges SO_4^{--} ions which then unite with the $PbSO_4$ to form $Pb(SO_4)_2$, lead disulfate. This, however, is quickly hydrolyzed to the original brown PbO_2, lead dioxide, and to H_2SO_4. The lead storage cell is very heavy and it is far from foolproof, but there is no better storage cell.

CHARGING REACTIONS

$$(92)\quad \begin{cases} PbSO_4 + 2\,H^+ + 2\,\epsilon \longrightarrow Pb + H_2SO_4 \\ PbSO_4 + SO_4^{--} \longrightarrow Pb(SO_4)_2 + 2\,\epsilon \\ Pb(SO_4)_2 + 2\,H_2O \longrightarrow PbO_2 + 2\,H_2SO_4. \end{cases}$$

The action in charging is just the opposite of that in discharging. It is chemical energy that is stored up, not electrical. In practice it is unwise to convert too thick a layer of the plates into lead sulfate, for regeneration becomes difficult, probably due to a lack of permeability. Nor should a discharged cell stand long before charging. Otherwise the lead sulfate coating becomes more compact. The normal voltage of such a cell, fully charged, is about 2 volts, but this falls on discharge. Thus there are two ways of learning the condition of the cell: measuring the density of the acid, or the potential fall between the poles. Since sulfuric acid is used up on discharge (the density of the liquid falling to 1.10), and since sulfuric acid is re-formed on charge (with a rise in density to 1.30) it is a simple matter to learn the cell condition. *Testing*

PREPARATION OF THE METALS

The eight processes (broadly treated) used in process metallurgy for obtaining relatively pure metals are: *Extraction from ores*

1. Reduction of oxides with carbon
2. Reduction with aluminum
3. Reduction with hydrogen
4. Roasting sulfides
5. Roasting sulfides with carbon
6. Electrolysis of fused salts
7. Amalgamation with mercury
8. Cyanide extraction

Sometimes a combination of two methods is applied to a single metal. Copper is prepared by first roasting the sulfide ore but the crude product must be purified for electrical uses by electrolysis of water solutions of a copper salt.

Carbonates of iron are converted into oxides on heating, and oxides are reduced by heating in contact with coke and limestone. American ores are already oxides, occurring in the Lake Superior region, Alabama, Colorado, and other areas. *Iron*

The modern blast furnace is of giant proportions. It is built of heavy steel plates lined with firebrick. The lower part where the heat is more intense is usually bound with water-cooled bronze blocks set in the brick wall. At the top of the "stack," as the furnace is called, are double bell-shaped doors which admit the ore, coke, and limestone of the "charge" without undue loss of gases.

An enormous volume of air, the blast, is blown through the furnace to burn the coke. It may seem that hot carbon is the active reducing agent in the blast furnace, but the carbon monoxide formed is even more effective because it is a gas and hence comes in better contact with the ore.

Slag

Impurities such as alumina and silica in the ore react with lime formed by heating the limestone of the charge to yield a low-melting slag, easily separated from the pig iron. The pasty particles of free iron left by reduction of Fe_2O_3 do not melt until they have dissolved enough impurities, such as sulfur, silicon, and carbon, to lower the melting point sufficiently. The melted iron runs into the crucible, while the lighter fluid slag floats on top.

Pig iron

The product of the blast furnace is termed *pig iron.* It is only about 92 to 94 per cent iron, the usual impurities being about 3 per cent carbon along with some silicon, manganese, sulfur, and phosphorus. Much of this iron is cast into "pigs" for convenience in handling and then shipped to the various markets. It is remelted at the foundries (by coke in a cupola) and cast into such shapes as are needed for radiators, stoves, wheels, and a thousand cheap useful articles. Pig iron is cast iron, but there are two types, depending on the rate of cooling. If poured into a metallic mold so that the liquid is chilled suddenly, the carbon does not have time to separate as graphite, but remains uniformly distributed as cementite, Fe_3C. The product is *white cast iron*, which is very hard and very brittle. If cooled more slowly in sand molds, *gray cast iron* results. In this form, part of the carbon has had time to separate as graphite. This product is not nearly so hard as the white variety and can be given some ductility if annealed for a few days.

Melting point

Pig iron melts at about 1150° C., while pure iron melts at about 1535° C. White cast iron is harder and less ductile than gray iron. Neither can be forged. Both are low in tensile strength, difficult to weld. Gray iron does not shrink from a mold as white iron or steel.

Cast iron has some useful properties not possessed by steel: its high fluidity and small contraction on cooling, which enable it at comparatively low temperatures to fill intricate molds; its ease of machining, due to the soft graphite flakes which lubricate the cutting tools; and its greater resistance to corrosion than steel. These qualities make it an irreplaceable metal. Admittedly, it has far less strength than steel and is rather brittle under shock. Pig iron may be turned into steel by burning out most of the carbon

(down to 0.2 per cent for mild steel) and also other impurities such as sulfur. Usually alloying elements are added.

Zinc oxides or carbonates need only heating with coal to reduce them to the metal. Sulfides must be heated with excess air to secure the oxide of zinc, which is then further heated with carbon to secure its reduction. In some smelters a sulfuric acid plant is attached to utilize the sulfur dioxide. *Zinc*

(93) $$2\,ZnS + 3\,O_2 \longrightarrow 2\,ZnO + 2\,SO_2,$$

(94) $$ZnO + C \longrightarrow Zn + CO.$$

The zinc oxide and coal are heated in fire-clay retorts. The zinc is volatilized and then condensed in receiving vessels. Carbon monoxide burns at the mouth of each receiver. Since zinc boils at 907° C. and the temperature of the reduction occurs at 1200° C., it is evident that the vapor of the metal must be formed. When the vapors are led into cold receivers at any temperature below 419° C. (melting point of zinc), only zinc dust is obtained, but as the hot vapors heat the receivers above 419° C. only the liquid form is condensed. This is cast into slabs called *spelter* which may be purified by electrolysis.

Until the recent development of chrome plating, the free element was not prepared in great quantity, as its alloy with iron is cheaper and more suitable for steelmaking. Most of this alloy, "ferrochrome," is made by carbon reduction of chromite ore in the electric furnace. A small amount of the carbon-free alloy is made by the thermit process described below. *Chromium and manganese*

Manganese, at present, is prepared as ferromanganese by reduction of a mixture of ores of iron and manganese. If below 30 per cent manganese, this alloy is called *spiegeleisen*.

Powdered aluminum, when ignited, reduces many metallic oxides with evolution of a great amount of heat. Temperatures of 3000° C. are obtained, so the metal is molten and may be cast in a mold. *Thermit process*

(95) $$Fe_2O_3 + 2\,Al \longrightarrow Al_2O_3 + 2\,Fe.$$

The heat of formation of Al_2O_3 is 392,600 cal. for the formula weight in grams, while the heat of decomposition of Fe_2O_3 is 197,000 cal.; hence, 195,600 cal. are freed. If a shaft on a ship at sea is broken, the ends can be welded together by surrounding them with a sand or clay mold and igniting a mixture of iron oxide and aluminum powder (thermit) in a funnel above. The melted iron runs

into the mold, melts the broken ends of the shaft, and produces a good union. In actual practice the shaft ends are preheated.

Copper

Most copper ores (but not all) are sulfides. Concentration by grinding, washing, and flotation is a common preliminary to smelting. Roasting in air burns out 80 per cent of the sulfur, but some must be left in order to form a *matte* of melted sulfides later.

In a huge reverberatory furnace the roasted product is heated by a long flame of burning powdered coal, blown in with compressed air. Limestone, $CaCO_3$, and magnesite, $MgCO_3$, are present to form removable slag with much of the iron, silica, and alumina. A molten matte of mixed sulfides of copper and iron is then removed to a converter where a blast of air burns out (*oxidizes*) the remaining sulfur and oxidizes the iron and other impurities. Addition of a little raw ore causes formation of a removable slag with the iron oxide.

Such crude copper is too low in electrical conductivity for certain uses and must be refined by electrolysis. Crude copper slabs as anodes (positive pole) and thin sheets of pure copper as cathodes (negative pole) in a bath of copper sulfate make this possible. Copper dissolves (on passage of current) at the anode and deposits in pure form at the cathode.

Magnesium

The leading process of preparing magnesium is electrolysis of melted magnesium chloride, with some added chlorides of calcium and sodium to lower the melting point. Magnesium ions in the melt are discharged at the cathode. The Pigeon process, very convenient for small plants, uses furnace reduction of magnesium oxide with silicon, or rather the cheaper ferrosilicon.

Aluminum

We owe the present process of making aluminum to Charles M. Hall who, as a student at Oberlin College, invented the process in 1886. After preliminary removal of iron oxide from the bauxite ore, the roasted hydroxide (now alumina, Al_2O_3) is added to the cryolite in the electrolytic cell. These cells are iron boxes lined with "compact" (retort coke). Several carbon anodes dip into melted cryolite, $AlF_3 \cdot 3NaF$, which is kept molten (about 1200° C.) merely by the heat of resistance. The carbon-lined iron cell is the cathode. The aluminum metal is discharged at the cathode and sinks to the bottom, to be tapped off from time to time. Oxygen is evolved at the carbon anodes slowly converting them into carbon dioxide. Since it is the dissolved alumina that is decomposed, the cryolite needs renewal less regularly. A current of 20,000 amperes,

with a 5- to 7-volt drop in each cell, produces 250 lb. of metal per day in each cell, equivalent to 12 kw-hr. (kilowatt hours) per pound.

To make 1 lb. of the metal, we must use 2 lb. of aluminum oxide, 0.8 lb. of carbon anodes, 0.1 lb. of cryolite, 0.1 lb. of fluorspar, and 12 kw-hr. of electricity.

Powder metallurgy

Metals of high melting point, such as tungsten, molybdenum, and tantalum, which cannot be readily melted, are prepared in small quantities by *powder metallurgy* with the aid of heat and pressure. The process is one of *sintering*, but not melting, the powder. Even with metals of lower melting point there are a number of advantages such as avoiding the usual final machining and grinding of small and complicated articles. The metal powders are prepared by reduction of oxides by electric deposition, or by atomization. Metals which are *immiscible* (do not mix) in the melted state, or which are widely different in volatility, may readily be alloyed by sintering their mixed powders.

PROPERTIES AND USES OF SOME METALS

Some metals are useful because they are light and others because they are heavy; some offer chemical activity and others provide resistance to corrosion; some are hard and others are malleable or ductile; some give us low melting points and others are useful because of their high melting points.

Magnesium

Magnesium is a silvery white metal melting at 651° C. and boiling at 1110° C. Its low density of 1.74 is interesting, for there is only one other commercial possibility, aluminum, among the light and strong metals to compete with it as an engineering material; and aluminum has a density of 2.70. (Beryllium, density of 1.84, is far too expensive for any but special uses.) It is important commercially that magnesium does not rust in dry air. A thin film of hydroxide, turning into the basic carbonate, does form in moist air, but it is coherent enough *to protect the metal* beneath. This film, however, is not adequate to prevent corrosion. At a temperature just below its melting point the metal can be drawn into wire.

Alloys

Magnesium alloys are now used in aircraft, in auto pistons, crank cases, forged propellers, amplifying apparatus, in the refining of nickel, and in the de-gassing of radio tubes. An alloy of 92 per cent magnesium with 8 per cent aluminum can be hardened by repeated heating to 420° C., then tempering four to seven days at 150° C.

Aluminum

The metal aluminum is the lightest of the very common metals, its density being only 2.7, while iron is three times as heavy. It melts at 660° C. and boils at 1800° C. It is rolled or hammered very easily at well below 150° C., yet thin sheets in the presence of a lubricant can be ground to a powder. It is an excellent conductor of heat and electricity. In wire of the same cross section and equal length it conducts electricity only two thirds as well as copper, but weight for weight it greatly excels copper in this respect. More than 450,000 miles of steel-cored aluminum cables are in use. The steel core is needed for tensile strength.

Corrosion, insulation

Chemically, aluminum is very active, yet it is tarnished only slightly in air. This is due to the formation of a thin coherent film of the hard, white oxide which protects the metal beneath. The smooth, bright surface of aluminum is a splendid reflector of radiant heat and consequently the metal in the form of foil is finding growing use as a heat insulator for locomotives, milk tanks, houses, and naval vessels. On a locomotive, 13 lb. of foil replaced 1200 lb. of other insulating material.

Aluminum is much used for auto parts, the rigid structure of airships, and structural work of various kinds. The powdered metal is mixed with oil or a volatile liquid and used as a paint for radiators and other metallic articles. Aluminum paint is largely used as a covering for oil-storage tanks in order to reflect much of the sun's heat radiation and thus lessen the usual losses of volatile constituents such as gasoline. In the form of cooking utensils it has a merited popularity. A golden bronze containing 92 per cent copper and 8 per cent aluminum is now much used for gears in automobile trucks and for other purposes where toughness and resistance to abrasion and corrosion are important.

Aluminum alloys

Duralumin, the most important of all aluminum alloys, containing 95 per cent aluminum, 4 per cent copper, 0.5 per cent manganese, and 0.5 per cent magnesium, is much used for airplane, airship, and automobile structures. It has nearly the strength of steel with about one-third of its weight. Heat treatment of certain aluminum alloys greatly improves their strengths.

Saving weight

Railway coaches using aluminum or its strong alloys for all but a few supporting parts (steel) are so much lighter that fuel costs go down and speeds go up. An aluminum Pullman sleeper weighs only one-half as much as the regular steel car of the same capacity. Aluminum dump cars save 11 tons of weight per car.

Zinc is a blue-white metal whose properties vary with its treatment. Cast zinc is moderately hard, but if heated to 100°–150° C., it becomes so malleable that it can be rolled to thin sheets which remain soft and malleable on cooling to ordinary temperatures. Galvanizing is done by cleaning the rust or scale from iron with dilute sulfuric acid and dipping the clean iron into melted zinc. On cooling, the zinc crystallizes in the familiar spangled design. About two ounces per square foot of surface are needed to give protection. *Zinc*

Rolled zinc is used for roofing, gutters, washing-machine parts, and battery elements. Alloyed with copper it forms brass, one of the most useful of all alloys. Zinc of 99.99 per cent purity is remarkably ductile and brilliant. It also yields better *die castings* than the less pure metal.

Tin is most malleable at 100° C., while at 200° C. it is so brittle that it can be ground to a powder. *Tin*

Tin is placed on thin sheets of iron or steel by mere dipping of the clean steel into melted tin. The sheet must first be pickled in acid to free it from all oxides of iron. The modern ideal in tin-plating is to secure as thin a coating of tin as possible, 1.5 per cent of the total weight, without holes or cracks. Electroplating of tin on sheet steel requires less tin than the dipping process.

Lead, a heavy metal of specific gravity 11.4, is soft and easily cut with a knife. The metal melts at 327.5° C. and boils at 1620° C. Just below its ordinary melting point (lowered by pressure) it can be squeezed into tubes. When freshly cut, its luster is bright, but the surface quickly tarnishes with formation of oxide, PbO, and basic carbonate, $2PbCO_3 \cdot Pb(OH)_2$. The coating protects the metal from further *oxidation*, although it detracts sadly from the real beauty of lead. *Lead*

Solutions of lead salts are extremely poisonous, so that water conducted through lead pipes may produce lead poisoning. This is particularly true of soft water which, with air, forms the somewhat soluble hydroxide, $Pb(OH)_2$. Hard water is much less dangerous because it soon forms a coating of insoluble carbonate and sulfate on the lead. Although lead is attacked by alkali, a very little sodium silicate in solution protects lead pipes so exposed. *Lead poisoning*

All lead paints begin with lead as raw material. Red lead is Pb_3O_4 and white lead, $2\ PbCO_3 \cdot Pb(OH)_2$.

Terneplate for roofing is made by dipping clean steel sheets into a molten lead containing 18 to 25 per cent tin. Without the tin, lead fails to coat steel on dipping.

Tungsten

Tungsten (W) has the highest melting point of all the metals, 3370° C. Tungsten is steel-gray, very hard, and very heavy, with a density of 19.3. The metal is obtained as a powder, on reduction of the oxide, and is worked into strong filaments or rods. Pure tungsten, after elongation by rolling and drawing while hot so as to develop long threadlike crystals, is soft and ductile.

Lamp filaments

Tungsten lamp filaments are saving the people of this country several hundred million dollars each year. The old carbon filaments used 3.25 watts per candle power, while the modern tungsten lamp uses less than 1.25 watts for the same illumination. We use over 900,000,000 of these lamps annually.

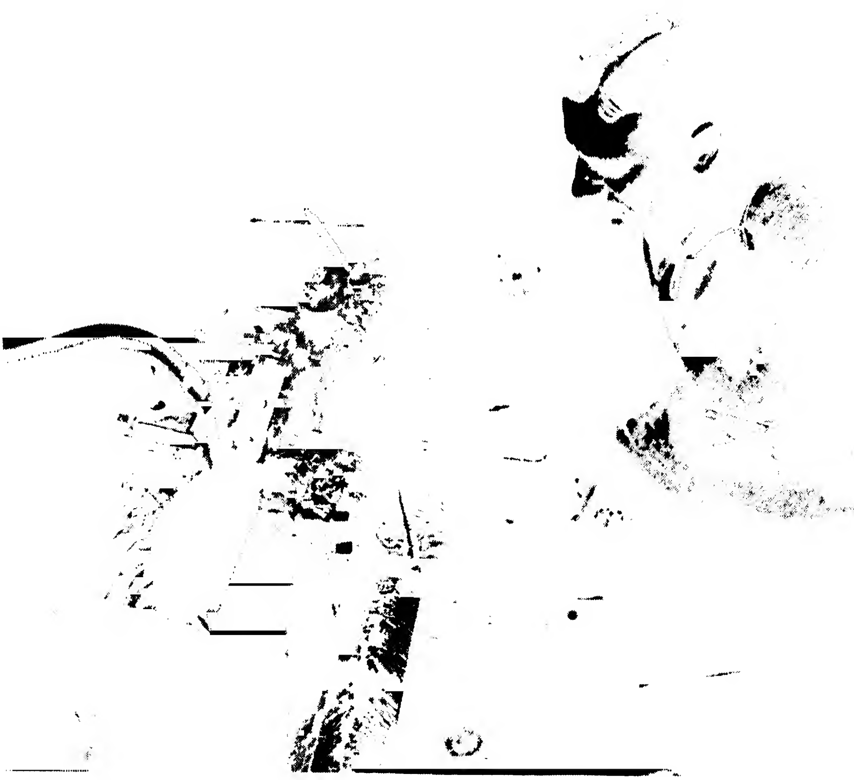


FIG. 46. HIGH-SPEED CUTTING TOOLS

Machine-shop work is greatly speeded up, in war for increased production and in peace for lower costs, by use of tungsten-molybdenum steel cutting tools that hold their sharp edges even when red-hot from friction. Cooling oils are needed nevertheless.

High-speed steel cutting tools contain 18 to 20 per cent tungsten. Since these tools can be worked on the lathe until red-hot without losing the necessary hardness, their use means an enormous saving in machine-shop time and expense (Fig. 46). Tungsten carbide (W_2C), with some cobalt, was developed in 1928 as the hardest cutting-tool material yet manufactured ("carboloy"). *Tungsten tools*

IMPORTANT ALLOYS

Alloys were invented in attempts to improve upon the properties of the metals. The most used alloy, of course, is steel. *Steel as an alloy*

Pure iron (not very common) is rather soft and tough, pig iron (crude) is hard and brittle, but steel is hard, elastic, and strong.

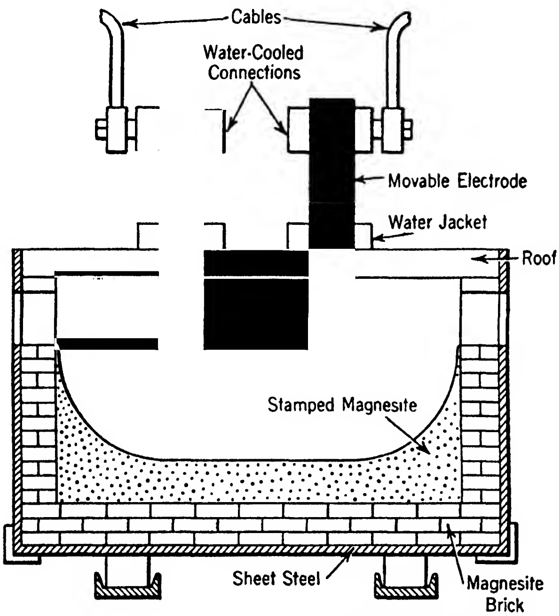


FIG. 47. ELECTRIC ARC OR RESISTANCE FURNACE WITH GRAPHITE ELECTRODES

Essentially the simplest steel is iron with 1 per cent or less of carbon, but most steels contain *alloying elements* such as chromium, manganese, nickel, tungsten, vanadium, to name a few. Many of the best alloy steels are made in an electric furnace (Fig. 47). Many alloys (brass, bronze, and so on) are formed by melting the metals together, others by reduction of mixed ores.

The bearing metals which surround revolving shafts are interesting examples of the introduction of harder particles in a softer *Bearing metals*

matrix. These *babbitts*, as they are often called, are alloys of tin and antimony, with some lead and copper.

High-speed, high-compression motors make severe demands upon bearing metals. Cadmium-nickel alloys containing from 0.75 to 3.00 per cent nickel have recently come into use in this field. Today, high-speed bearings may be lined with cadmium-silver-copper, cadmium-zinc, or cadmium-nickel, possibly electroplated with indium to increase resistance to oil corrosion.

Corrosion-resistant alloys

Monel metal (Cu–Ni) and the 18–8 chrome-nickel stainless steels of low carbon content are much used to resist corrosion. Manganese-bronze propeller blades for steamships contain Mn, Cu, Sn and Zn. This alloy is resistant to salt-water corrosion. Stellite, a Co–Cr–W–C alloy of great hardness, is so noncorrodible that it is used in surgical instruments and other fine products.

Low-melting alloys

The low-melting alloys, such as Wood's metal or Rose's metal, have an important application. The automatic sprinklers dotting the ceilings of great department stores or factories are merely water pipes closed by a plug of such an alloy. When a fire starts, these plugs melt very soon and a spray of water is released. Wood's metal melts at 158° F. or 70° C. Solder is another low-melting alloy of value. Each metal in the solid solution lowers the melting point or freezing point of the other; hence the mixture melts at a lower temperature than the constituents. This is the same effect as that produced by salt in water.

Plumbers' solder (62 per cent lead and 38 per cent tin) is useful because on cooling a large mass of lead-rich crystals separates, forming a mushy mass that can be "wiped" on a pipe joint. It begins to solidify at 240° C. but is not entirely solid until 180° C. is reached.

Some hard alloys

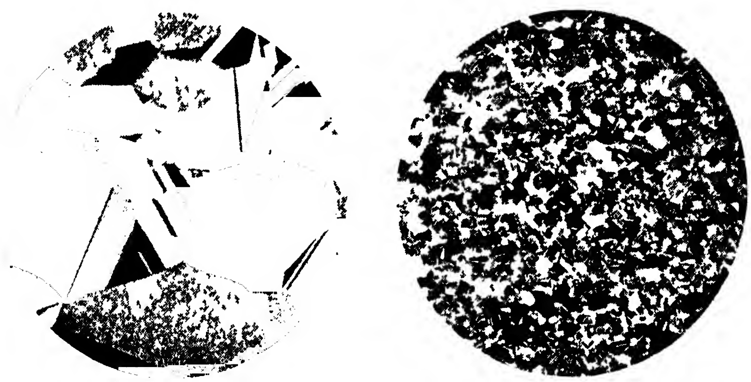
Cast iron (not pure iron) and the bronzes are very hard. One of the best bronze castings contains 10 per cent tin, 2 per cent zinc, and 88 per cent copper. Bronze (Cu–Sn) is the most ancient of the hard alloys. Copper is a soft metal and yet its alloy with a very little silver and chromium, "cupaloy," is harder than ordinary steel. Special heat-treatments are necessary in developing this great hardness. Lead containing several per cent of calcium is too hard to be cut with a hack saw. High-manganese steels make excellent jaws for rock crushers.

Beryllium alloys

Only 2 per cent beryllium in nickel gives it the tensile strength of steel. Beryllium is to copper what carbon is to iron: a powerful

agent for increasing hardness, elasticity, and strength. The alloy steels are among the strongest alloys known.

Beryllium has other useful alloying properties. Be–Ni valve springs are almost necessary for the highest speed airplanes because of their vibration resistance. The nonsparking beryllium-alloy tools have had some use where explosive vapors were present. The effect of nickel on the structure of a beryllium-copper alloy is shown by the photomicrograph in Fig. 48. *Microscopic structure*



Courtesy American Brass Company

FIG 48. MICROSCOPIC STRUCTURE OF ANNEALED ALLOY

Coarse grain beryllium-copper alloy (left) annealed from 800° C. Same alloy (right) with 0 25 per cent nickel annealed from 800° C shows finer grain. Magnification, 75x.

Steel is essentially iron containing from 0.1 to 1.0 per cent carbon, although most steels contain other alloying elements to give special properties. Steel contains no more than 2 per cent carbon because of brittleness. Aside from its great strength and elasticity certain steels can be magnetized. *Steel*

By the Bessemer process (actually discovered by William Kelly, an American, in 1852) many impurities are burned out of molten pig iron by a blast of air. At each "blow," with its brilliant flame, 15 tons of steel are made in 15 minutes. As the converter pours the liquid into molds or ladles, carbon and the alloying elements are added. *Bessemer*

The open-hearth process is slower, less spectacular, but makes better steel. Fifty tons of steel are made every eight hours. In this process oxygen needed to burn out impurities is secured from the air, from iron ore and rusty scrap. Preheated air and fuel gas supply the heat, reflected from the arched ceiling of the open hearth. *Open hearth*

Alloy steels

We could not make automobiles for less than several times their present cost were it not for the modern development of alloy steels. Crankshafts, driving shafts, transmission shafts, axles, frames, and gears are stronger, tougher, lighter, and stand shocks and strains better because they are made of alloy steels.

Silicon, aluminum, manganese, and titanium are added to steels at the moment of casting as "scavengers," to remove oxygen. Steel so cleansed is as much as 40 per cent stronger. But some of these elements, and a number of others, are also added to form valuable alloys with the steel. Alloy steels are often made in the electric furnace.

Manganese steel (12 to 14 per cent Mn), if suddenly cooled, is *hard, and tougher than any other steel alloy.* It is used in grinding machinery, rock crushers, and burglarproof safes. With a little added molybdenum it is suitable for heavy-duty gears.

Tool steels

Tungsten steel (such as 18 per cent tungsten, 4 per cent chromium, and 1 per cent vanadium) is self-hardening; that is, a tool of this material hardens on cooling in air without being quenched in oil or water. It also holds its temper when the tool is worked until red-hot from friction. (See Fig. 46.)

Molybdenum (fortunately produced chiefly in our own country) is replacing part of the tungsten in high-speed tools.

Nickel steel (3.5 per cent Ni) resists corrosion, is hard, elastic, and very suitable for armor plate, wire cables, and propeller blades. The automobile industry is a leading user.

Stainless steel

Polished stainless steel (18 per cent chromium-8 nickel) is highly resistant to corrosion. It is much used on the exteriors of office buildings, for certain chemical equipment in industry, and for other products. The corrosion resistance is increased by the addition of from 1 to 3 per cent molybdenum.

The modern oil refinery, with its high-pressure tubes and stills, is a large user of special alloy steels. The automobile industry is the largest purchaser of steels in general.

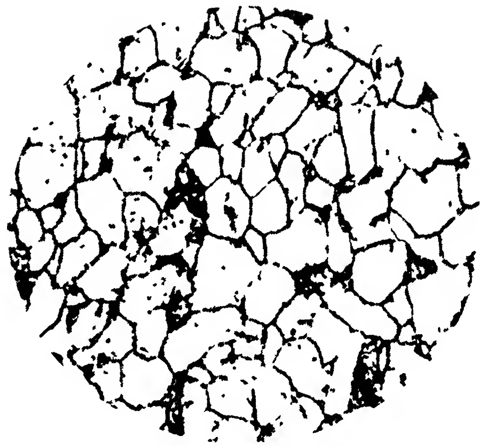
Heat treatment of steel

The two most important modifications of iron are *alpha-iron* (like pure wrought iron), which is soft, ductile, and magnetic; and *gamma-iron*, stable above 900° C. The latter is denser and only slightly magnetic.

Sudden cooling of white-hot steel gives a supersaturated solution of carbon in alpha-iron (*martensite*), which is hard and brittle. Slow cooling gives time for thin plates of Fe_3C to form, alternating

with alpha-iron masses, the two together forming *pearlite*. Because of the large amount of alpha-iron, this steel is soft and ductile. This structure is well illustrated in Fig. 49.

Annealing by reheating a quenched steel to 200° to 350° C. allows some crystal growth, some separation of hard cementite, Fe_3C, and some release from strains. Any intermediate degree of hardness can be secured by tempering. The effect of quenching and tempering becomes clear from a comparison of the two photomicrographs of Fig. 50.

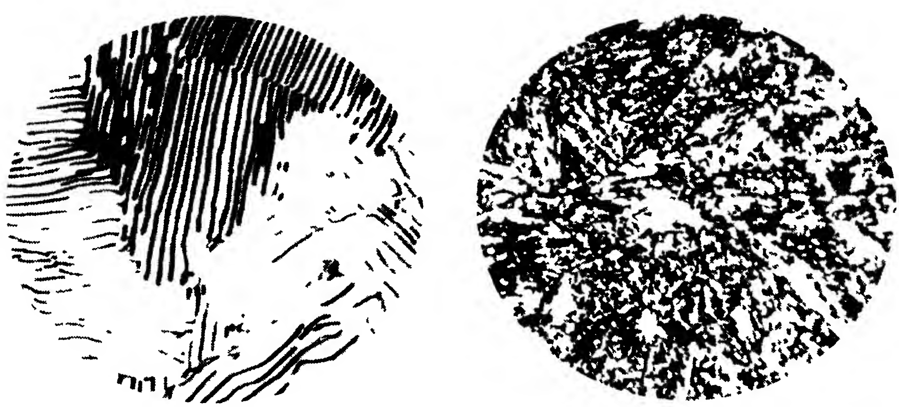


Courtesy H B Pulsifer

FIG. 49. SIMPLE STEEL WIRE

Photomicrograph of a cold-drawn steel wire, 0.1 per cent carbon. Magnification, 250. The white areas of pure iron (ferrite) are surrounded by dark areas of pearlite containing the iron carbide, Fe_3C, and iron.

In *casehardening*, a low-carbon steel is held for several hours in charcoal and bone ash or suitable gas at 1000° C.; carbon is taken up by the surface, yielding a steel that is hard on the surface and tough in the interior. Gears must be casehardened.



Courtesy J F. Vilella

FIG. 50. COMPARISON OF ANNEALED AND QUENCHED STEELS

The effect of heat-treatment of steel is shown at the left. A coarse structure is obtained by cooling this 0.89 per cent carbon steel from 877° C. at the rate of 3° C. per minute. The finer structure of the same steel on the right is obtained by sudden water cooling from 877° C. and then tempering one hour at 550° C. Magnification of polished and etched surface, 2000.

SOME COMPOUNDS OF THE METALS

Sodium carbonate, Na_2CO_3

By the Solvay process, concentrated sodium chloride solution is saturated with ammonia and carbon dioxide. At first, ammonium bicarbonate, $(NH_4)HCO_3$, is formed and this reacts with sodium chloride to form sodium bicarbonate. The latter salt, only sparingly soluble in the salt solution, precipitates, and after removal is heated to form sodium carbonate.

$$NH_4OH + H_2CO_3 \longrightarrow (NH_4)HCO_3 + H_2O, \tag{96}$$
$$(NH_4)HCO_3 + NaCl \longrightarrow NaHCO_3 + NH_4Cl, \tag{97}$$
$$2\,NaHCO_3 \longrightarrow Na_2CO_3 + H_2O + CO_2. \tag{98}$$

Part of the sodium bicarbonate, $NaHCO_3$, produced in the early stage of the Solvay process is used in making baking powders. This is the "bicarbonate of soda" of the kitchen cabinet. Industry calls Na_2CO_3 *soda ash.*

Hydrates

Sodium carbonate, Na_2CO_3, is a white solid, very soluble in water, yielding alkaline solutions by hydrolysis. On concentrating these solutions various hydrates crystallize out according to the temperature. On standing in the air the *decahydrate*, $Na_2CO_3 \cdot 10\,H_2O$, effloresces, due to its high vapor tension, and crumbles to a white powder, which is the monohydrate, $Na_2CO_3 \cdot H_2O$. For some purposes the decahydrate is liked because it dissolves more rapidly. The common names for sodium carbonate are washing soda and sal soda.

The production in the United States in one recent year was 105,000 tons of natural soda and nearly 3,000,000 tons of Solvay soda. It is essential to many important industries, such as the manufacture of glass, soap, paper, leather, enamels.

Calcium oxide, CaO

Calcium oxide, known as "quicklime," is usually obtained by heating limestone to about 900° C.

$$CaCO_3 \rightleftharpoons CaO + CO_2. \tag{99}$$

Calcium oxide is a white solid melting at about 2750° C. and boiling at higher temperatures. Pure calcium oxide takes up 31.1 per cent of its own weight of water, but the usual impurities lessen this amount. As the oxide hydrates to form the hydroxide, much heat is evolved.

On long exposure to the air, quicklime becomes "air slaked," changing into the hydroxide, $Ca(OH)_2$, and the carbonate, $CaCO_3$. Lime has uses to the extent of more than 4,000,000 tons in this

country in the preparation of paper, rubber, sand-lime bricks, calcium carbide and acetylene, lime-sulfur sprays, calcium arsenate dust for cotton crops, Bordeaux mixture, and in the clarification and softening of water.

Calcium hydroxide, $Ca(OH)_2$

Slaked lime, $Ca(OH)_2$, is a white powder formed by careful addition of water to quicklime, CaO. It is slightly soluble, 100 cc. of water dissolving 0.17 g. of $Ca(OH)_2$ at 18° C. At higher temperatures it is less soluble. The little that is in solution ionizes rather well, but when considerable quantities of a base are required, it is customary to use a suspension of the solid in water known as "milk of lime." As fast as the little in solution is used up, some of the suspended hydroxide goes into solution.

$$\text{(100)} \qquad \underset{\text{Undissolved}}{Ca(OH)_2} \rightleftharpoons \underset{\text{Solution}}{Ca^{++} + OH^- + OH^-}.$$

This is the cheapest base known and has many industrial uses. It is important for making mortar, plaster, alkalies, bleaching powder, purifying illuminating gas, removing hair from hides, and clarifying sugar solutions. Naturally some of these uses duplicate those for the oxide.

Mortar

In earlier days mortar, to the bricklayer, was a mixture of slaked lime, sand, and water which hardened slowly on drying. Now we add Portland cement for quick setting. The chemist knows that carbon dioxide of the air slowly penetrates the wet mortar and reacts with the calcium hydroxide to form solid *interlacing crystals of calcium carbonate*, $Ca(OH)_2 + CO_2 \longrightarrow CaCO_3 + H_2O$. The sand, preferably with sharp edges, merely gives bulk, rigidity, increases porosity, and reduces shrinkage. Possibly the colloidal particles of calcium hydroxide "set" in some way suggestive of the drying of glue or water glass, or interlacing crystals of calcium hydroxide may form. Wall plaster differs from mortar only in containing hair as a binder. It is porous enough to allow considerable circulation of air through the walls.

Plaster of Paris

Gypsum, $CaSO_4 \cdot 2\,H_2O$, is heated a short time at 200° C., causing the loss of three-fourths of its water, leaving the *hemihydrate*, $2\,CaSO_4 \cdot H_2O$, or plaster of Paris. This residue, when mixed with the proper amount of water, sets to a white solid in from 5 to 15 minutes. The white finishing coat of common plastered walls contains plaster of Paris. Statuary is often made from it, because the material expands slightly on setting, thus taking a

very sharp impression of the mold. For commercial work, such as white plaster, stucco, interior wall bricks, and wallboard, the setting process is retarded by the addition of glue or a product made by digesting hair with sodium hydroxide.

Softening of water

In boilers a hard scale of salts (calcium sulfate, calcium carbonate) is left by the evaporation of feed water. This insoluble scale is a poor conductor of heat and causes a serious waste of fuel. A layer $\frac{1}{8}$ in. thick causes a loss of 10 to 12 per cent of fuel.

Hardness of water also causes an appalling waste of soap. No lather can form and consequently no cleansing can be achieved until all the hardness has been precipitated at the expense of the soap. Thus, with sodium oleate, representing a typical soap, we have this wasteful water-softening reaction.

$$2\ Na \cdot Oleate + CaSO_4 \longrightarrow Na_2SO_4 + Ca \cdot (Oleate)_2.$$

The insoluble calcium soaps form sticky curds that are only an annoyance.

Precipitating hardness

One of the cheapest softeners is slaked lime, $Ca(OH)_2$. This sounds to the public like making matters worse, for it is often said that "lime causes the hardness of water." However, the following equations show the explanation of the paradox. In each case the $Ca(OH)_2$ precipitates the salt (hardness) and leaves the pure water H_2O. The vertical arrow following a compound means that the compound is insoluble and "comes down" or precipitates.

(101) $Ca(HCO_3)_2 + Ca(OH)_2 \longrightarrow 2\ H_2O + 2\ CaCO_3\downarrow$, (insoluble)

(102) $Fe(HCO_3)_2 + Ca(OH)_2 \longrightarrow 2\ H_2O + CaCO_3\downarrow + FeCO_3\downarrow$, (insoluble)

(103) $MgCO_3 + Ca(OH)_2 \longrightarrow CaCO_3\downarrow + Mg(OH)_2\downarrow$. (insoluble)

To precipitate hardness due to sulfates, crude "soda ash," Na_2CO_3, is used.

(104) $CaSO_4 + Na_2CO_3 \longrightarrow Na_2SO_4$ (soluble) $+ CaCO_3\downarrow$ (insoluble)

Magnesium compounds

Magnesium oxide, MgO, derived from the carbonate (magnesite) at high temperatures, is an important refractory lining for furnaces. Open-hearth furnaces require it. Magnesia pipe covering for heat insulation is the basic magnesium carbonate, $3\ MgCO_3 \cdot Mg(OH)_2 \cdot 3\ H_2O$, formed by hot precipitation from any soluble magnesium salt with sodium carbonate. Asbestos, a natural competitor, is $CaMg_3(SiO_3)_4$. Its value in brake linings

and when mixed with Portland cement for asbestos shingles is great.

Magnesia cement is formed when a wet mixture of the oxide and chloride, $MgCl_2$, hardens. Sawdust or wood-meal fillers may be added. Such compounds as this basic magnesium chloride make wall plaster, floorings, or stucco for outside walls.

Lithopone

The important pigment lithopone (a combination of barium sulfate and zinc sulfide) is made by mixing solutions of barium sulfide and zinc sulfate.

(105) $$BaS + ZnSO_4 \longrightarrow BaSO_4 + ZnS.$$

Both products are white and insoluble. The precipitate first formed has no body or hiding power, but when washed, dried, heated to dull redness, suddenly quenched in cold water, ground wet, washed, and dried again, it yields an excellent pigment. Lithopone is a brilliant white with better hiding power than zinc oxide. It is unexcelled for interior painting, for first coats, linoleum, floor oilcloth, and rubber compounding.

Research has improved the quality so much that we now use over 160,000 tons of lithopone per year for both outdoor and indoor paints, and even mixtures with white lead.

Brick and pottery

Kaolin or china clay is the weathered residue from feldspar. Its composition is represented by $H_2Al_2(SiO_4)_2 \cdot H_2O$, while that of feldspar is $KAlSi_3O_8$. Common clay owes its color to impurities, usually oxides of iron. In burning brick or tile, the impurities fuse and act as a bond for the clay particles. Firebrick, with its great infusibility, must be made from rather pure clay.

Porcelain is a fired mixture of kaolin and feldspar. The more fusible feldspar binds the kaolin particles and makes the mass translucent. When such ware is dipped into a water suspension of ground feldspar, dried, and burned again, it receives a glaze of feldspar.

Portland cement

The Romans ground together volcanic ashes and lime to form a cement that, when mixed with water, set to a solid. Some specimens of this *pozzuolana cement* are still in good condition. Today we do practically the same thing in grinding together blast-furnace slag (granulated) and lime. But the best of these hydraulic cements is the modern Portland cement in which the essential raw materials are clay and limestone. There must be present the proper equivalents in combined form of CaO, SiO_2, and Al_2O_3. The average composition of Portland cement is about as follows:

CaO	63 per cent	Fe_2O_3	2.5 per cent
Al_2O_3	6.5 per cent	MgO	2.5 per cent
SiO_2	21.5 per cent	SO_3	1.5 per cent

The raw materials, clay and limestone, are finely ground, fired in a rotary kiln to incipient fusion (the edges of lumps just melting), and the clinkered product is ground again. These rotary kilns are nearly horizontal, from 60 to 200 ft. long, and 6 to 9 ft. in diameter. They are heated with powdered coal. Nearly 8,000,000 tons of powdered coal are used as fuel in the cement industry annually.

Cement manufacture

The materials enter at the cooler end of the rotating kiln, slowly working down to the end where the flame enters. The product is a calcium aluminate and calcium silicate. About 3 per cent of gypsum ($CaSO_4$) is added to retard setting. A little iron oxide and magnesium oxide are allowed, but there is a definite range of concentrations for all the constituents. The clinker is ground to produce the cement powder of 200-mesh fineness.

Hardening

Common cement is probably a mixture of calcium silicates, calcium aluminate, and an excess of lime; consequently it is highly basic. Hardening is due to *hydration*, *hydrolysis*, and *crystallization*. The calcium aluminate reacts with the mixing water to give the following reaction:

$$(106) \qquad Ca_3(AlO_3)_2 + 6\,H_2O \longrightarrow 3\,Ca(OH)_2 + 2\,Al(OH)_3.$$

The calcium hydroxide formed as in the above equation slowly crystallizes and cements together the particles of calcium silicate. The aluminum hydroxide fills in. The tricalcium silicate present, $3\,CaO \cdot SiO_2$, hydrates rapidly, giving early strength. $CaSO_4$ is added to delay this action.

White lead paint

A paint is prepared by grinding some heavy insoluble pigment of sufficient covering power with linseed oil or other *drying oils*. Finally thinners are added. The drying of paint is not really a loss of water, but oxidation of the oil with formation of a tough, solid film. This oxidation of the oil is hastened by catalytic dryers made by boiling manganese dioxide with linseed oil. Among the pigments used are red lead, Pb_3O_4, barium sulfate, $BaSO_4$, lead chromate, $PbCrO_4$, zinc oxide, ZnO, cadmium sulfide, CdS, lithopone, $BaSO_4 \cdot ZnS$, iron oxides, and titanium oxide, TiO_2; but the one most used is white lead or basic lead carbonate, $Pb(OH)_2 \cdot 2\,PbCO_3$. In the United States alone nearly 200,000 tons of white lead are used annually.

Problems

1. Calculate the percentage by weight of magnesia (MgO) in magnesite ($MgCO_3$). *Ans.* 47.81%.

2. How much would 1 ton of quicklime weigh after being completely slaked with water? *Ans.* 1.32 tons.

3. Calculate the percentage of aluminum in white clay (kaolin). *Ans.* 22.56%.

STRATEGIC MATERIALS

Limited production

Materials necessary to national strength of which we produce none or only a part within our continental boundaries are in the main the following:

STRATEGIC MATERIALS FOR INDUSTRY

Name	*Imported from*	*Amount Used per Year*	*Uses*
Natural rubber	East Indies	600,000 tons	Tires, hose, many articles; synthetic rubber is replacing it.
Tin	East Indies and Bolivia	75,000 tons	Tin cans, alloys (bronzes, etc.)
Chromium	Cuba, Turkey, Philippines, Rhodesia	500,000 tons of concentrates	Stainless steel, armor plate, etc. New concentration of Montana ore will soon give us independence.
Manganese	Cuba, Brazil	1,500,000 tons of rich ore	Cleansing steel from sulfur; very hard steels. Our low-grade ores are helping.
Linseed oil	Argentina	350,000,000 lb.	Drying oil for most paints
Coconut oil	The Tropics	800,000,000 lb.	Soapmaking
Nickel	Canada		Stainless steel; plating
Quinine	East Indies		For malaria; synthetic atabrine is helping.
Silk	Japan	Formerly $200,000,000	Clothing, parachutes, bags for smokeless powder; nylon substitutes well.
Quartz crystal	Brazil	All we could get; but not enough	Optical uses; thin vibrating plates for radio equipment; new sources found in the United States.
Tanning materials	South America		Leather
Kapok	Tropics		Life jackets, mattresses, etc.; milkweed floss may help.
Hemp	Philippines		Rope
Jute	India	600,000,000 lb.	Burlap bags and string

Changes in war

In addition in 1941 and 1942 war priorities were placed on copper, steel, aluminum, magnesium, sugar, coffee, tungsten, antimony, plastics, nylon, and a host of common materials. We are large producers of some of these.

Technical Drawing

THE LANGUAGE OF ENGINEERING

Early drawings

At the dawn of history man demonstrated a natural urge to draw by scratching upon the walls of his caves rude drawings of animals and other subjects of which he liked to be reminded. Later a mastery of crude tools made it possible for him to erect simple structures and devise elementary articles which he needed in his daily life. At some point in this development he realized that drawings could be very useful in planning and recording his ideas concerning things he wished to build. We do not have examples of these earliest drawings, made undoubtedly upon stone, but later developments show us that drawings of some character must have been used.

Temples of stone

The Bible states that Solomon's Temple was "built of stone made ready before it was brought thither," indicating that drawings of some kind showing the exact sizes and shapes of each stone or timber must have been used during construction. Perhaps the earliest known construction drawing in existence is the plan view of a fortress drawn by the Chaldean engineer, Gudea, and engraved upon a stone tablet as shown in Fig. 52. Obviously the ancient Greek temples, so complex in arrangement and refined in detail as, for example, the Parthenon, could not have been constructed without the most accurate drawings to convey the designer's ideas to the builders.

Technical drawing as a science

In 30 B.C. Vitruvius wrote a treatise on architecture in which he referred to projection drawings for structures. But the theory of projections of objects upon imaginary planes of projection to obtain *views* was not developed until the early part of the fifteenth century, by the Italian architects, Alberti, Brunelleschi, and others.

Near the end of the eighteenth century the French mathematician, Gaspard Monge, introduced the two planes of projection at right angles to each other, and the new science became an established academic study. This provided the basis for *descriptive geometry*, which embraces all of the theoretical background to modern technical drawing.

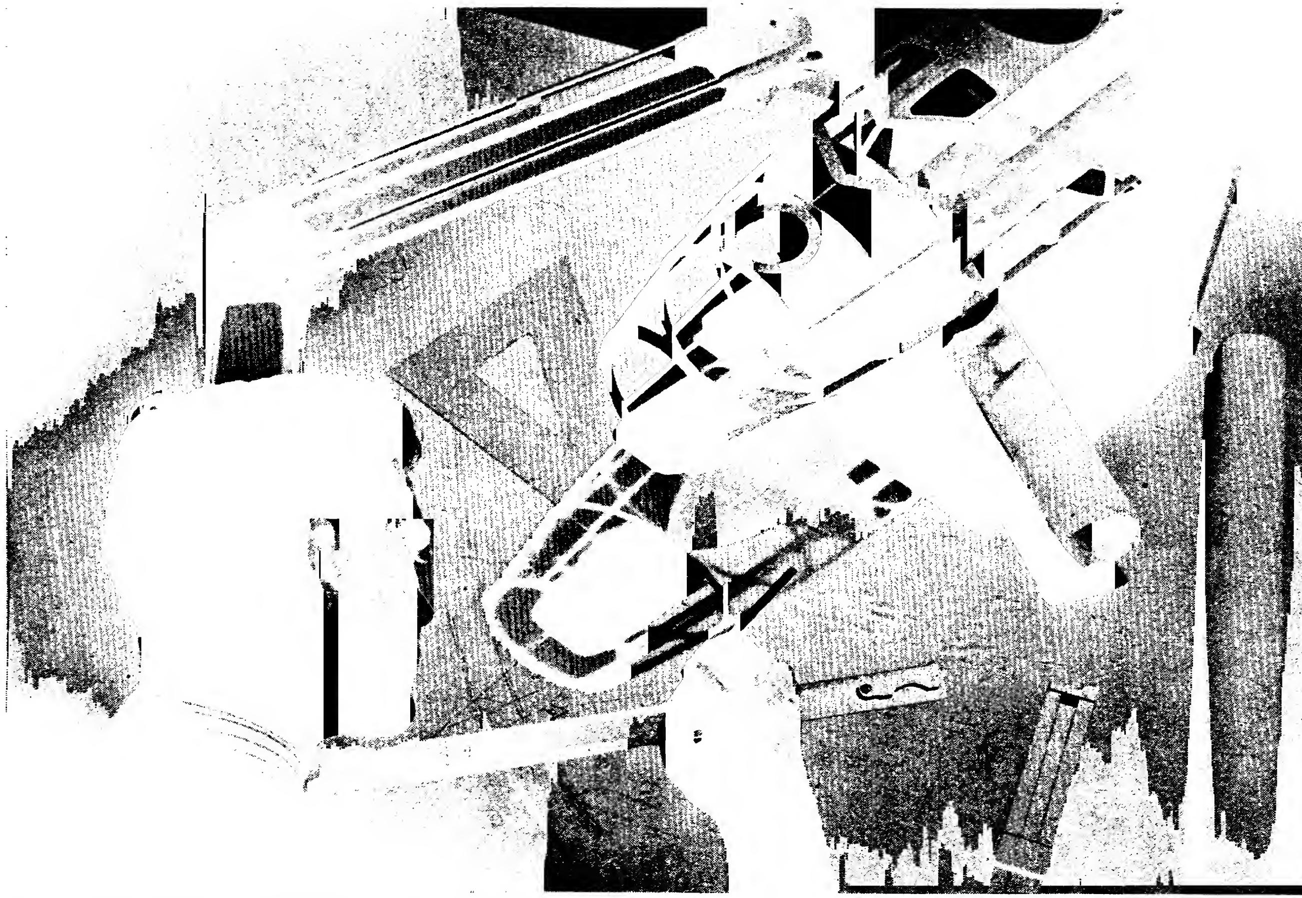
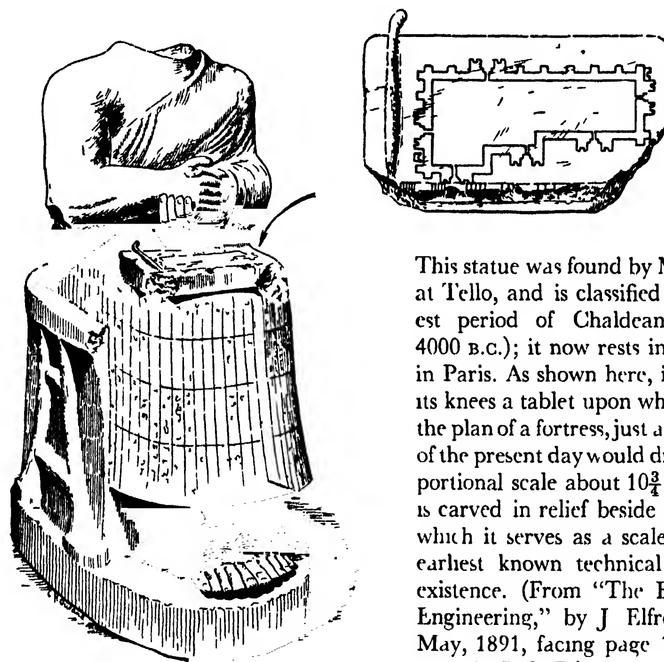


Fig. 51. Engineering draftsman at work with modern equipment

With the advent of this present industrial age came an increased need for exact methods of representation, by means of drawings, of all of the enormously complicated and extensive details of our machines and structures.

Modern technical drawing

Technical drawing is the universal language of engineering, and is as indispensable today as it will always be if technology continues in the activities of men. By means of it, the shape, size, finish, color, and construction of any object, no matter how complex,



This statue was found by M. de Sarzec, at Tello, and is classified in the earliest period of Chaldean art (about 4000 B.C.); it now rests in the Louvre, in Paris. As shown here, it holds upon its knees a tablet upon which is carved the plan of a fortress, just as an architect of the present day would draw it. A proportional scale about $10\frac{3}{4}$ in. in length is carved in relief beside the plan, for which it serves as a scale. This is the earliest known technical drawing in existence. (From "The Beginnings of Engineering," by J. Elfreth Watkins, May, 1891, facing page 309. *Transactions*, A. S. C. E.)

FIG. 52. EARLIEST KNOWN TECHNICAL DRAWING

can be described accurately and clearly. It is the language used by engineers to develop and record their ideas and to transmit them to the workers who carry out the designs. Evidently, everyone connected in any way with engineering or construction must understand this universal language. A typical modern technical draftsman at work is shown in Fig. 51. Because of the enormous growth of the airplane industry among others, such draftsmen, thoroughly skilled in using the language of technical drawing, are much in demand. The worker needs only an understanding of the principles of technical drawing so that he can "read a blueprint."

Drawing as a Language

The engineer must have a thorough understanding of the principles and in addition must be able actually to produce drawings himself when the need arises.

What the engineer must know

The engineer has need for the development of skill in two phases of technical drawing: first, he must be able to execute rapid, clean-cut freehand *technical sketches* (Figs. 191 and 192), by means of which he can quickly develop a constructive idea and transmit his thoughts to others; second, he must acquire a certain amount of skill in executing a *mechanical drawing* (Figs. 206 and 207) by means of which he can bring his ideas into exact focus and record them for the use of others.

The section of this book devoted to technical drawing covers the minimum fundamentals of the subject, but college students of engineering will supplement this material with more extensive study and practice.

INSTRUMENTAL DRAWING

You have frequently heard the apology: "I can't even draw a straight line." This is really a stupid remark to make, for no one can make a straight line without the aid of a guiding edge. As we shall see later, even a good freehand technical sketch is not composed of rigidly straight lines. Mechanical drawings, of course, are made with the aid of precision drafting instruments, the uses of which are described below.

Drawing equipment

The principal items of drawing equipment used by the engineer are shown in Fig. 53. In this brief introduction to technical drawing, however, you may wish to economize in the purchase of equipment. The drawing instruments should, if possible, be of high grade; serviceable sets may be purchased in a wide range of prices. You should consult an instructor or some other experienced draftsman or a reliable dealer before purchasing drawing instruments.

Equipment list

The complete list of equipment shown in Fig. 53 is as follows:

1. Drawing board (approx. 20″ × 24″)
2. T-square (24″, transparent edge)
3. Drawing instruments
4. 45° triangle (8″ side)
5. 30° × 60° triangle (10″ long side)
6. Lettering triangle
7. Triangular architect's scale
8. Irregular curve
9. Protractor
10. Drawing pencils (F, 2H, 4H)
11. Sandpaper pad or file
12. Ruby pencil eraser and art gum
13. Erasing shield
14. Art gum
15. Pen staff
16. Gillott's 303 and 404 pen points
17. Black drawing ink
18. Drawing paper

Penciled mechanical drawings are made upon detail paper, tracing paper, or tracing cloth. In industry the latter two are generally used for pencil drawings from which blueprints can be made. Detail paper is used for drawings that may require considerable correcting and changing. The finished drawings are then "traced" in ink on tracing paper or tracing cloth if blueprints are needed. Although detail paper may be purchased in white or even in light green, the popular color is buff, a pleasing light tan.

The paper should be placed close enough to the left-hand edge of the board to minimize the error resulting from the slight "give" of the T-square blade, and far enough from the top to permit the T-square head to maintain complete contact with the edge of the board (see Fig. 54).

Fastening the paper

You should fasten one upper corner and then pivot the sheet about this corner until the top edge of the sheet "lines up" with the top edge of the T-square; then fasten a second corner If the paper is small and of stiff character, two fastenings at the top may be sufficient, otherwise four or more fastenings may be necessary

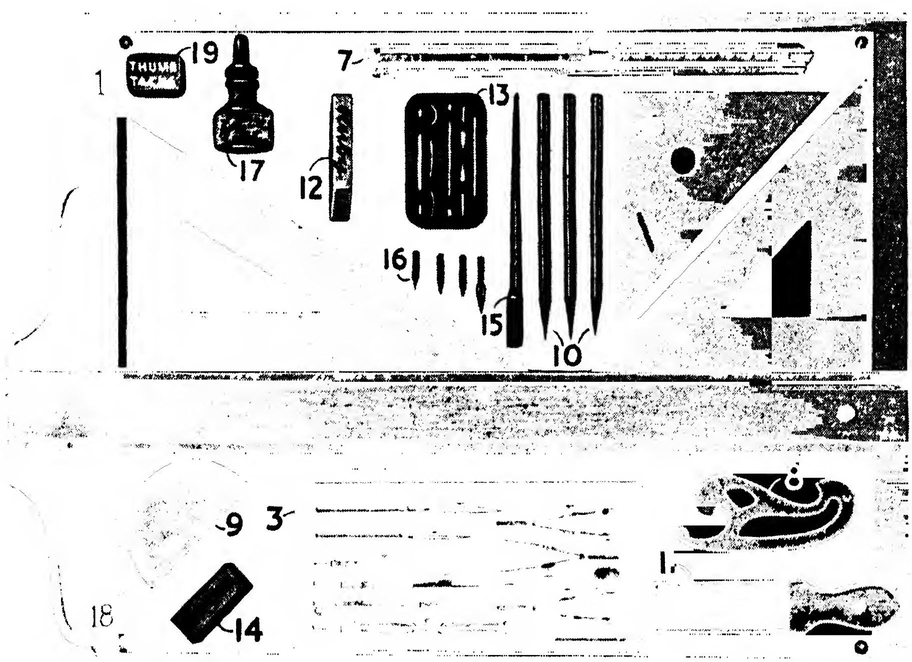


Fig. 53. Equipment used in technical drawing

Aligning the paper

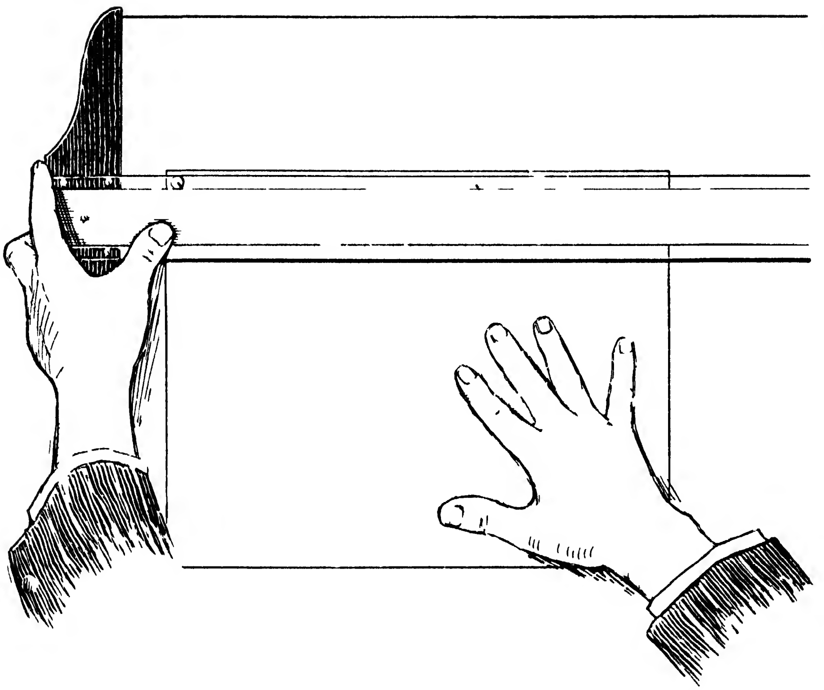
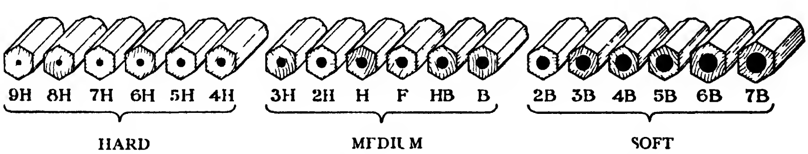


Fig. 54. Fastening paper to the board

Drawing pencils are made in a wide range of grades: from 9H — the hardest, to 7B — the softest. The scale of numbers and the classification of pencils according to use are shown in Fig. 55. However,

Grades of drawing pencils



The harder pencils in this group (left) are used where extreme accuracy is required, as on graphical computations, charts and diagrams. The softer pencils in this group (right) are used by some for line work on engineering drawings, but their use is restricted because the lines are apt to be too light.

These grades are for general-purpose work in technical drawing. The softer grades (right) are used for technical sketching, for lettering, arrowheads, and other freehand work on mechanical drawings. The harder pencils (left) are used for line work on machine drawings and architectural drawings. The H and 2H pencils are widely used on pencil tracings for blueprinting.

These pencils are too soft to be useful in mechanical drafting. Their use for such work results in smudged, rough lines which are hard to erase, and the pencil must be sharpened continually. These grades are used for art work of various kinds, and for full-size details in architectural drawing.

Fig. 55. Grades of pencils

the hardness of lead alone should not determine the choice of a pencil to be used because there are other variables, including the hardness or softness of the paper, the humidity of the air (which affects the paper), and the personal preference of the draftsman. In general, all construction lines (or "blocking-in" lines) should be drawn lightly with a fairly hard pencil, say 4H, 5H, or 6H; and all main required lines and lettering should be drawn *black* with a fairly soft pencil, such as F, H, or 2H.

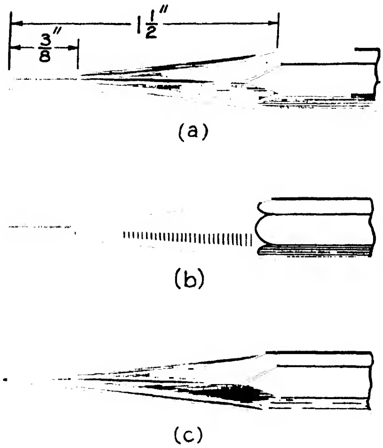


FIG. 56. PENCIL POINTS

Sharpening the pencil

Always sharpen the unlettered end of the pencil as in Fig. 56, either with a pocket knife or with a draftsman's pencil sharpener, so that the wood is removed, leaving about ⅜ in. of the lead uncut. In either case you then "dress down" the lead to a sharp conical point as in Fig. 57, and wipe the lead clean with a cloth. While drawing, you should keep your pencil *sharp*. It is a good idea to keep the sandpaper pad or file close by for frequent pointing of the pencil.

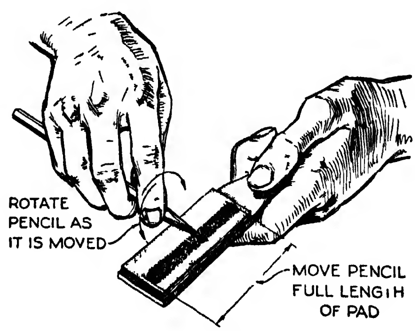
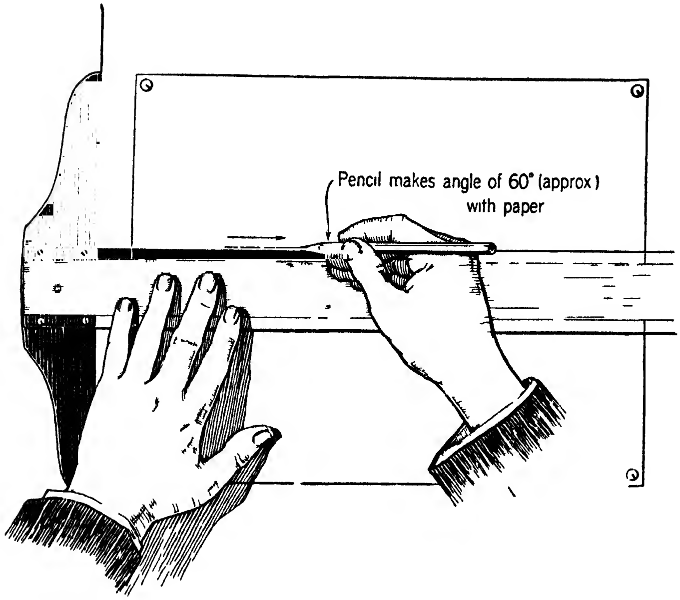


FIG 57 POINTING THE PENCIL

LINEWORK

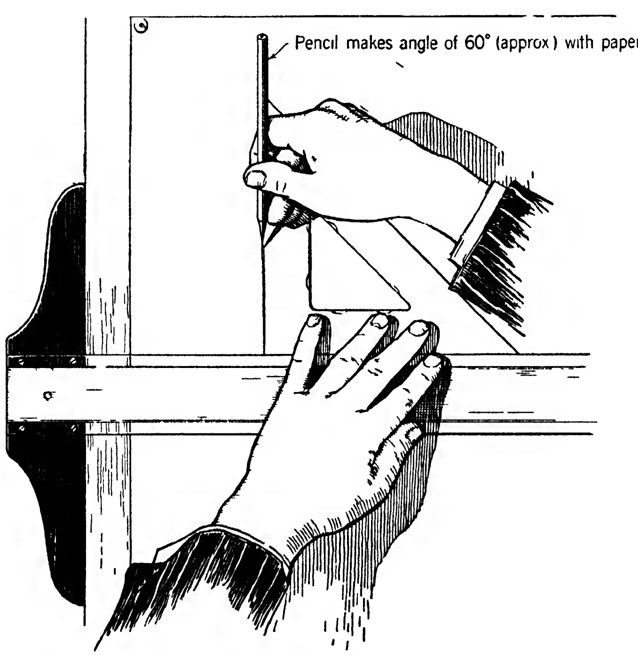
Horizontal lines

Always draw horizontal lines from left to right along the top edge of the T-square (Fig. 58). Left-handed persons should, in general, reverse all instructions in terms of "right" and "left," which follow. Press the T-square firmly against the "working edge" of the board, and lean the pencil at an angle of approximately 60° with the paper, with the little finger gliding lightly along the surface of the T-square blade. In order to produce lines of uniform width, rotate the pencil slowly, so that it will not wear off flat in one place.



Horizontal lines, drawn from left to right

FIG 58 DRAWING A HORIZONTAL LINE



Vertical lines, drawn upward

FIG 59 DRAWING A VERTICAL LINE

We use the 45° and 30° × 60° triangles (Figs. 60, 61, and 62) to draw vertical and inclined lines. You will realize that a 180° angle can be divided into four equal 45° angles with the 45° triangle, or into six equal 30° angles with the 30° × 60° triangle.

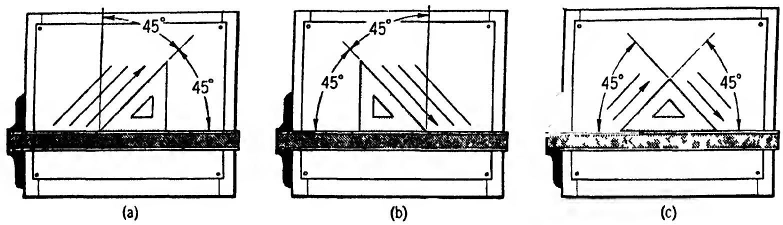


FIG. 60 DRAWING LINES WITH 45° TRIANGLE

Vertical lines

Vertical lines are drawn upward with either triangle along the left side, as shown in Fig. 59. Press the head of the T-square firmly against the drawing board, and then shift the left hand to the position shown in Fig. 59 where it holds both the T-square and triangle firmly in position. Here again, the pencil is leaned at an

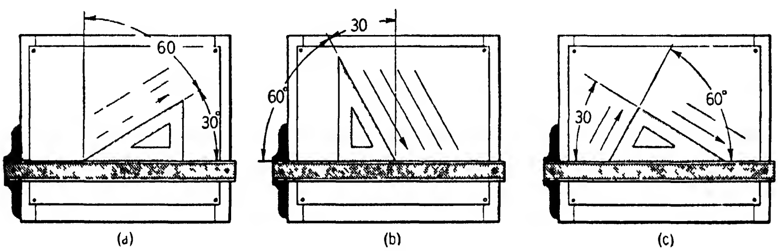


FIG. 61 DRAWING LINES WITH 30° × 60° TRIANGLE

angle of approximately 60° with the paper, and the pencil is rotated slowly throughout the extent of the line.

Inclined lines

You will soon learn to draw inclined lines at a large variety of angles with the 45° and 30° × 60° triangles, either singly or in combination. The method of drawing lines with the 45° triangle

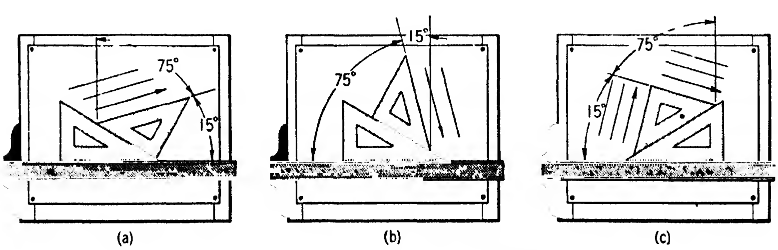


FIG. 62 DRAWING LINES WITH TRIANGLES IN COMBINATION

is shown in Fig. 60, the use of the 30° × 60° triangle is shown in Fig. 61, and the use of both triangles in combination is illustrated in Fig. 62. The direction in which the lines should be drawn is clearly indicated in each case. Good draftsmen follow these procedures exactly. By using the 45° and 30° × 60° triangles singly

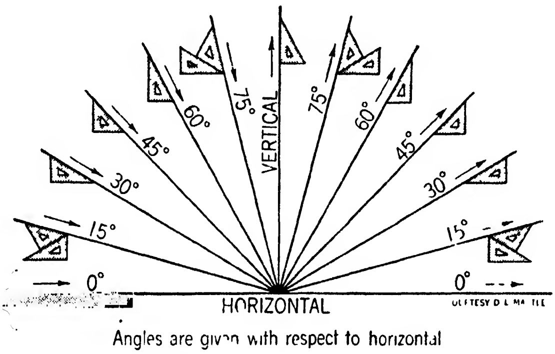
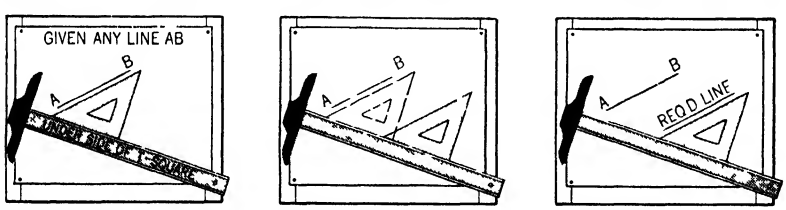


FIG. 63. STANDARD INCLINED LINES

and in combination, you can draw a complete set of radial lines with uniform 15° intervals, as shown in Fig. 63.

Parallel lines

Parallel lines at the standard inclinations indicated above may be readily drawn by sliding the triangles along the T-square and repeating the lines, as shown in Figs. 60–62. If you wish to draw a line parallel to *any* given line (Fig. 64), move the T-square and triangle as a unit until the hypotenuse of the triangle lines up



(a) MOVE T-SQUARE & TRIANGLE TO LINE UP WITH AB (b) SLIDE TRIANGLE ALONG T SQUARE (c) DRAW REQUIRED LINE PARALLEL TO AB

FIG. 64. DRAWING PARALLEL LINES

with the given line *AB* in *a*; then, with the T-square held firmly in position, move the triangle along the T-square away from the line as in *b*, and draw the required line as shown in *c*. Another triangle could be substituted for the T-square.

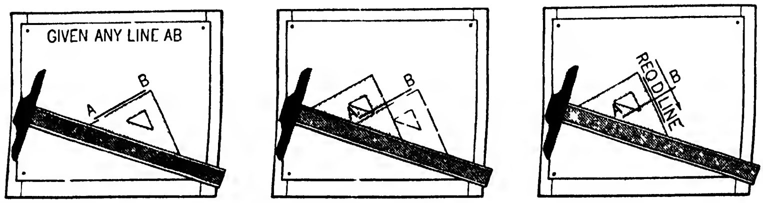
Perpendicular lines

If a given line slopes at any standard angle, illustrated in Fig. 63, a line also may be drawn perpendicular to it with the triangles used as shown above. For example, in either Fig. 60, 61, or 62,

the lines in *b* are perpendicular to those in *a*, and the two sets of lines in *c* are mutually perpendicular.

To draw a line perpendicular to any given line by the *adjacent-sides method* (Fig. 65), move the T-square and triangle as a unit

Using the legs of a triangle

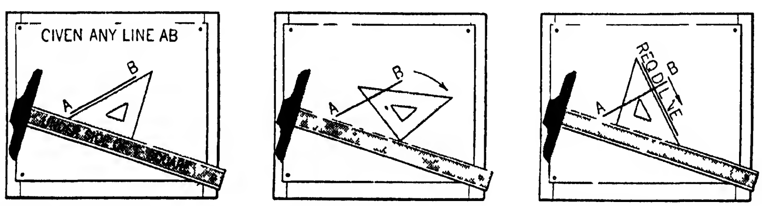


(a) MOVE T SQUARE & TRIANGLE TO LINE UP WITH AB (b) SLIDE TRIANGLE ALONG T SQUARE (c) DRAW REQUIRED LINE PERPENDICULAR TO AB

FIG. 65. DRAWING PERPENDICULAR LINES — ADJACENT-SIDES METHOD

until a side of the triangle "lines up" with the given line as in *a*; then with the T-square held firmly in position, slide the triangle across the line as in *b*, and draw the perpendicular as in *c*. A perpendicular to a given line can also be drawn by the *revolved-triangle*

Revolving a 45° triangle

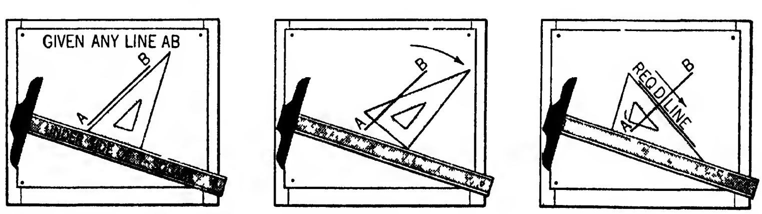


(a) MOVE T SQUARE & TRIANGLE TO LINE UP WITH AB (b) REVOLVE TRIANGLE ABOUT 90° CORNER (c) DRAW REQUIRED LINE PERPENDICULAR TO AB

FIG. 66. DRAWING PERPENDICULAR LINES — REVOLVED-TRIANGLE METHOD (45° TRIANGLE)

method, the T-square and triangle being manipulated as shown in Fig. 66 and Fig. 67. We choose this method when the hypotenuse is needed to draw a longer line.

Revolving a 30° × 60° triangle



(a) MOVE T-SQUARE & TRIANGLE TO LINE UP WITH AB (b) REVOLVE TRIANGLE ABOUT 90° CORNER (c) DRAW REQUIRED LINE PERPENDICULAR TO AB

FIG. 67. DRAWING PERPENDICULAR LINES — REVOLVED-TRIANGLE METHOD (30° × 60° TRIANGLE)

SCALE DRAWING

The scales

In technical drawing we make all measurements with a scale (never called a *ruler*). A drawing of an object is made *full-size* if the size of the paper permits, or to a *reduced scale* if necessary, such as half-size, quarter-size, or eighth-size. For example, a machine part may be drawn half-size ($\frac{1}{2}$ in. = 1 in.); a building may be drawn $\frac{1}{48}$ size ($\frac{1}{4}$ in. = 1 ft.); or a map may be drawn $\frac{1}{1200}$ size (1 in. = 100 ft.).

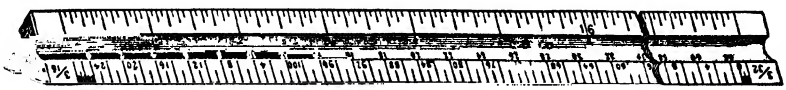


FIG 68 ARCHITECT'S SCALE

Engineer's scale

The two most common scales are the *architect's scale* (Fig. 68) and the *engineer's scale* (Fig. 69). The former is used wherever the unit of measurement is the inch divided into sixteenths, and the latter is used wherever measurements are made in the decimal system. Inches are represented by the symbol ″ and feet by the symbol ′. For example, eight feet three and five-eighths inches is written 8′–3$\frac{5}{8}$″, while eight feet and five-eighths of an inch is

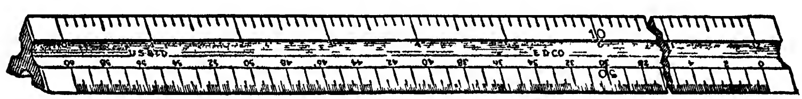


FIG 69 ENGINEER'S SCALE

8′–0$\frac{5}{8}$″. On drawings where all measurements are in inches, they are understood to be in inches, and the symbol ″ is omitted.

Professional draftsmen usually prefer flat scales rather than triangular scales, but a set of flat scales is needed to provide all the scales that may be found on one triangular scale; hence the triangular scale is preferred where cost is a factor.

The architect's scale

The architect's scale is used for architectural, structural, and machine drawings because those drawings are made by the inch-divided-into-sixteenths system, as distinguished from the decimal system. In machine drawing, however, decimal dimensions are also commonly used along with the regular measurements; therefore an engineer's scale or a conversion table will be convenient.

Full-size scales

The architect's full-size scale is divided first into inches, then halves, quarters, eighths, and finally sixteenths, as in Fig. 70 *a*, the division lines diminishing in length with each decreasing divi-

sion. The use of the full-size scale is illustrated in the same figure, where it should be noted that the beginning of the scale is at zero and not at the end of the stick.

Half-size

To set off a measurement half-size, the full-size scale is used and each dimension is divided mentally by 2 as in Fig. 70 *b*. The $\frac{1}{2}$-in.

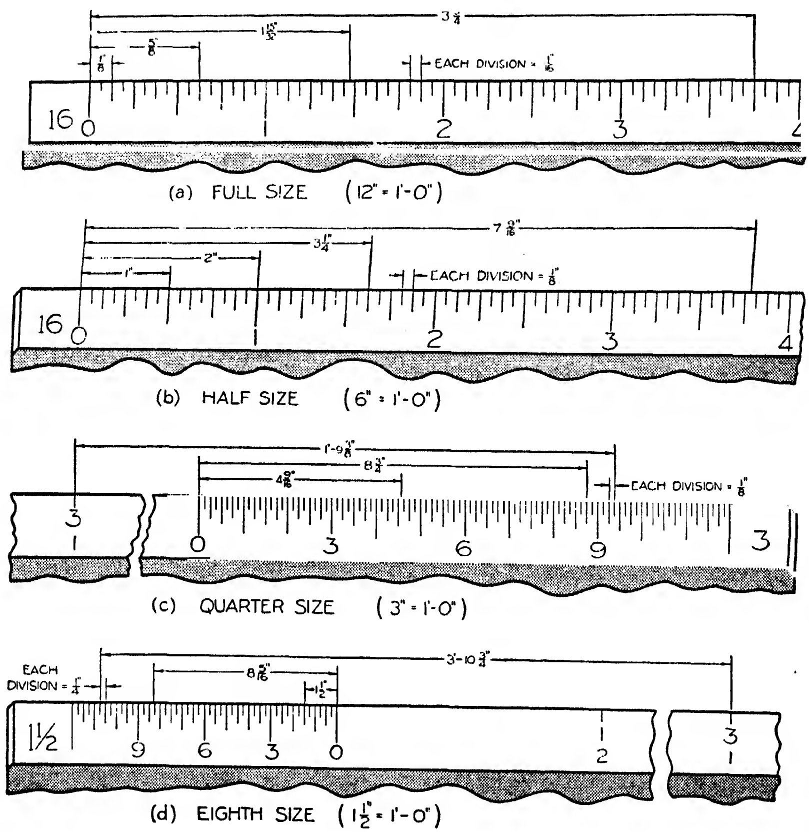


FIG. 70. READING THE ARCHITECT'S SCALE (SHOWN ACTUAL SIZE)

scale should not be used for half-size measurements, as this scale is intended for drawing to a scale of $\frac{1}{2}$ in. = 1 ft., or $\frac{1}{24}$ size.

Reduced scales

All of the remaining scales on the architect's scale are reduced scales, in which inches or fractions thereof stand for feet. The reduced scales, marked 3″, $1\frac{1}{2}$″, 1″, $\frac{3}{4}$″, $\frac{1}{2}$″, $\frac{3}{8}$″, $\frac{1}{4}$″, $\frac{3}{16}$″, $\frac{1}{8}$″, and $\frac{3}{32}$″ are each intended to equal 1 ft. per main division.

Quarter-size

To set off a measurement to quarter-size the 3-in. scale is used in which 3 in. = 1 ft. (Fig. 70 *c*). The subdivided portion to the right of zero represents 1 ft., and is divided into inches, then half-

inches, then quarter-inches, and finally, eighth-inches. To set off a measurement of more than 1 ft., the *feet* are set off to the *left* of zero, and the *inches* to the *right* of zero, as for example, the dimension $1'$–$9\frac{3}{8}''$ in Fig. 70 *c*.

To set off a measurement to one-eighth size the $1\frac{1}{2}$-in. scale is *Eighth-size*
used, in which $1\frac{1}{2}$ in. = 1 ft. (Fig. 70 *d*). The subdivided portion to the left of zero represents $1'$, but is actually $1\frac{1}{2}''$; hence $1\frac{1}{2}$ in. = 1 ft., or one-eighth size. To set off more than 1 ft., for example, $3'$–$10\frac{3}{4}''$ (Fig. 70 *d*), the dimension $3'$ is set off to the right of zero, and the dimension $10\frac{3}{4}''$ to the left of zero; the required distance is that between these two extreme marks. This measurement is *one-eighth* of $3'$–$10\frac{3}{4}''$. The remaining scales are used in the same manner.

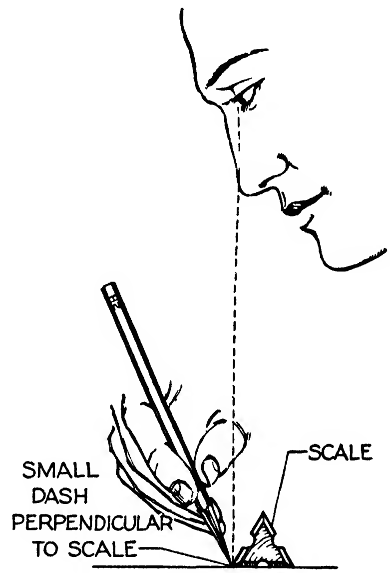


FIG. 71. ACCURACY IN USE OF SCALE

The habit of accuracy is an objective every engineering student should strive to attain. Accurate drafting depends largely upon the correct use of the scale. Measurements should not be taken directly off the scale with the dividers or compass. Always place the scale along the line to be measured, and with the eye directly above the correct graduation on the scale (Fig. 71) make a short dash with a sharp pencil at right angles to the scale as in Fig. 72 *a*.
If extreme accuracy is required, make a tiny prick mark at the *Accuracy*
exact graduation with a needle point or with one leg of the dividers, as shown in Fig. 72 *b*.

FIG. 72. ACCURATE MEASUREMENTS WITH THE SCALE

You must avoid *cumulative errors* in the use of the scale. If a series of measurements is to be made along one line, all should be set off at one setting of the scale, by adding each measurement to the

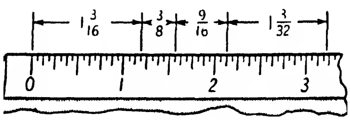


FIG. 73. SUCCESSIVE MEASUREMENTS

preceding one (FIG. 73). Such distances should never be set off individually by moving the scale to a new position each time, since even slight errors may accumulate and give rise to a large error.

USE OF INSTRUMENTS

Drawing instruments

A case of drawing instruments suitable for school or professional drafting is shown in Fig. 74. This set contains the following instruments:

- 1 Compass, with a pencil attachment, a pen attachment, and a lengthening bar
- 1 Dividers
- 2 Ruling pens
- 1 Bow pencil
- 1 Bow pen
- 1 Bow dividers
- 1 Case of leads for compass

In technical drawing, neatness, accuracy, and speed are essential; and high-grade instruments are always to be used if possible.

Use of compass

Circles and arcs with a radius of approximately 1″ or over are drawn with the compass. You can and should handle the compass entirely with one hand. When the compass is lifted from the case, the legs are squeezed apart with the thumb and forefinger, the needle point set at the center of the circle, and the compass adjusted to the required radius previously set off with a scale on the center line — all with one hand as in Fig. 75 *a*. Start drawing the circle on the left-hand side (*b*), lean the compass slightly forward, and draw the arc in a clockwise direction with the handle held between the thumb and forefinger (*c*).

The compass point

If you want to excel in your draftsmanship, use great care to sharpen the compass lead properly on the sandpaper pad or file, as shown in Fig. 76 *a*, *b*, and *c*. Adjust the needle point carefully as shown in *d*; avoid deep holes in the paper by exerting only a light pressure on the compass. For ink work the legs of the compass should be "broken" or adjusted so that the lower sections

FIG. 74. SET OF DRAWING INSTRUMENTS

are parallel, in order to bring both nibs of the pen attachment in contact with the paper (Fig. 75 *d*).

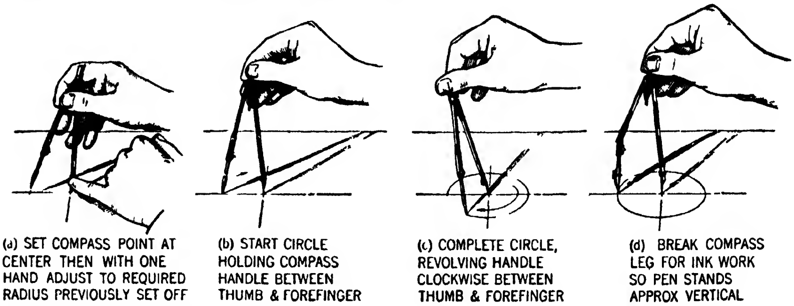


FIG. 75. USING THF COMPASS

Use of bow pencil or pen

Circles and arcs of about 1″ radius or less are drawn with the bow pencil or bow pen (Fig. 77). First, the needle point is set at the exact center (*a*) and the radius adjusted by turning the thumb-

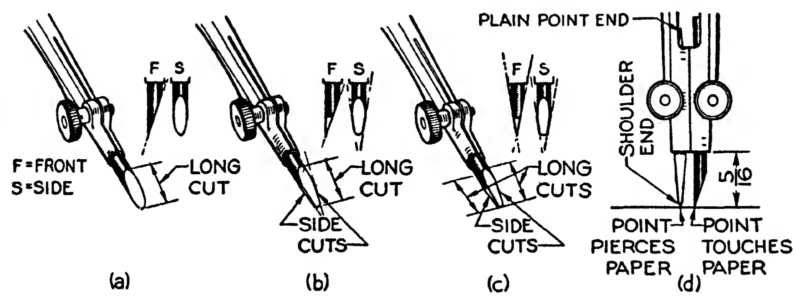


FIG. 76. COMPASS LEAD POINTS

screw between the thumb and third finger (*b*). Then the circle is started on the left-hand side and is drawn in a clockwise direction as shown in Fig. 77 *c*.

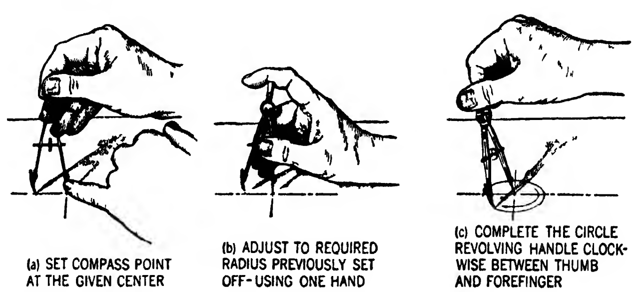


FIG. 77. USING THE BOW PENCIL

Use of dividers

The dividers are used for subdividing lines into equal spaces and for transferring distances that are approximately 1″ or over. To divide a line into equal parts, say three, as in Fig. 78, lift the dividers, squeeze the legs apart between the thumb and forefinger until the points are separated an estimated one-third of the total distance, and set the point at one end of the line as in *a*. Then the

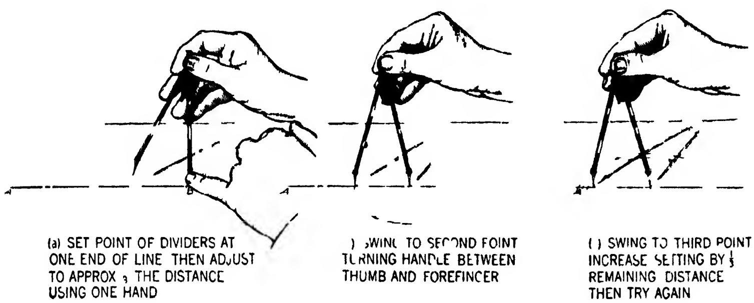


FIG 78 DIVIDING A LINE INTO EQUAL PARTS

dividers are twirled between the thumb and forefinger to the second point (*b*) and finally to the third point (*c*). Here the spacing of the dividers is decreased or increased by one-third the total error, large or small as the case may be, and the process is repeated until the spaces appear as equal thirds.

Use of bow dividers

Where small divisions are needed, the bow dividers should always be used instead of the large dividers, because of their greater rigidity. The bow dividers are adjusted with one hand, as shown for the bow pencil in Fig. 77 *a*, and are used in a manner similar to that described for the large dividers (Fig. 78).

Use of irregular curve

The correct use of the irregular curve requires skill, especially if the lines are to be drawn in ink. First, sketch a light freehand line smoothly through the points; then match the various segments of the curve in succession to the freehand curve (Fig. 79).

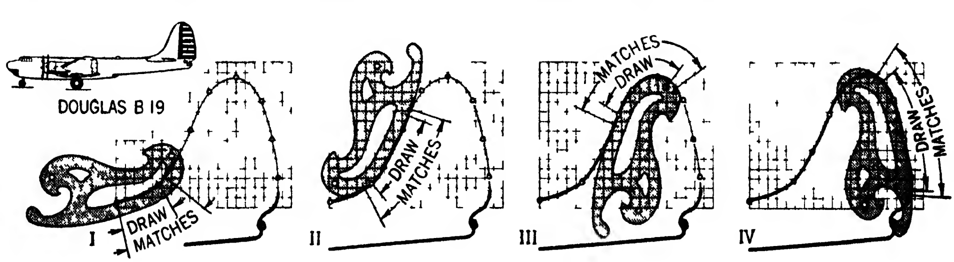


FIG 79. SETTINGS TO DRAW A SMOOTH CURVE

Care must be taken to *overlap* the matched portions at each setting; otherwise abrupt changes in direction are sure to result.

DRAFTING TECHNIQUES

Keeping drawings clean

The utmost cleanliness in drawing should become habitual, for the draftsman can always keep his work clean if he really tries. First, your hands must obviously be kept clean, and second, your drawing tools must be frequently wiped and cleaned. Art gum is excellent for cleaning triangles and T-square; water is not recommended. Of course, you should never sharpen a pencil over a drawing, and your sleeves or arms should not be allowed to rest

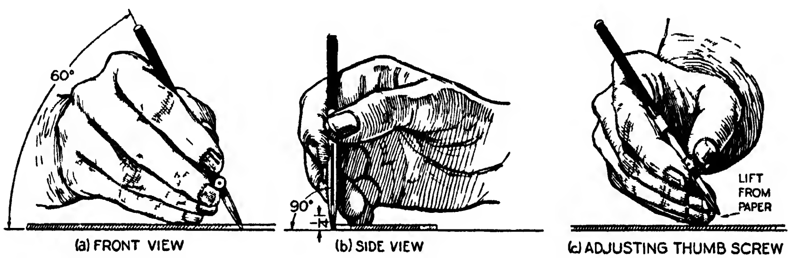


FIG. 80 USE OF RULING PEN

upon a penciled area, else smudging will result. So far as possible nothing should be allowed to come in contact with the penciled surface except the T-square and triangles, and they should be lifted to be moved, whenever possible.

Using ink

The ruling pen (Fig. 80) is used to ink ruled lines and is never used freehand. It is filled to a height of about $\frac{1}{4}$ in., as shown in *b*, with the *quill* of the ink stopper. The thumbscrew controls the width of the line and is adjusted with one hand as shown in *c*.

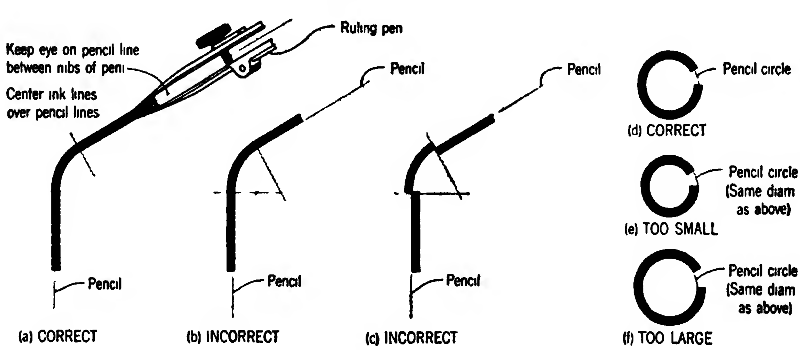


FIG 81 INK LINES OVER PENCIL LINES

Always draw ink lines exactly centered over the pencil lines and not to one side (Fig. 81). You should lean the pen at an angle of about 60° with the paper (Fig. 80 *a*), keeping the thumbscrew on the side away from the T-square or triangle. The correct position of the pen and the resulting correct line are shown in Fig. 82 *a*; ***Ink lines***

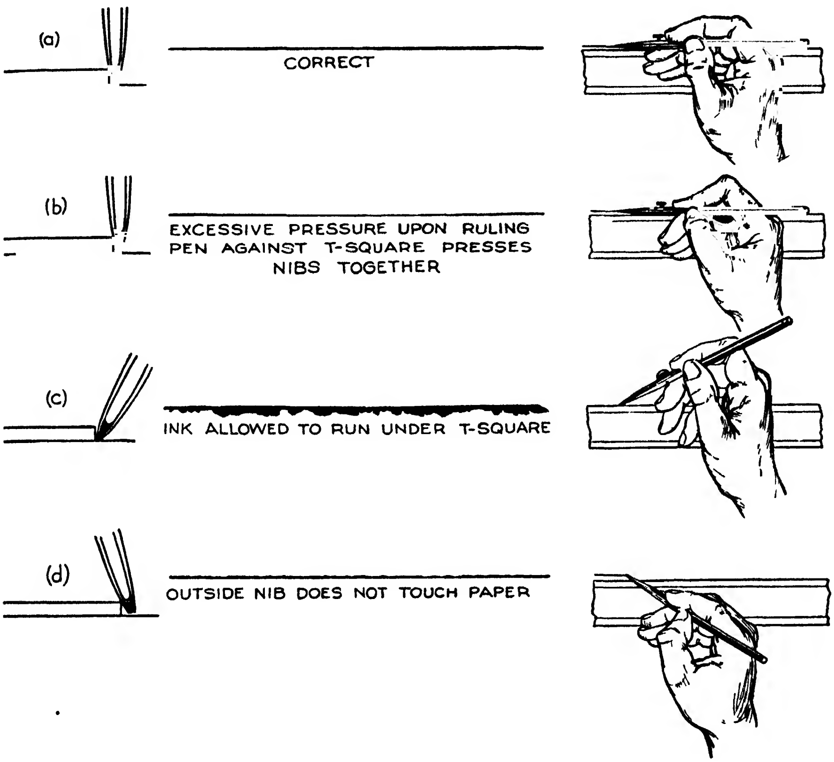


FIG. 82. POSITION OF RULING PEN

incorrect uses are illustrated in sketches *b*, *c*, and *d*. The ruling pen should be cleaned frequently by inserting a blotter or a piece of cloth between the nibs; *it should never be laid down with ink in it.* When you need to draw large circles in ink, fit the ink attachment to the large compass. The bow pen should be used for inking circles of less than about 1 in. radius.

When you make an error in inking, your first act should be to blot the area quickly with a blotter to absorb excess ink; then after the spot has dried, you can with patience erase the spot or error with the Ruby pencil eraser. Draftsmen have no use for a gritty "ink" eraser, which is apt to damage the paper. ***Erasures***

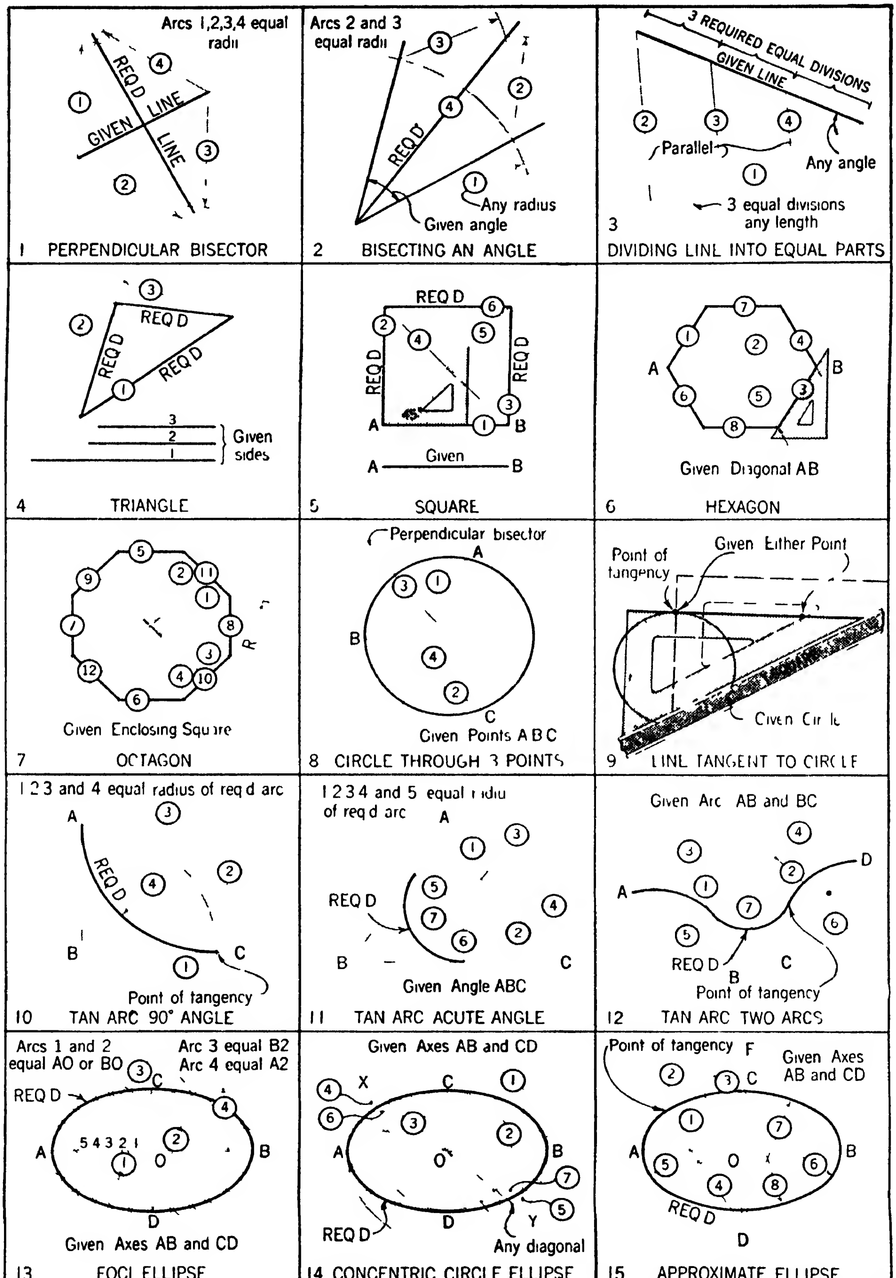


FIG. 83. COMMON GEOMETRICAL CONSTRUCTIONS

In technical drawing there are a number of applications of plane geometry which are useful and which are used repeatedly (Fig. 83). Some of these may be drawn with the compass and a straight edge as in pure geometry. However, the draftsman has at his disposal drafting instruments which can be used in many cases to better advantage, and these methods are shown where applicable. *Geometrical constructions*

Given and *required* lines are generally heavy and should be drawn with a sharp medium-grade pencil so as to produce a black line. Construction lines should be extremely light and should be drawn with a sharp hard pencil. In Fig. 83 the small encircled numbers show the order in which the lines should be drawn in each case. *Weight of lines*

Technical drawings are composed chiefly of symbolic lines (Fig. 84). These lines are designed to provide a reasonably natural appearance of the object represented, but they are strictly symbols and should be drawn to conform to fixed standards, as illustrated. A pencil drawing is composed roughly of only two line thicknesses, which may be described as "thin" and "medium," while an ink drawing also may have "heavy" lines and some variations between these. It is important on both pencil and ink drawings, and especially the latter, to obtain a clear *contrast* between the various weights of lines. Each line in its own class should be drawn uniformly throughout the drawing. *The graphic alphabet*

The greater part of commercial drafting is executed in pencil. Most blueprints are made from pencil tracings, and all ink tracings are preceded by pencil drawings. It should therefore be evident that skill in drafting implies chiefly *skill in pencil drawing.* "Technique" is a style or quality of drawing imparted by the individual draftsman to his work. It is characterized by bright, clean linework and lettering. On pencil drawings all lines are equally *black* with the single exception of the construction line, which should be extremely light. The several types of black pencil lines should vary in width but not in color. *Pencil tracings*

The proper order of penciling a working drawing is illustrated in Fig. 85. The drawing is carried through two main stages: first, the construction stage, embraced in spaces I and II, in which the drawing is "blocked-in" completely with a hard pencil, such as 4H to 6H, with extremely light lines; and second, the finishing stage, shown in spaces III and IV, in which the lines are heavied-in and the lettering added with a softer pencil, such as 2H or 3H. *Order of penciling*

A definite order should be followed in inking a drawing or

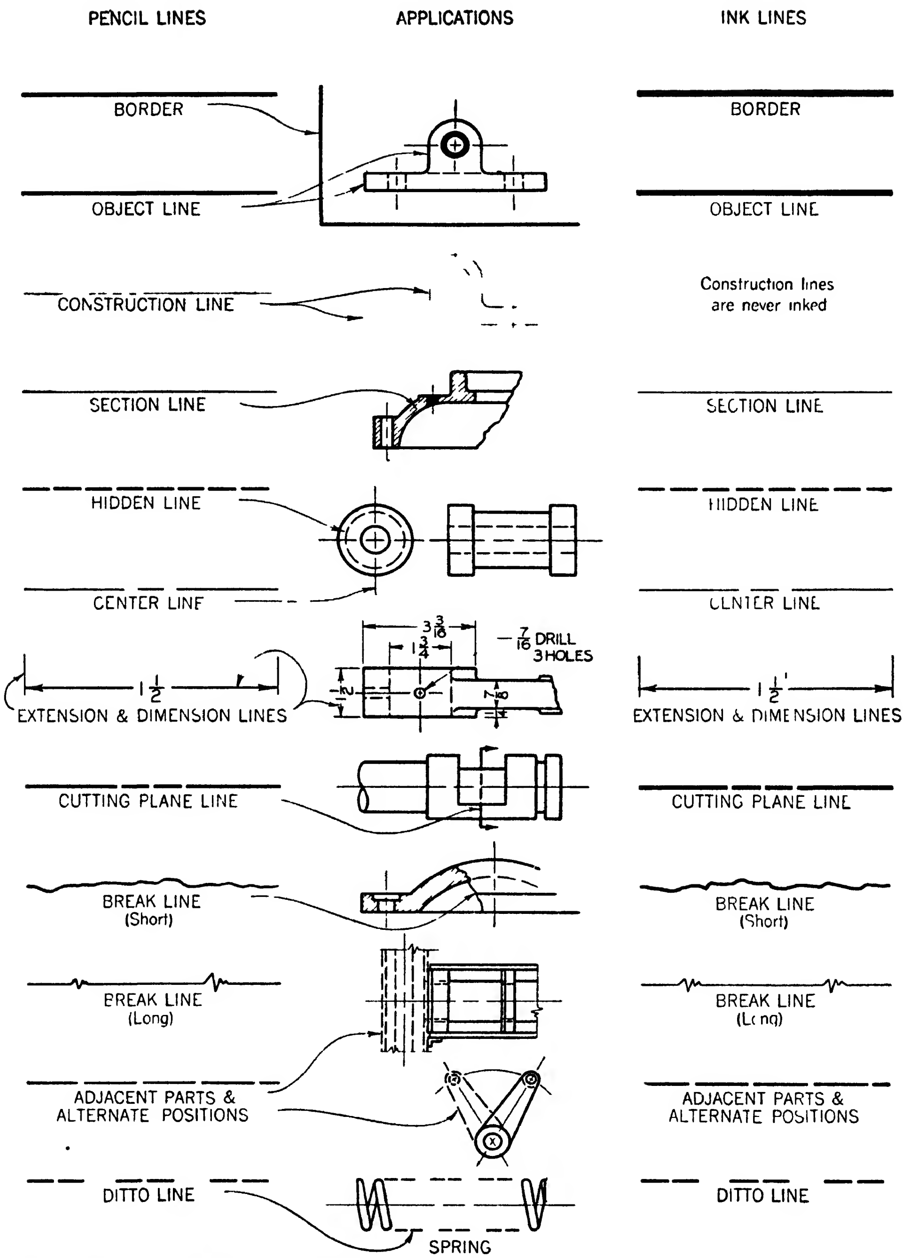


FIG 84. THE GRAPHIC ALPHABET

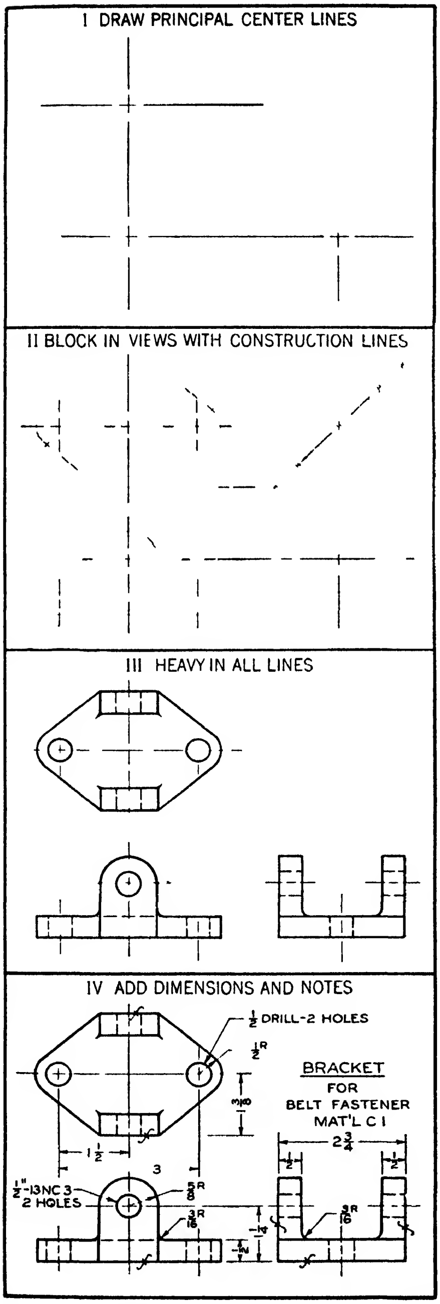


Fig. 85. Order of penciling

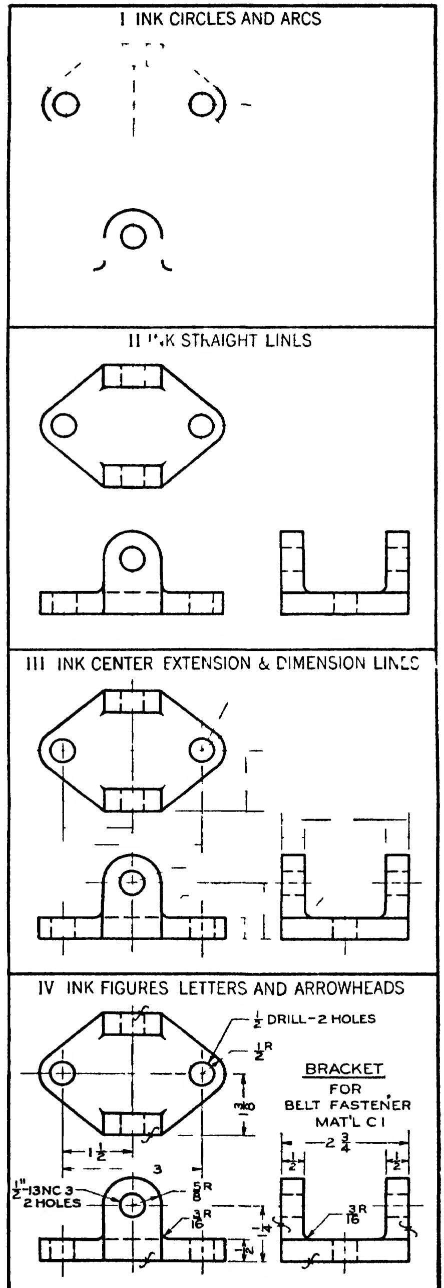


Fig. 86. Order of inking

tracing, as illustrated in Fig. 86. Before the inking is started, all points of tangency should be located, establishing the extent of all arcs. All arcs and curves are inked first, followed by the straight lines. Lettering and arrowheads are then added to complete the tracing.

TECHNICAL LETTERING

Technical drawing is universally recognized as a basic study in all branches of engineering, and as such is taught as a "tool subject" in virtually every technical school in the world. It occupies this position in technical curriculums because the fundamentals embraced are prerequisite to the engineering studies which follow and to the practice of engineering itself.

Importance of lettering

In a similar way *skill in lettering* is prerequisite to engineering training and practice, and more especially to technical drawing. The ability to letter is almost as important to the engineer as the ability to write longhand is to the average person.

History

Before the invention of printing by Gutenberg in the fifteenth century, all letters were made by hand, and ornamented according to the artistic taste of the letterer. Early type was modeled upon the accepted letter-forms in each country, but modern newsprint capital letters are essentially the same as the *Old Roman* letters, developed approximately two thousand years ago. The present small letters, called *lower-case* letters because they were found in the lower case of type by compositors, which are used in all of our books and newspapers, were developed in the Italian Renaissance from earlier capital letters.

Single-stroke gothic letters

During the latter part of the nineteenth century the development of engineering made evident a need for a simple legible letter which could be made with single strokes of an ordinary pen. Our present single-stroke gothic letter is due largely to the work of C. W. Reinhardt, chief draftsman in the 1890's for the *Engineering News.*

A complete working drawing must not only describe, by means of views, the *shape* of the object, but by means of figured dimensions it must give the *size* of the object. Also, information which is needed in the construction of the part will be given by notes and specifications. Well-executed lettering contributes more than any other single factor to the attractiveness and legibility of a drawing.

How to start

The position of the hand when you are lettering should be natural and uncramped, as in Fig. 87. The forearm should always rest upon the drawing board or table top. Pencil lettering should be made with a medium-soft pencil with a sharp point. You should frequently turn the pencil in order not to wear the point flat in any one place. For ink lettering the Gillott's 404 steel point, or its equivalent, is recommended for the usual $\frac{1}{8}''$ lettering; for smaller letters the Gillott's 303 is satisfactory; and for letters over $\frac{1}{8}''$ the various ball-pointed pens may be used to advantage. When you are doing ink lettering, you should wipe the pen point clean before each dip of fresh ink.

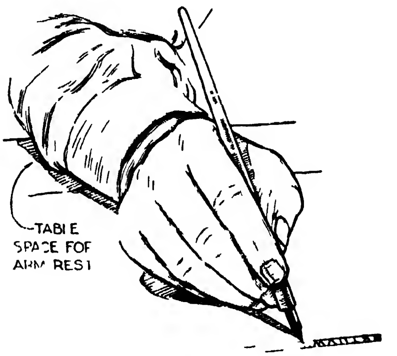


FIG 87 POSITION OF HAND IN LETTERING

Steps to follow

Any normal person can learn to do at least acceptable lettering if he is persistent in practice and in his will to improve. There are three steps in learning to letter: (1) learning the strokes and proportions of the letters, (2) learning correct spacing methods, (3) persistent practice.

Both vertical and inclined letters are used on technical drawings, according to the preferences of the various companies or of individual draftsmen. You will be wise to learn both styles and finally adopt the one which suits you better. Of the two, the vertical should be learned first.

VERTICAL LETTERING

Vertical upper case letters

The vertical capital letters are shown in Fig. 88. The letters are built up by a series of strokes, as shown below each of the large letters. The arrows indicate the direction in which each stroke should be drawn. With the exception of the letters I and W, all letters are either five or six units wide, such simple widths being easily remembered. The letters O, C, D, G, and Q are based upon the true circle. You must study and practice until you have developed a clear and accurate mental image of each letter; this is absolutely essential for success in lettering. It is especially suggested that you learn the forms of the letters by sketching them, six squares high, on cross-section paper.

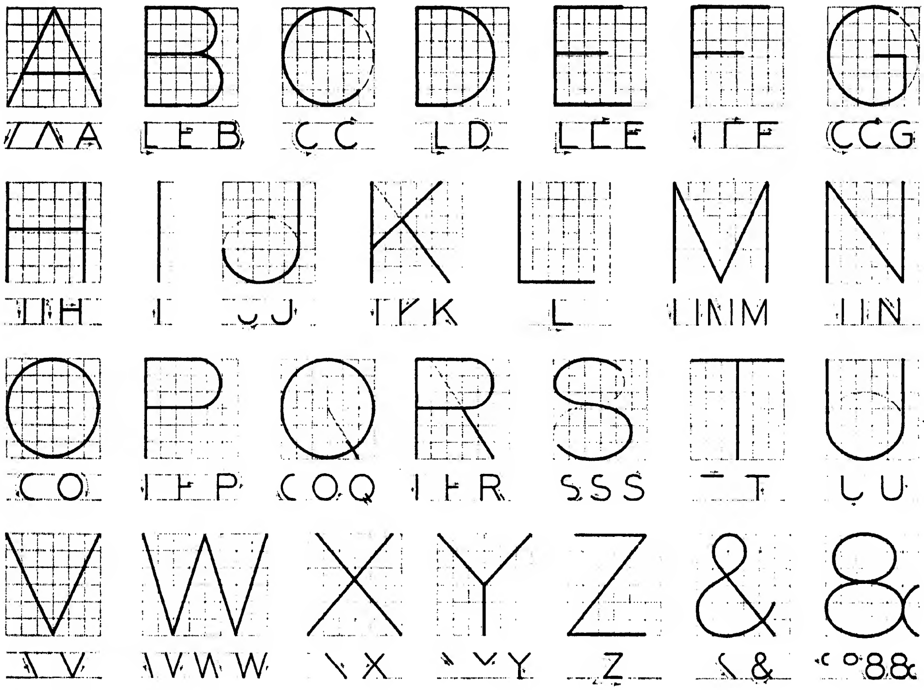


FIG. 88. VERTICAL CAPITAL LETTERS

Guide lines

Light horizontal and vertical guide lines, drawn with a sharp hard pencil, will assist you by limiting the height of the letters and by helping you to keep them vertical. (See Fig. 89.) The vertical guide lines help to keep the letters vertical but are not intended to assist in spacing. The clear space measured vertically between lines of capitals may vary from two-thirds of the height to the full height of the letters.

Guide-line spacing

Guide lines are spaced by means of the scale or by using bow dividers, as in Fig. 90. A convenient height-spacing for most letter-

Lines of lettering

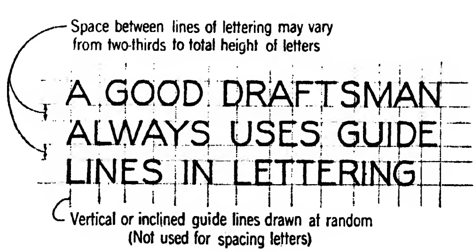


FIG. 89. GUIDE LINES FOR CAPS

ing is $\frac{1}{8}''$, with spaces of $\frac{1}{8}''$ between lines, as shown in Fig. 90 *a* and *b*. Where the spacing between lines is equal to the height of the letters, you will also find the use of bow dividers very convenient, as only one setting of the points is necessary (*b*). The

Use of dividers

(a) EQUAL SPACING WITH SCALE (b) EQUAL SPACING WITH DIVIDERS (c) UNEQUAL SPACING WITH SCALE (d) UNEQUAL SPACING WITH DIVIDERS

FIG 90 SPACING OF GUIDE LINES

scale may be inclined to the guide lines to decrease vertical distances between guide lines, even though the spacing between lines does not equal the height of the letters (*c*). The bow dividers may still be used, with a single setting of the points, in cases where the spacing between lines does not equal the height of the letters (*d*).

Guide-line devices

Guide lines may be drawn conveniently with the aid of the Ames Lettering Instrument (Fig. 91 *a*) or the Braddock-Rowe Triangle (*b*). With either instrument, guide lines are drawn by inserting a sharp pencil in the systematically arranged holes and moving the instrument along the T-square. Numbers are provided to indicate the groups of holes that give various heights of letters in thirty-seconds of an inch.

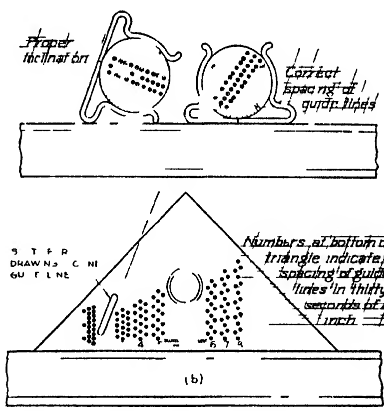


FIG 91 GUIDE-LINE INSTRUMENTS

Spacing of letters

Uniformity in spacing letters and words is a matter of balancing spaces by eye. *The background areas between letters must appear approximately equal* (Fig. 92). Certain letters, such as the L and T, are of such shape that they may be overlapped to produce good spacing. Words should be spaced a distance apart equal to the letter O.

To letter to a stop-line as in Fig. 93 *a*, space off each letter from right to left by eye, and then letter forward to the stop-line.

Centering lines of lettering

To space a line of letters symmetrically about a center line (Fig. 93 *b*), count the letters and spaces between words, and place the middle letter or space at the center; then letter the right half, followed by the left half.

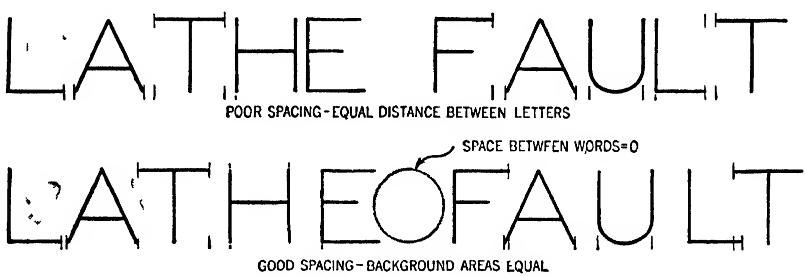


Fig. 92. Spacing letters and words

A strip of paper, as shown in Fig. 93 *c*, may also be used to letter a trial line, to be placed for reference immediately above the line to be lettered.

Vertical numerals

The importance of correct formation of the numerals is understood when it is realized that all *numerical values on technical drawings*

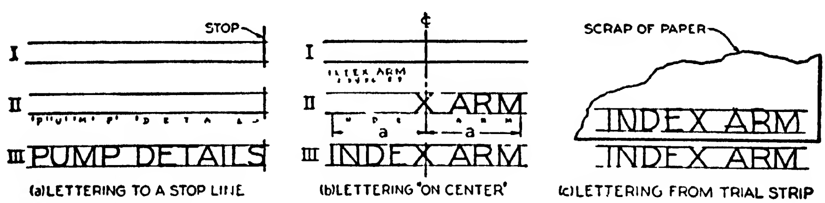


Fig. 93. Spacing

are expressed by various combinations of the ten basic numerals shown in Fig. 94. It is therefore worth your time to study carefully the proportion and spacing of these numerals. The background areas should be kept "open," and in fractions a clear space must be left between the numerals and the division line. All numerals

Fractions

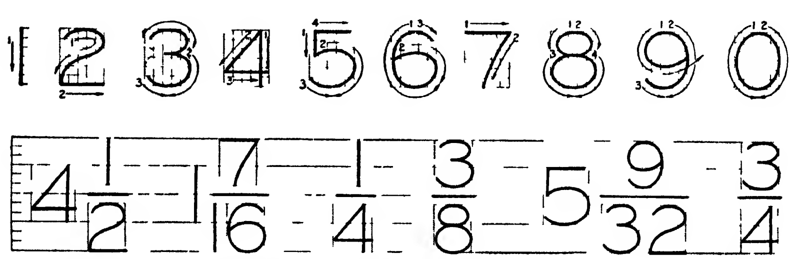


Fig. 94. Vertical numerals and fractions

are five units wide except the I and the 4. A *fraction* is one and two-thirds to twice the height of the whole number. The numerator and denominator are each two-thirds the height of the whole number.

Guide lines for numerals

A simple method of spacing guide lines for fractions is to use the bow dividers and set off spaces a equal to each other as illustrated by Fig. 95. The guide lines are then drawn through the points with the aid of the T-square or triangle.

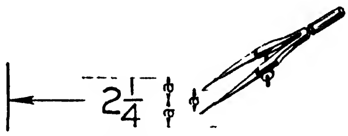


FIG. 95 SPACING WITH BOW DIVIDERS

Stability

If the upper portions of certain letters and numerals are as wide as the lower portions, the characters will appear to be top-heavy or to lack *stability*. To correct this optical illusion the upper portions are reduced in size, thus producing a more pleasing appearance (Fig. 96). Stability in the letters B, E, F, and H is improved by placing the horizontal crossbars slightly above center.

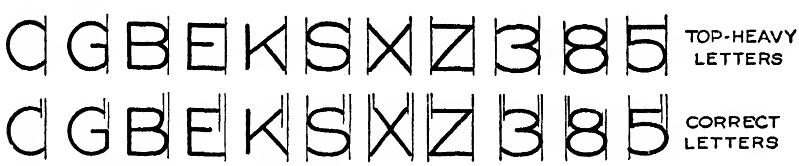


FIG. 96. STABILITY OF LETTERING

INCLINED LETTERING

Inclined letters

The order and direction of the strokes and the proportions of inclined letters correspond to those for vertical letters. The circular parts of vertical letters, both upper case and lower case, become ellipses for inclined letters. Inclined capitals, or upper-case letters, are shown in Fig. 97 and the inclined lower-case letters are shown in Fig. 100. In making inclined letters, you should exercise care to make the inclined strokes slope uniformly and to keep the sides of the letters *A*, *V*, *W*, *X*, and *Y* sloping symmetrically about the inclined center lines. A comparison of correct and incorrect inclined capitals is shown in Fig. 98.

Guide lines for inclined letters

Light horizontal and inclined guide lines are used to regulate inclined letters (Fig. 99). Except for the inclined guide lines, all spacing is comparable to that for vertical capitals (page 178). The inclined guide lines are drawn at random (not used for spacing) at a slope of 2 on 5, which is approximately $67\frac{1}{2}°$ with horizontal

(Fig. 99). The slot on the Braddock-Rowe Triangle shown in Fig 91 *b* is inclined at $67\frac{1}{2}°$ with the horizontal and is convenient for making inclined guide lines.

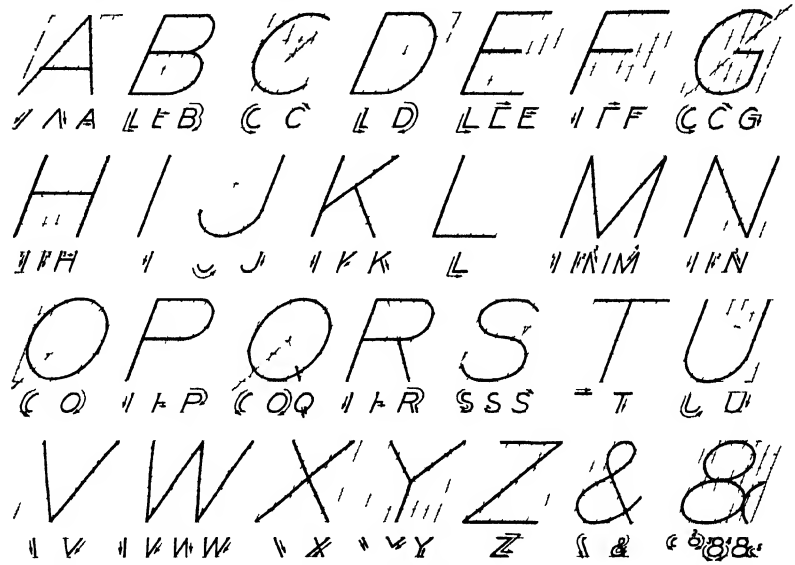


Fig 97 Inclined capital letters

Reinhardt letters

Inclined lower-case letters, known as the "Reinhardt letters," are shown in Fig 100 This letter comes from the same letter forms as our handwritten script and is, therefore, the easiest letter for most people to make acceptably It is easy to read because it

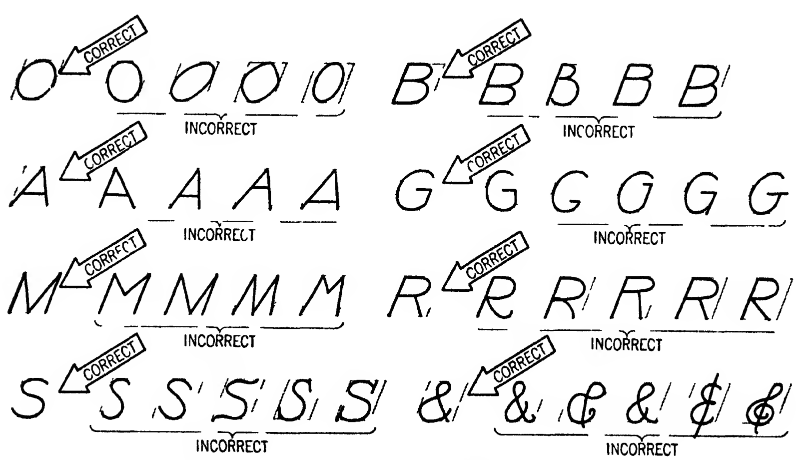


Fig 98 Correct and incorrect inclined capitals

closely resembles book and newspaper type, which is predominantly lower case, and it may be executed more rapidly than the other forms. Hence, this letter is widely used by engineers in all

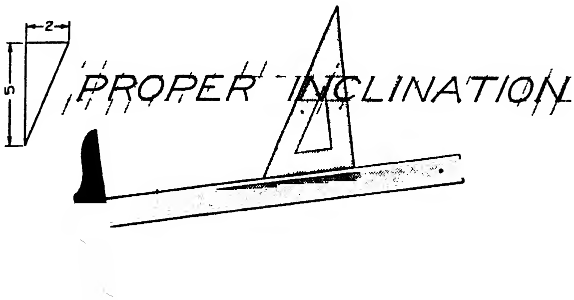


FIG 99 INCLINED GUIDE LINES

fields. The *bodies* of the letters are two-thirds the height of the corresponding capital letters, and the *ascenders* or *descenders* extend upward or downward one-third of this height (Fig. 100).

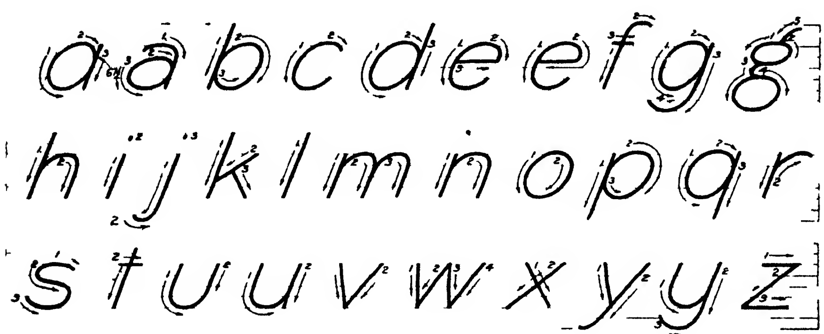


FIG 100 INCLINED LOWER-CASE LETTERS

Amount of slope

This alphabet is composed of combinations of only two forms, the straight line and the ellipse, the long axes of the ellipses sloping

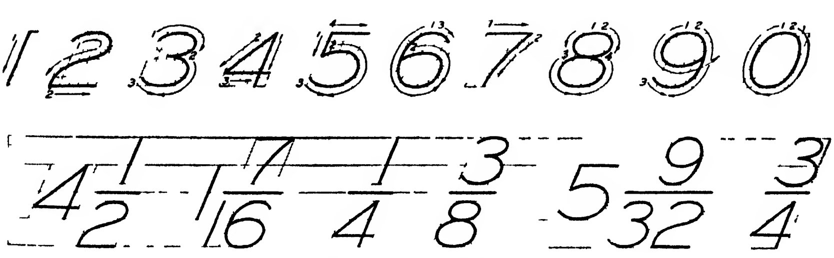


FIG 101 INCLINED WHOLE NUMBERS AND FRACTIONS

at 45° as in the inclined capital letters. The slope of the letters is $67\frac{1}{2}°$ with the horizontal.

Inclined numerals

The inclined numerals are shown in Fig. 101. They are different from the vertical numerals only in that they are inclined.

SHAPE DESCRIPTION

Views and line drawings

A mechanical unit may be represented attractively, though superficially, by means of a photograph (Fig. 102). For technical purposes, such as the development of the original design and the conveying of the designer's ideas to the shop, *line drawings* are

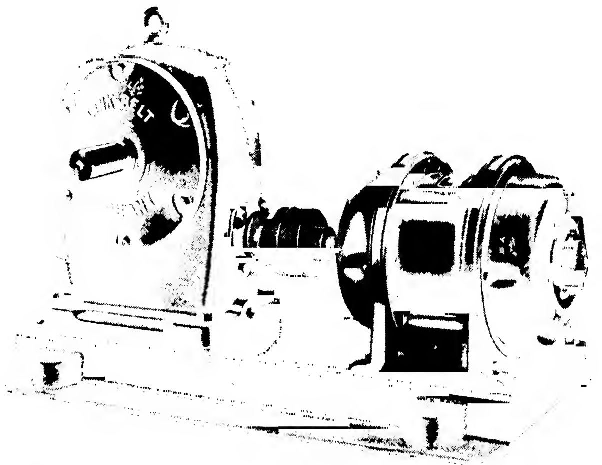


FIG. 102. REDUCTION GEAR DRIVE (PHOTOGRAPH)

necessary. The object must be represented by means of *views* showing the various sides, so that the true dimensions of the object will be shown. A three-view drawing of the "Reduction Gear Drive" is shown in Fig. 103, in which you should note that the three principal dimensions —*length*, *height*, and *depth*— appear in their true sizes (or proportionately, if drawn to a reduced scale). A white line print showing three views of a worker is shown in Fig. 104, which illustrates the front view, top view, and right side view. Notice, however, that the *right side view* of the man shows *his* left side.

Viewing the object

Not only are assemblies represented by views, as in Fig. 103, but every individual part of a machine must be separately represented by means of its views, systematically arranged. For example, the "Holder for Timing Roller on an Offset Press" placed in an imaginary *glass box*, as shown in Fig. 105, is viewed by the observer

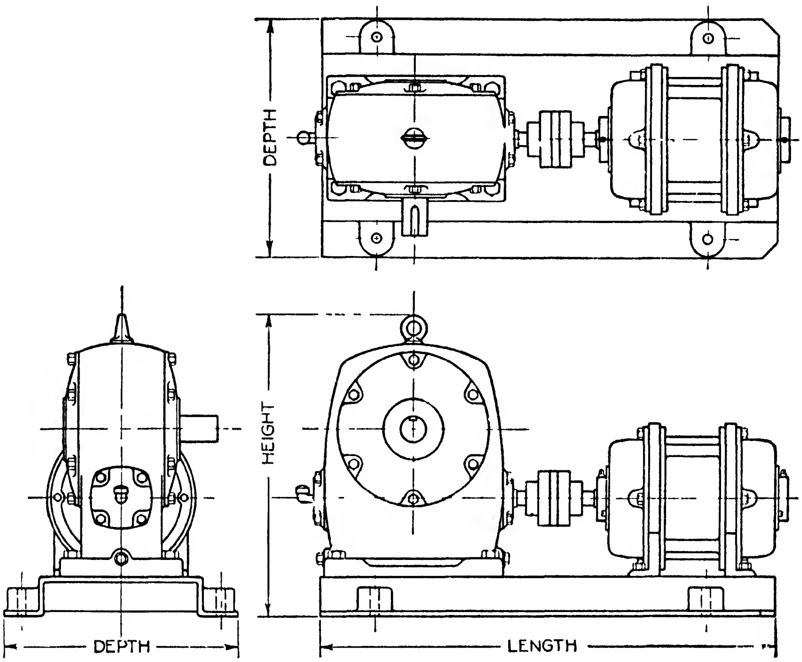


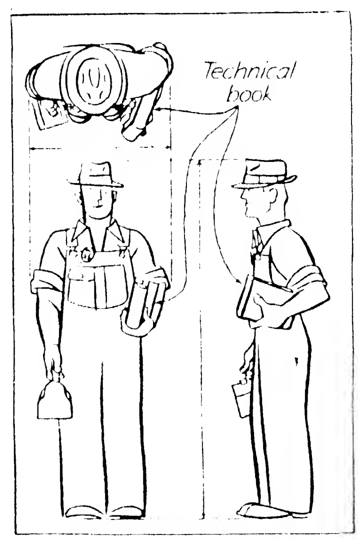
FIG. 103. THREE VIEWS OF A REDUCTION GEAR DRIVE

Three-view drawing

looking perpendicularly toward the front of the object to obtain the *front view*, perpendicularly toward the top of the object to obtain the *top view*, and perpendicularly toward the right side of the object to obtain the *right side view*. In Fig. 105 the arrows indicate the respective *lines of sight* for the three principal views, and these arrows are mutually perpendicular.

Principal dimensions

For purposes of graphical analysis, all objects can be assumed to have three principal dimensions — width, height, and depth. Two of these dimensions are shown true size in any view. As illustrated in Fig. 105, the front view always shows width and height,



By Russel Hamilton, courtesy Publishers' Weekly

FIG. 104. THREE VIEWS OF A TECHNICIAN

the top view shows width and depth, and the right side view shows height and depth. For each view the observer is, theoretically, at an infinite distance from the object, so that in a given view the lines of sight to the various points on the object will be mathematically parallel.

Possible views

There are, obviously, six principal directions in which an object may be viewed in this manner; hence six regular views of any object can be obtained, though three are usually adequate.

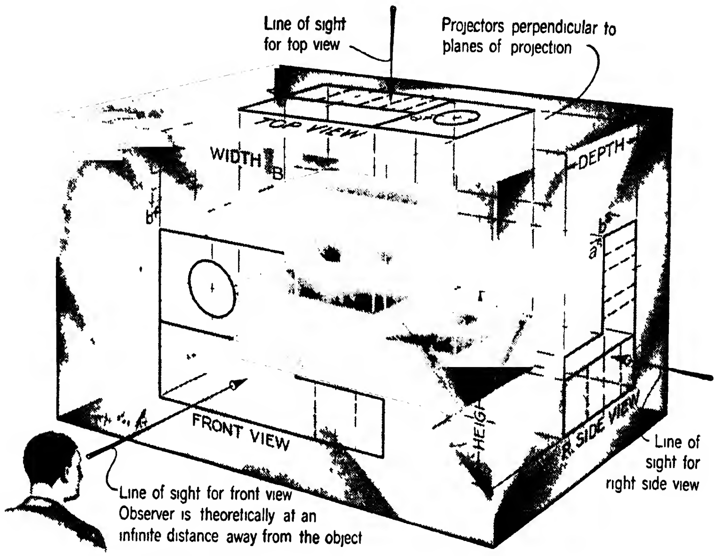


Fig. 105 The glass box

Projections

A view of an object is known technically as a "projection," and we call the system of views described above *multiview projection.* A projection is a view of an object "projected" upon an imaginary *plane of projection.*

In Fig. 105 the *projectors,* or lines of sight, are parallel to each other and perpendicular to the plane of projection for that view. This is equivalent to saying that any view may be obtained by dropping perpendiculars from all *edges and contours** of the object to the corresponding plane of projection. The piercing points of these projectors, being infinite in number, form lines on the plane of projection. Thus, from a point A on the object in Fig. 105, perpendiculars strike the front plane of projection at a^f, the top plane

* An edge is the intersection of two surfaces; a contour is an outline of a body seen from one position.

of projection at a^t, and the right side plane of projection at a^s. Likewise, the projections of the point B are obtained at b^f, b^t, and b^s; and if all points between A and B are similarly projected, the front view of the line AB is seen to be a^fb^f, the top view a^tb^t, and the right side view a^sb^s, which happens to be a point. If this procedure is continued with all of the edges and contours of the object, six complete views may be obtained.

Hidden lines

One of the most important advantages in the use of multiview projection is our ability to show edges and contours which normally would be invisible in a given view, by means of lines composed of dashes, or *hidden lines*. In Fig. 106 the edge AB would not

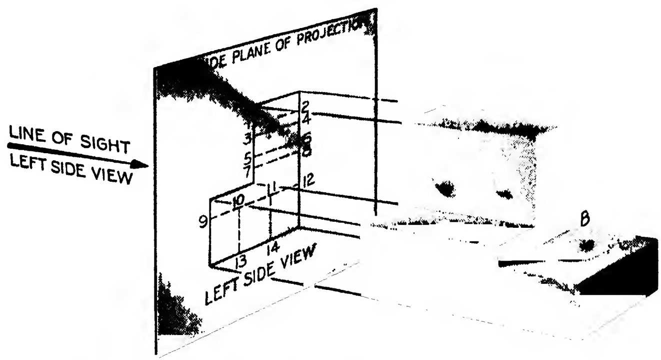


FIG 106. HIDDEN EDGES AND CONTOURS

be visible to the observer for the left side view, and therefore its projection 9–12 is represented as a hidden line. Likewise, the contours of hole C are invisible, and are projected as hidden lines 10–13 and 11–14. (Note the T's formed by the dashes at 10 and 11.) Hole D is projected as hidden lines 1–2 and 7–8; hole E is projected as hidden lines 3–4 and 5–6. You should similarly visualize the projection of each of the other hidden lines in the top and side views of Fig. 105.

Unfolding the glass box

This unique system of projection has been devised to represent the mutually perpendicular views of a three-dimensional object on a single plane, such as a sheet of drawing paper. To accomplish this the glass box is unfolded until all planes coincide with the front plane of projection, as in Fig. 107. The top, bottom, and side planes are "hinged" to the front plane, which remains stationary, and the back plane is hinged to the left side plane. The edges about which the planes are revolved are called *axes* or *folding lines.*

Projection lines The lines extending around the glass box from one view to another are the *projections of the perpendiculars* from points on the object to the planes of projection. For example, the projector A–7 (Fig. 107) is projected on the top plane at 3–5 and on the side plane at 11–8. Since 3–5 and 11–8 are both equal in length to the projector A–7, the side view is seen to be the same distance from

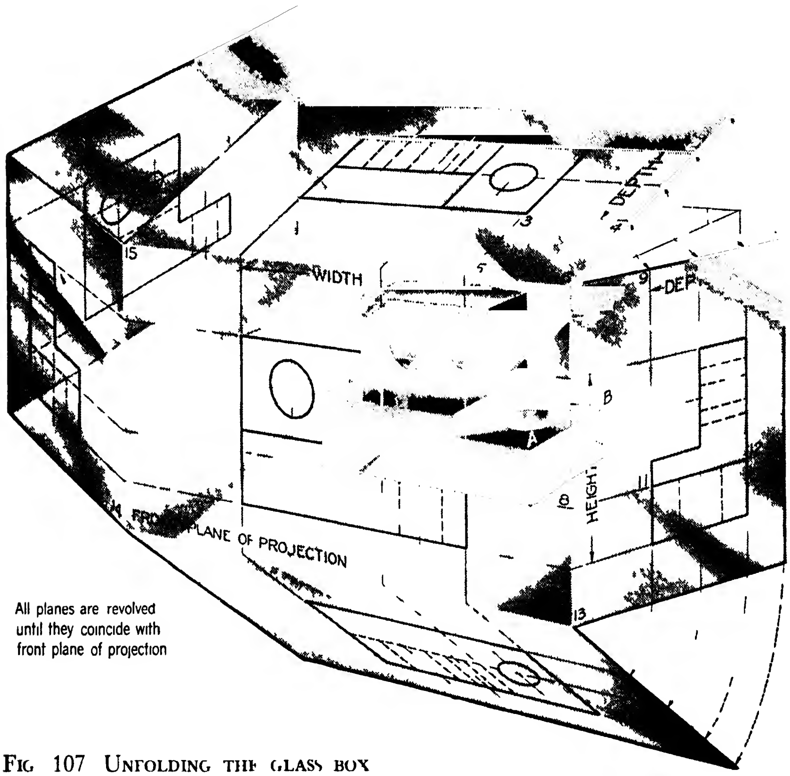


FIG. 107 UNFOLDING THE GLASS BOX

the front plane as is the top view. Likewise distance 6–4 equals 6–9 and 4–2 equals 9–10, which is to say that depth in the top view is the same as depth in the side view.

Positions of views The positions of the six planes, after they have been revolved into one plane, are shown in Fig. 108. The top view is always directly over the front view, the bottom view is directly under the front view, and the width of the object is shown in all three. The right side view is always directly to the right of the front view. The left side view is directly to the left of the front view. The back view is directly to the left of the left side view, and the

height is shown in all four views. The depth is shown in the four views that surround the front view.

It is now evident (Fig. 108) that the separation of the top and right side planes produces arcs 4–9 and 2–10 which could be used on a drawing to transfer the dimension from the top to the side view and vice versa. This same relation occurs at all four corners of the front plane of projection (Fig. 108).

Construction lines

The draftsman must thoroughly understand the principles of projection, but he need not and should not show the planes of

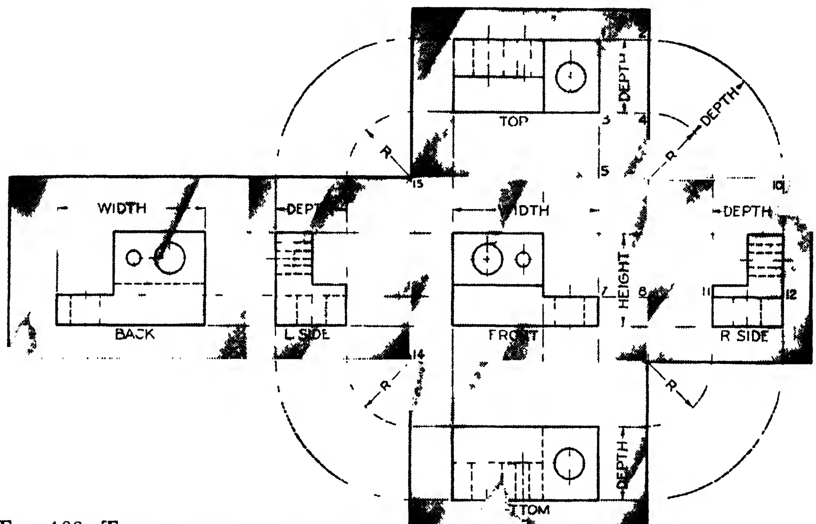


FIG 108. THE GLASS BOX UNFOLDED

projection on a drawing. The six views, with the planes omitted, and drawn sufficiently near each other to look well, are shown in Fig. 109. The arcs of projection are omitted and the simpler 45° construction lines are substituted to transfer the depth dimension between the four views adjacent to the front view. Many draftsmen omit even this construction, preferring to transfer the depth dimensions with the dividers or scale.

Spacing the views

Although the views must always "line up" as shown in Fig. 109, the distances between views are adjusted by the draftsman. *The distance between any two adjacent views is independent of the distance between other adjacent views.* For example, the distance between the front and top views is not necessarily equal to the distance between the front and right side views. However, a balanced appearance is highly desirable.

Relation of surfaces

In Fig. 109 it should also be observed that the back face of the object appears in the top view at 1–2, in the right side view at 3–4, in the bottom view at 6–5, and in the left side view at 8–7; in all four cases, on the side farthest from the front view. Conversely, the front face of the object appears in the top view at 9–10, in the

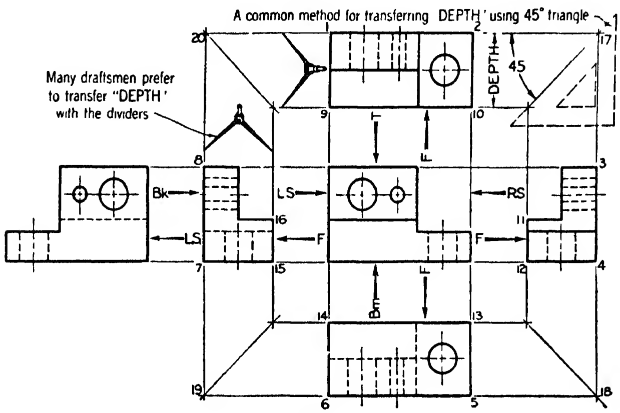


Fig. 109. The six views on paper

right side view at 11–12, in the bottom view at 14–13, and in the left side view at 16–15; in all four cases, on the side nearest to the front view.

Reciprocal views

Notice also in Fig. 109 that *adjacent views are reciprocal*. For example, if the front view is visualized as the object itself, the right side view is obtained by looking toward the right side of the front view in the direction of the arrow *RS*, which is the direction of sight to obtain the right side view. Likewise, if the right side view is visualized as the object itself, the front view is obtained by looking toward the left side of the right side view in the direction of the arrow *F*.

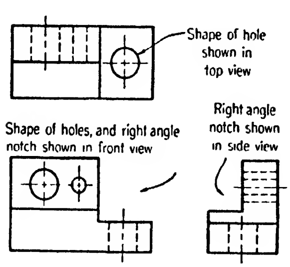


Fig. 110. The three views

Needed views

Clearly, all of the six possible views are not needed to describe completely the shape of the object. The draftsman therefore draws only the *necessary views* which are those that together provide the clearest description of the shape of the object. The three *standard views* of the "Holder for Timing Roller" are shown in Fig. 110, and the features shown in each view are indicated. If a not-too-strict interpretation

of the rule of necessary views is followed, as is often the case in commercial drafting, we might omit the right side view, or the top view, on the assumption that shapes not shown completely are understood to be rectangular in form.

Possible side views

However, you should note carefully that *many different side views could be drawn* to accompany rectangular top and front views, as in Fig. 111. If only two views of a simple rectangular block are

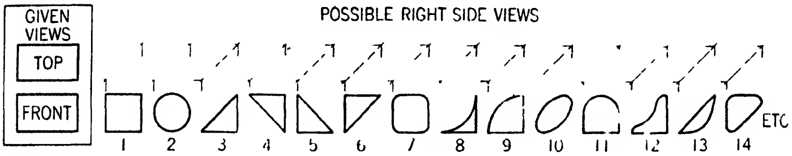


Fig. 111. Possible side views

drawn, it is understood that the object is rectangular and not any of the odd shapes shown in this illustration. It becomes, therefore, a matter of judgment on your part as to the number of views that should be drawn for each given object, and the only rule to follow is to include those views which in your opinion show the shape of the object most clearly.

Other types of views

Complicated shapes may require more than three views, and in many cases special views, such as *partial views* (Fig. 206), *auxiliary views* (Fig. 160), or *sections* (Fig. 139). Always select those views

Space saving arrangement

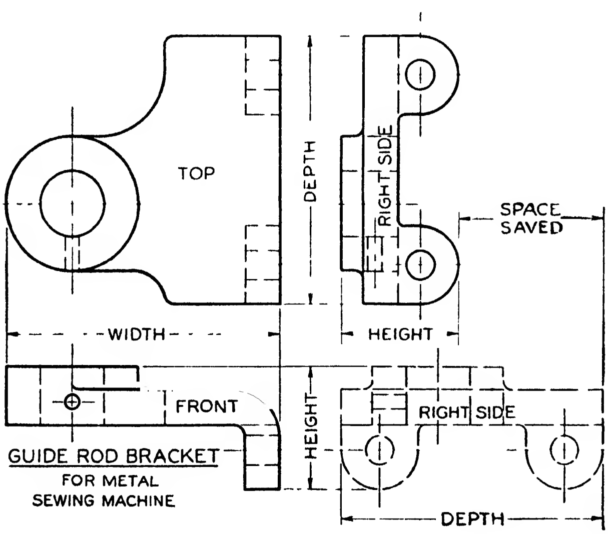


Fig. 112. Side view beside the top view

which show essential shapes or contours, and give preference to those with the least number of hidden lines. In Fig. 110 the right side view and the top view were chosen in preference to the left side and bottom views because of the numerous hidden lines in the latter.

In drawings of flat wide objects, such as the "Guide Rod Bracket" in Fig. 112, it is preferable to place the side view opposite the top view instead of opposite the front view in order to use the space on the paper to better advantage. Here the side plane is considered hinged to the top plane of projection instead of to the front plane of projection.

SURFACES AND EDGES

In studying multiview projection we find it useful to classify objects according to their various types of surfaces and edges and to examine each in turn.

Normal surfaces

A normal surface is a plane which is parallel to one of the planes of projection. It will appear in true size and shape on the plane to which it is parallel, and as a line (i.e., edgewise) on adjacent planes of projection. For purposes of analysis, we may consider the "breakdown," or steps in machining the block of cold-rolled steel, shown in Fig. 113, in which all surfaces are normal surfaces. In *space I*, normal surface *A* is parallel to the top plane and appears true size in the top view at 2–3–7–6, as the line 9–10 in the front view, and as the line 17–18 in the side view. Normal surface *B* is parallel to the side plane and appears true size in the side view at 17–18–20–19, as the line 3–7 in the top view, and as the line 10–13 in the front view. Normal surface *C* is parallel to the front plane and appears true size in the front view at 9–10–13–14–16–15–11–12, as the line 5–8 in the top view, and as the line 17–21 in the side view. All other surfaces in the block may be similarly visualized, and you should identify each individually. Observe carefully the changes of shape in the views produced by each machining operation, including the introduction of hidden lines, new visible edges, and the "dropping out" of certain lines as the result of a new cut.

Surfaces in projection

In the top view of *space I*, normal surfaces 1–2–6–5, 2–3–7–6, and 3–4–8–7 appear as adjacent rectangles. Lines 2–6 and 3–7 indicate that there must be three surfaces, but we must examine these surfaces in the front view also to determine the height of each. In the front view, surface 9–10 is seen to be the highest, and

surfaces 11–12 and 13–14 are at the same lower level. In the side view these two latter surfaces appear as *one line* 19–20. Surface 11–12 might appear as a hidden line 19–20 in the side view, but surface 13–14 appears as a visible line 19–20, which covers the hidden line and takes precedence over it.

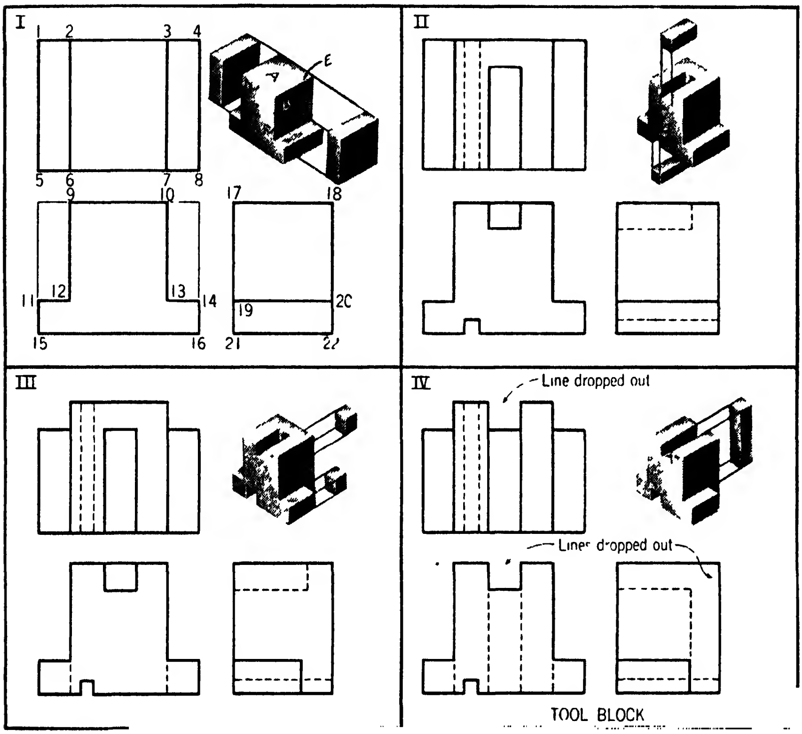


FIG. 113. MACHINING A "TOOL BLOCK" — NORMAL SURFACES AND EDGES

A normal edge is a line of intersection of two plane surfaces which is perpendicular to a plane of projection. It will appear as a point on the plane of projection to which it is perpendicular and as a line in true length on adjacent planes of projection. In Fig. 113 *I*, edge *D* is perpendicular to the side plane of projection and is shown as the point 17 in the side view. It is parallel to the front and top planes of projection and is shown true length at 9–10 in the front view and as 6–7 in the top view. Edges *E* and *F* are perpendicular respectively to the front and top planes of projection, and their views may be similarly analyzed. *Normal edges*

An inclined surface is a plane which is perpendicular to one plane of projection but not parallel to any plane of projection. It will show as a *Inclined surfaces*

Steps in machining inclined surfaces

line on the plane to which it is perpendicular and will not show true size in any view. Inclined surfaces are represented by the steps in "Machining a Locating Finger" shown in Fig. 114. In *space I*, inclined surface *A* is perpendicular to the top plane of projection and is shown in the top view as the line 5–3. It is shown as a foreshortened surface in the front view at 7–8–11–10 and in the side view at 12–13–16–15.

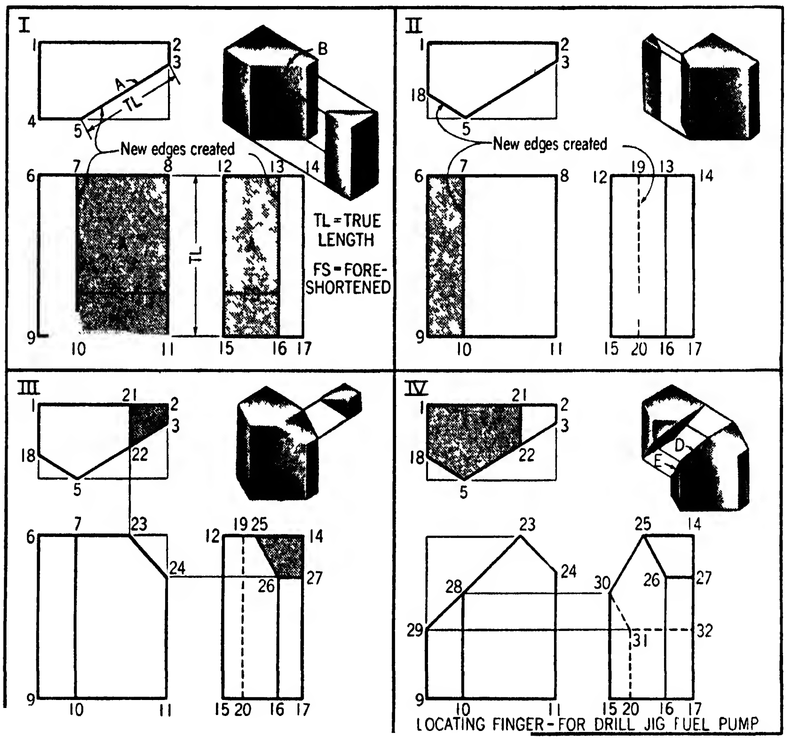


FIG 114 MACHINING A "LOCATING FINGER" — INCLINED SURFACES

Each successive cut in Fig. 114 is perpendicular to a plane of projection and produces an inclined surface, and each step should be carefully studied. In *space IV* the final cut produces an inclined surface perpendicular to the front plane of projection showing as the line 29–23 in the front view, as a foreshortened surface in the top view at 1–21–22–5–18, and as a foreshortened surface in the side view at 25–14–32–31–30.

Inclined edges

An inclined edge is a line of intersection of two plane surfaces which is parallel to one plane of projection and not parallel to the others. In Fig. 114 *I*, inclined edge *B* is parallel to the top plane of projection, and its

true length is shown in the top view at 5–3. It is shown foreshortened in the front view at 7–8 and in the side view at 12–13. Each succeeding cut produces additional inclined lines, each of which you should analyze similarly.

Oblique edges

An oblique edge is a line of intersection of two plane surfaces which is oblique to all planes of projection. Such edges may be produced by the intersection of inclined planes as in Fig. 114, but none appear until two inclined planes intersect as in *space III.* Here edge C is oblique to all planes of projection and is shown foreshortened

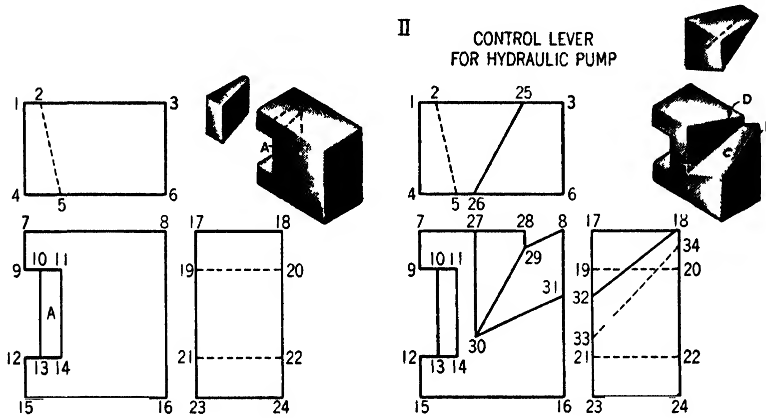


Fig. 115. Machining a "control lever" — inclined and oblique surfaces

in the top view at 22–3, in the front view at 23–24, and in the side view at 25–26. In *space IV* two additional oblique lines D and E are produced, and you should identify each line in the three views.

Oblique surfaces

An oblique surface is a plane which is oblique to all planes of projection. It will show as a foreshortened surface (never as a line) in all views. In Fig. 115 an inclined surface and two normal surfaces are produced by the first cut in *space I*, but in *space II* an inclined surface B and an oblique surface C are produced by the V-shaped cut. The oblique surface C shows as a foreshortened surface in all views: in the top view at 25–3–6–26, in the front view at 29–8–31–30, and in the side view at 32–18–34–33. Edges D and E are inclined edges, and F is an oblique edge not showing in true length in any view.

Cylindrical surfaces

Cylindrical surfaces are very common, because of the ease with which they can be produced on the lathe, the drill press, and other

machines using the principle of rotation either of the "work" or of the cutting tool. These surfaces may be classified as *normal*, *inclined*, or *oblique* cylindrical surfaces depending upon the position of the center line of the cylindrical surface with respect to the planes of projection. Normal cylindrical surfaces are illustrated in Fig. 116. In *space I* of this figure the removal of the two upper

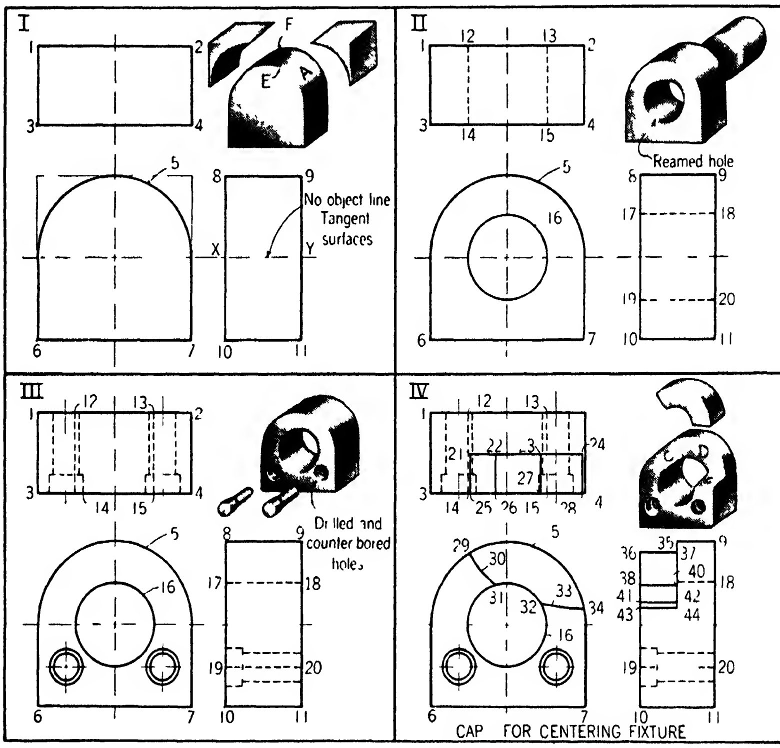


FIG. 116. MACHINING A "CAP" — CYLINDRICAL SURFACES

corners produces the cylindrical surface *A* which appears in the top view as the surface 1–2–4–3, in the front view as arc 5, and in the side view as the surface 8–9–Y–X.

Holes

In *space II* a reamed hole shows in the front view as circle 16, in the top view as surface 12–13–15–14, and in the side view as surface 17-18–20–19. In *space III* two drilled and counterbored holes are added, producing four cylindrical surfaces and two normal surfaces. The normal surfaces are the bottoms of the counterbores. In *space IV* a cylindrical cut is added, producing two cylindrical surfaces which show in the front view as arcs 30

and 33, in the top view as surfaces 21–22–26–25 and 23–24–28–27, and in the side view as surfaces 36–37–40–38 and 41–42–44–43. In addition, normal surface *D* is produced, showing in the front view as surface 29–34–32–31, in the top view as line 21–24, and in the side view as line 35–44.

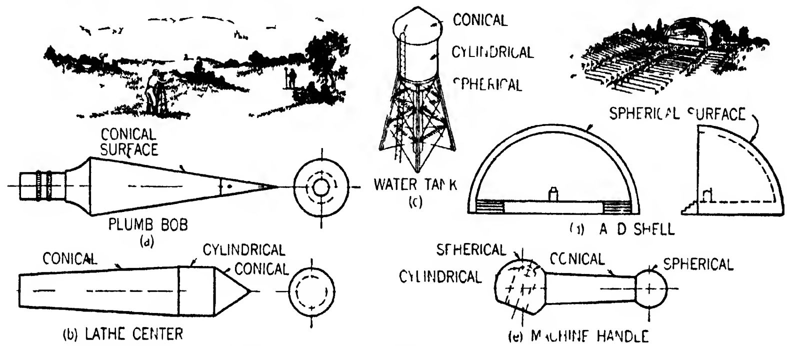


FIG. 117. CURVED SURFACES IN ENGINEERING

Other curved surfaces

Circles and circular arcs are also produced by cylindrical cuts. In *space I* of Fig. 116 arcs *E* and *F* (see pictorial), are shown in the top view at 1–2 and 3–4 respectively, in the front view at 5, and in the side view at 8–X and 9–Y respectively. Other common geometric shapes encountered in engineering include the cone and the sphere illustrated in Fig. 117.

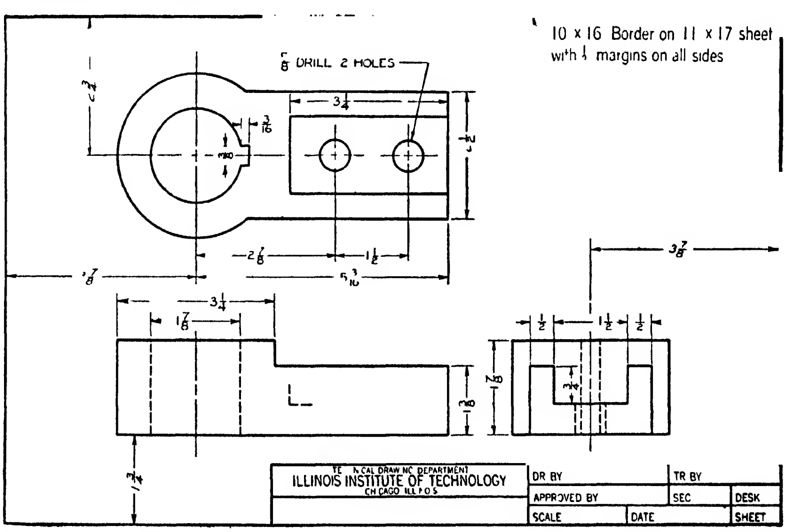


FIG. 118. CONTROL LEVER

Complete the detail drawing, omitting spacing dimensions only.

Problems

In Fig. 118 and Fig. 119 are shown two layouts to be completed. In Fig. 118 add any missing lines needed to complete the views. Omit spacing dimensions (inclined numerals) and the note in the upper right hand corner, but include the dimensions on the views. In Fig. 119 the views are almost completely blocked in with construction lines. Complete this drawing, omitting the pictorial drawing and the spacing dimensions. Be sure to construct all tangent points accurately. (See Fig. 83.) Do not dimension this drawing until you have studied the section on dimensioning, page 214.

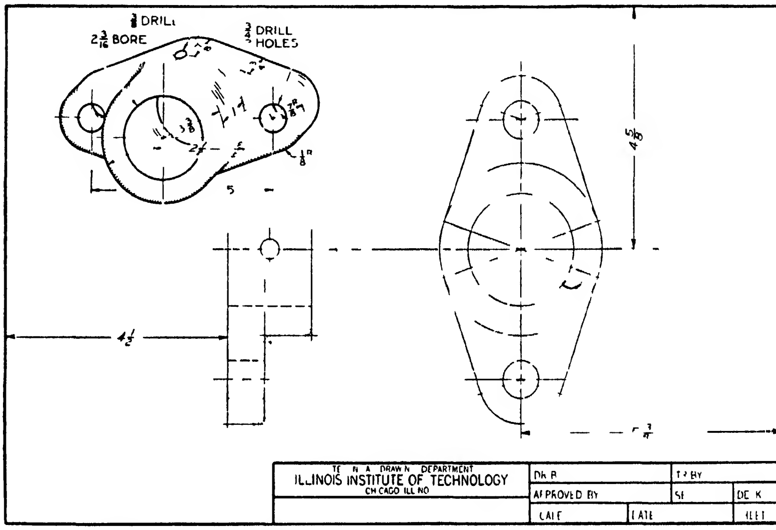


FIG. 119. BEARING

Complete the detail drawing, omitting the pictorial and the spacing dimensions.

Machine problems

A wide selection of practical machine problems is shown in Figs. 120 to 137 inclusive. For each problem it is best first to make a freehand sketch of the necessary views on cross-section paper. Let each square on the paper represent some dimension so that your sketch will be to scale. Next, obtain your instructor's approval of your choice of views; then make a mechanical drawing, using the same sheet layout as shown in Fig. 118. Do not add dimensions until you have studied dimensioning, page 214.

SECTIONING

Natural sections

Sectioning an object is a procedure with which everyone has had experience, although he may not have realized it. We have all viewed with horror in the carnival side show the sawing of a woman in half. We have bitten into an apple and have thus made a section through a wormhole. We have wondered when purchasing a watermelon how it looks inside, and we then satisfied our curiosity about the interior structure of this luscious fruit by making a *cross section* through the melon with a long-bladed knife.

the front view the cutting plane appears as a line, called a *cutting-plane line*. The arrows indicate the direction of sight for the sectional view. This type of section is a *full section*, so called because the cutting plane passes entirely through the object.

Section lining

In drawing a full section you will see that the sectioned areas are those portions that have actually been in contact with the cutting plane, and these are crosshatched with very sharp parallel section lines spaced by eye to look well. The lines are spaced from about $\frac{1}{16}''$ apart to about $\frac{1}{8}''$ apart depending upon the size of the drawing, and are usually drawn at 45° with the horizontal.

Visible lines

However, in a sectional view we also show the visible lines behind the cutting plane, as in Fig. 140. The disconnected sections (Fig. 140 *a*) must be connected by adding the lines representing the vertical ribs, the rear rims of the holes, and other visible lines

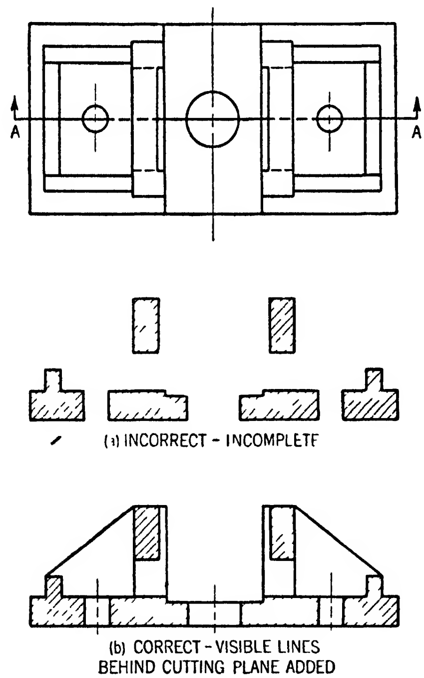


FIG. 140. VISIBLE LINES BEHIND THE CUTTING PLANE

back of the cutting plane as shown at *b*. Note that all section lines run in the same direction on any section or series of sections of a single object.

Sometimes, wishing to retain a portion of the exterior in the

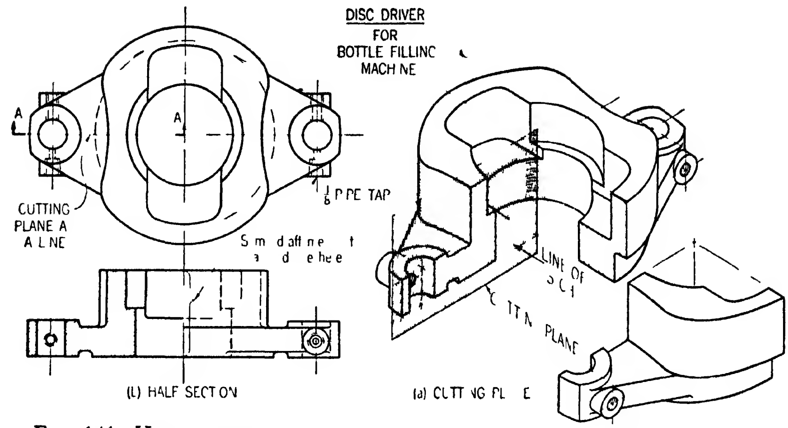


FIG 141 HALF SECTION

Half sections sectional view, we section only a part of the object A *half section*, so called because the cutting plane passes only halfway through the object, is shown in Fig 141 Another method of sectioning only

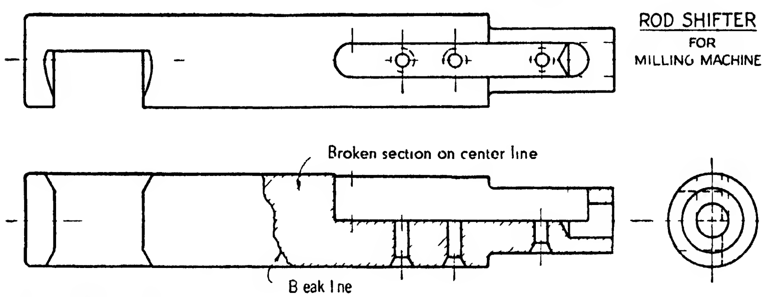


FIG 142 BROKEN SECTION

part of a view is to introduce a *break line* to separate the sectioned area from the rest of the view, this produces a *broken section* (Fig 142)

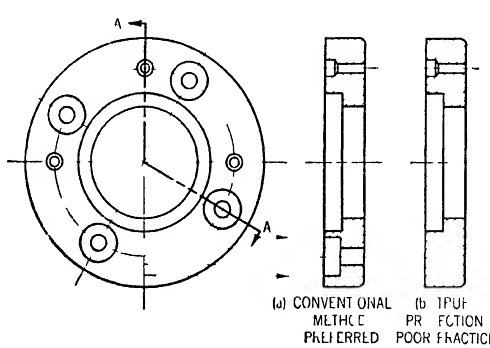


FIG 143 CONVENTIONAL VIOLATION OF PROJECTION

Violations of projections

In practice it has become customary to violate the rules of projection in order to obtain a clearer drawing. For example, in Fig. 143 *b* the true sectional view does not give an accurate idea of the symmetry of the arrangement of holes about the center.

Webs and similar thin parts not sectioned

Revolving features into the section

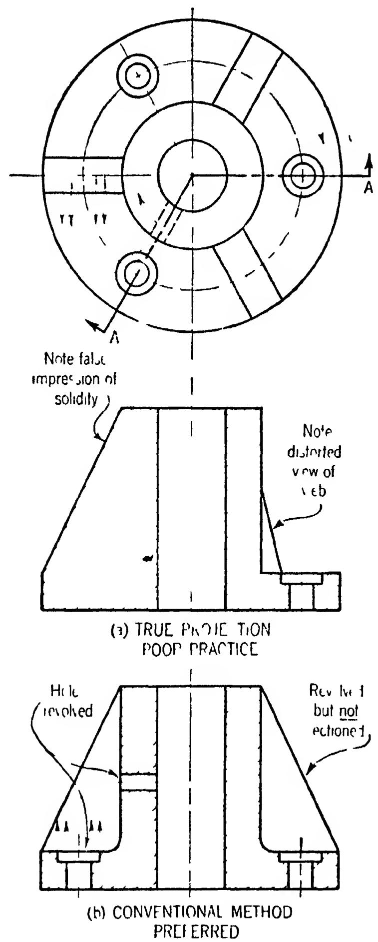


Fig. 144. Webs in section

In order to show one of the large counterbored holes in the section, the hole in the front view is conceived to be revolved into the vertical center line so that it may appear in the section, as shown in Fig. 143 *a*. The cutting plane line makes this clear.

Another common violation of projection relates to webs in section. In Fig. 144 *a* the web on the left, shown in section, gives a false impression of mass. Furthermore, the rear web on the right

Webs in section

is seen obliquely, resulting in a very awkward and misleading representation. The preferred section is shown in Fig. 144 *b*, in

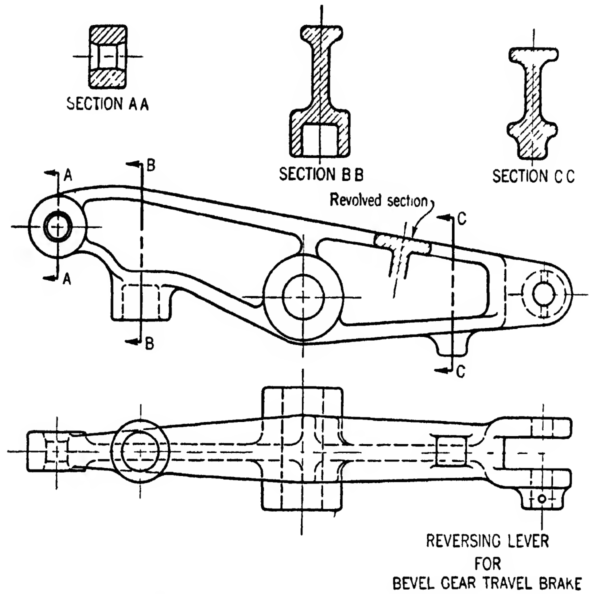


FIG. 145. DETAIL SECTIONS

which the web on the right has been revolved, two holes on the left have been revolved, and neither of the webs is crosshatched.

Representing construction materials

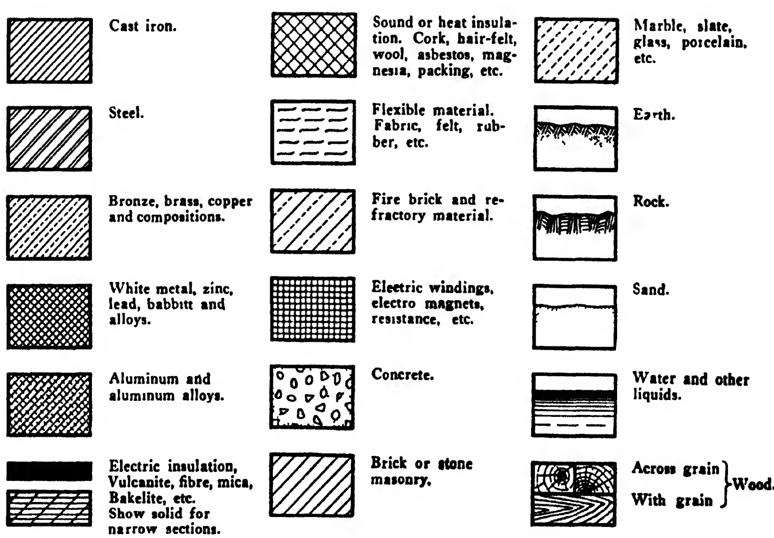


FIG. 146. AMERICAN STANDARD SECTION LINING

Sometimes it is not possible or convenient to show sections in the regular views. For example, see Fig. 145 where the sections A–A, B–B, and C–C could not be superimposed upon one another in a right side view. Instead, *detail sections* are placed at convenient spaces on the drawing and identified with their several cutting

Detail sections

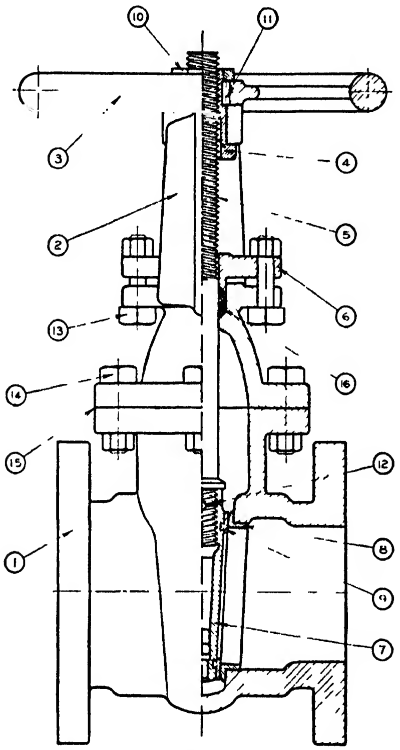


FIG. 147. SECTIONAL VIEW OF GATE VALVE ASSEMBLY — A HALF SECTION

planes by means of notes. Another method, little used in practice, is to revolve the section in place, as shown near the right-hand end of the lever in Fig. 145. It will be observed that none of the detail sections shown could have been *revolved sections* without covering important lines of the drawing.

Section lining

Often on detail drawings (drawings of single pieces) the section lining shown on the preceding illustrations, which is the standard symbol for cast iron, is used regardless of the actual material. Material specifications are usually given in a more detailed manner by means of notes. However, if desired, the general type of material can be indicated by different symbols, as shown in Fig. 146.

Assembly sections

A half section in an assembly drawing is shown in Fig. 147. Notice that section lines for different adjacent parts run in opposite directions, and that the different symbols for the several kinds of material are shown. Some violations of projection are

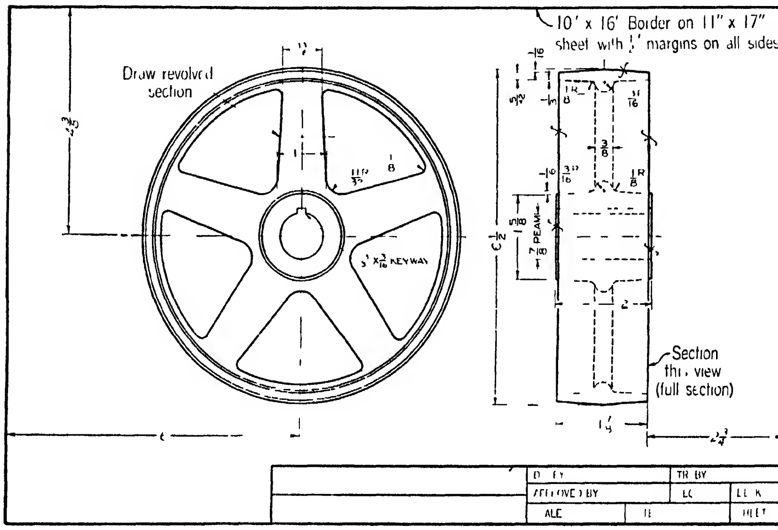


FIG. 148. PULLEY

Complete the drawing, omitting spacing dimensions and instructional notes.

illustrated with respect to the bolts, nuts, pins, threaded shaft, and similar forms which are not sectioned. These parts have no interior details worth showing and are therefore left in exterior view.

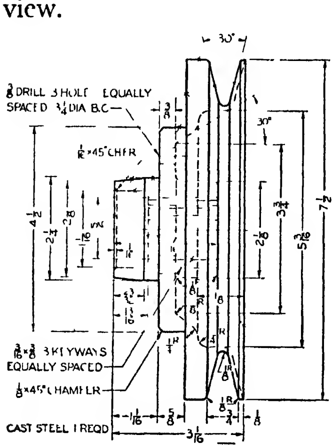


FIG. 149. SHEAVE

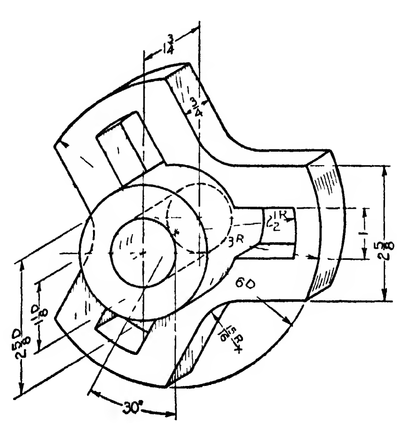


FIG. 150. ROTARY STOP

Problems

In Fig. 148 is given a practical problem in which the sheet layout of Fig. 118 is used again. Draw the views as indicated, but omit all spacing dimensions (inclined numerals) and additional notes. Add dimensions to the views only after you have studied the section on dimensioning. See p. 214.

The problems in Figs. 149–156 inclusive are selected to include a variety of sections. Choose the necessary views and sections needed and make a sketch of these on cross-section paper. Then proceed to make mechanical drawings on

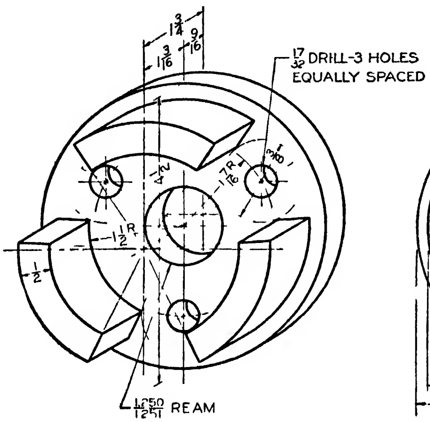


Fig. 151. Cup washer

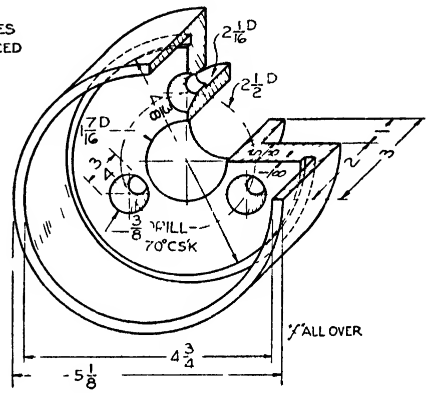


Fig. 152. Special bearing

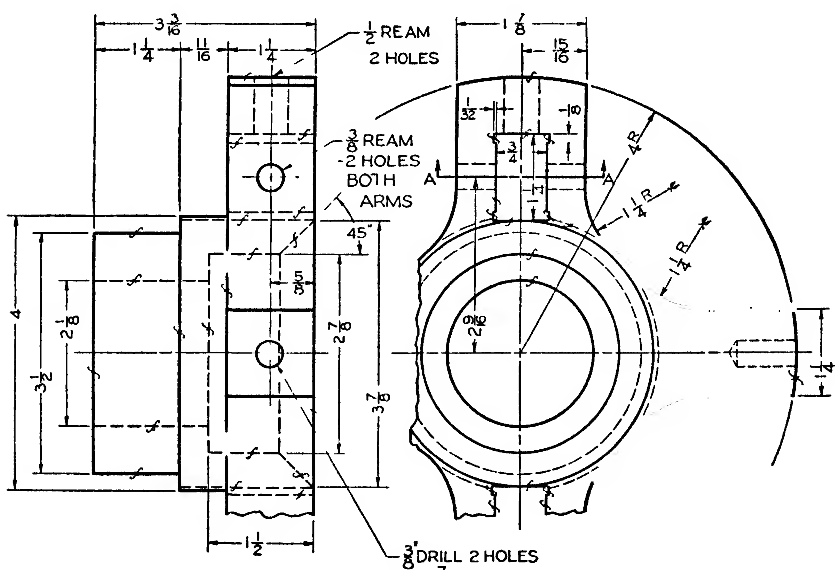


Fig. 153. Web for lathe clutch

Given: Partial front and left side views. Required: Complete front view and right side view in full section, and draw detail section A–A.

detail paper or vellum, using the sheet layout of Fig. 148. Do not dimension the drawing until you have thoroughly studied dimensioning.

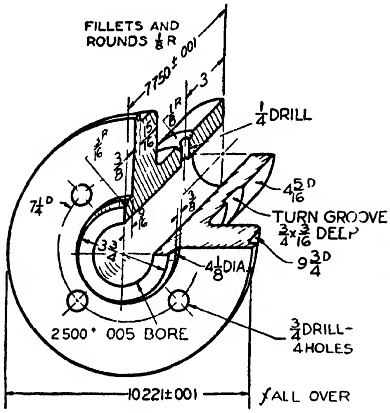


Fig. 154. Flanged bearing

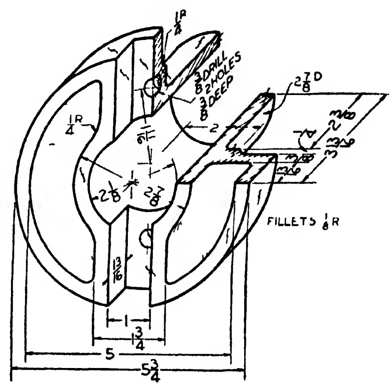


Fig 155 Stock guide

AUXILIARY VIEWS AND REVOLUTION

Regular views Up to this point we have considered the various *regular* views of an object of essentially rectangular shape. Usually two or three such views are sufficient to explain clearly the shape of the object, but occasionally this is not the case.

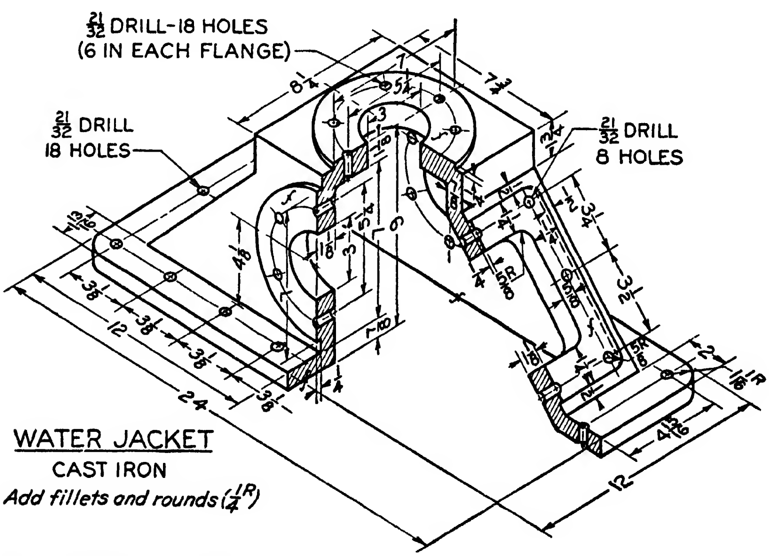


Fig. 156. Water jacket

For example, in Fig. 157 a clear description of the shape of the semicircular face is not shown in the regular views. In both the front and side views the circular arc appears as an ellipse because it is clear that the plane of the surface is not parallel to a plane of projection. In order to obtain a true view of such a surface it is necessary to view the object in a different direction, and this means that we must set up a different plane of projection — one which will be parallel to the inclined surface. A view thus obtained is called an *auxiliary view.*

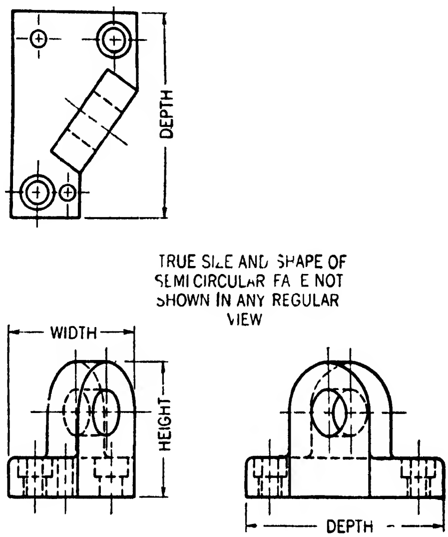


FIG. 157. INADEQUACY OF REGULAR VIEWS

If we imagine an auxiliary plane in our glass box (Fig. 158 *a*) which is parallel to the semicircular face of the object, such a plane will be perpendicular to the top plane of projection. Then if we revolve the auxiliary plane about a "hinge line" (line of intersection of auxiliary plane and top plane) until it coincides with the top plane, we will have two adjacent views of the object (*b*) obtained in a similar manner to the regular views. Observe that the true height of the object appears in the auxiliary view. *Auxiliary plane*

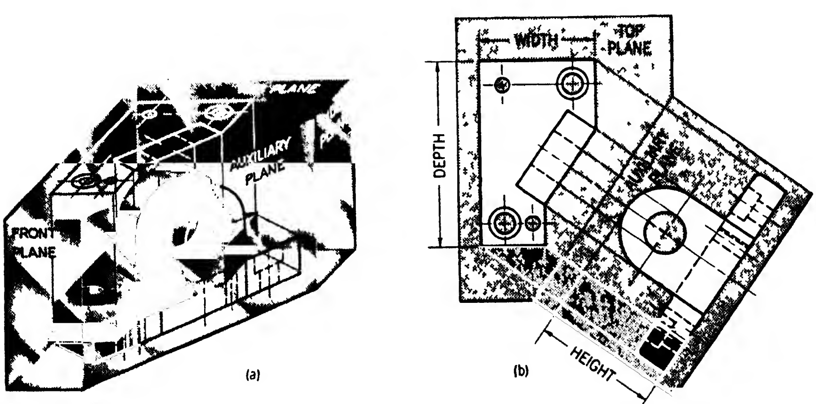


FIG. 158. THE AUXILIARY PLANE

Projection

We can now add this auxiliary view to our drawing as shown in Fig. 159. The length L in the auxiliary view is obtained by direct projection from the top view as indicated by the arrows.

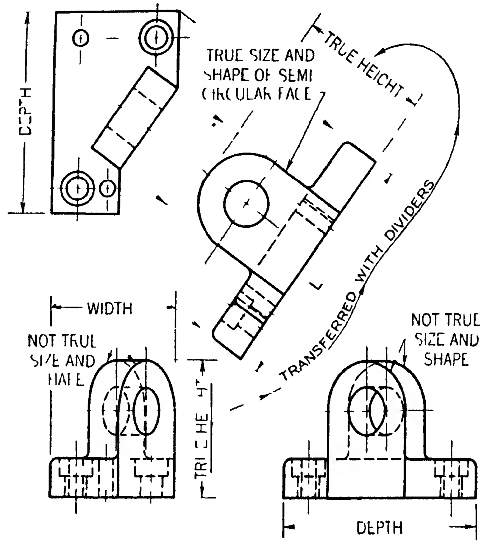


FIG. 159 AUXILIARY VIEWS ADDED

Any auxiliary view is projected from some other view, and one dimension of an auxiliary view can always be established thus by direct projection The other dimension in the auxiliary view

Saving time and effort

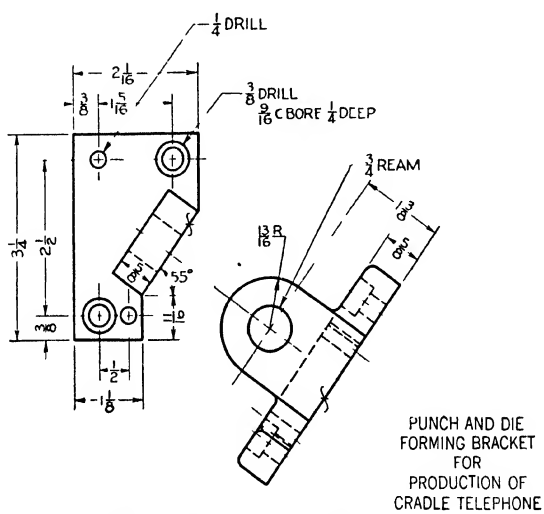


FIG 160 MINIMUM REQUIRED VIEWS

(height) was taken from the front view, but it might have been taken from any view which had been *projected from the top view*, the latter being the view we projected from to get our auxiliary view.

Minimum views

Your next question may well be: "If the front and side views are not useful in describing the object, why keep them?" In this you would be correct; these views now serve no purpose and should be omitted, as in Fig. 160. We now have a drawing which is complete because the three major dimensions *width*, *height*, and *depth* are shown, and because the shape of the object is completely described in these two views. That we have a complete drawing is proved by the fact that we can now place all dimensions on the views, and the object can then be built in the shop.

Revolution

Most people seem to try to make something difficult out of *revolution*. As a matter of fact it is very simple and is another step in our argument concerning auxiliary views. In the case of auxiliary views our given object has some feature which we do not see in its true size and shape through one of the regular planes of the glass box; so we merely set up an auxiliary plane parallel to the important feature and obtain a true view of that part. In revolution, instead of changing your position to suit the object, you revolve the object to suit your position. In either case you must look perpendicularly toward the surface in question. What difference does it make whether you move around to view this face perpendicularly, or you move the object around until you obtain the same view? In Fig. 161 the top view is simply redrawn in the "revolved" position so that the front view will show the true circular shape. Notice that the dimension (in this case — *height*) parallel to the imaginary axis of revolution is not changed.

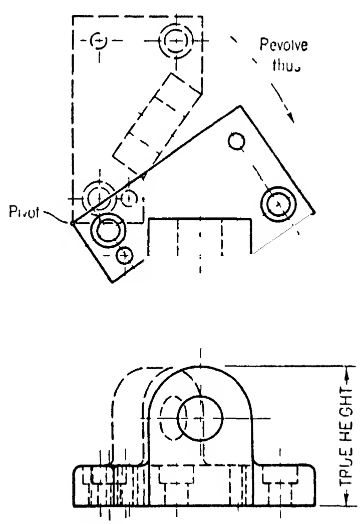


FIG. 161. REVOLUTION

Problems

Two typical auxiliary-view problems are given in Figs. 162 and 163. These can be drawn conveniently on a size 11″ × 17″ sheet as shown in Fig. 148. If you wish to dimension these problems, you should first study the section that follows on dimensioning.

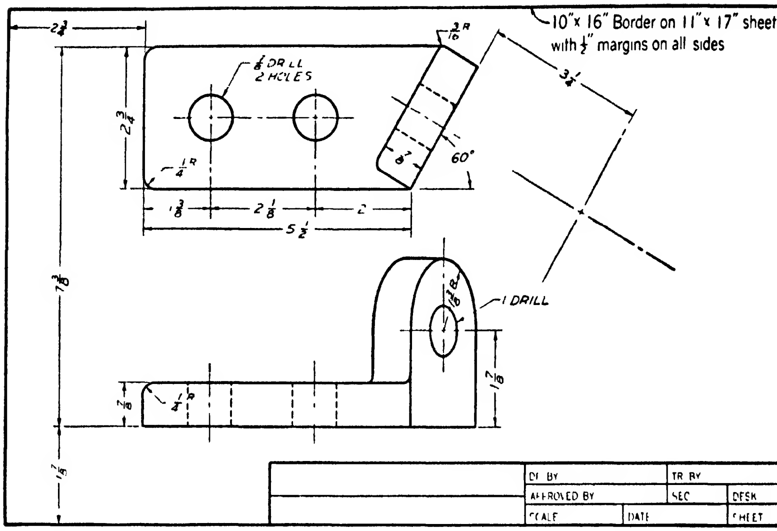


FIG. 162. ANGLE BRACKET

Complete the detail drawing, omitting spacing dimensions and note.

DIMENSIONING

Shape and size description

If a drawing is to be complete so that the object may be made from it exactly as intended by the designer, it must show not only a complete *shape description* but also a complete *size description*. It is the designing engineer's job to decide upon the shape of a new part, the material it is to be made of, and the kinds of fits involved

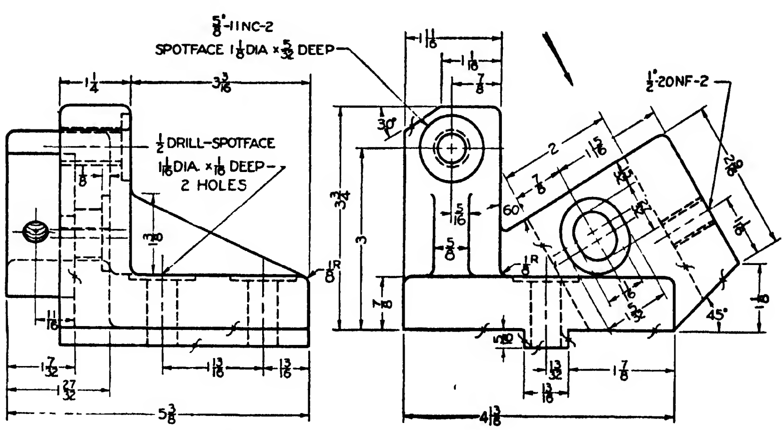


FIG. 163. SUPPORT BRACKET

Given: Front and left side views. Required: Front and right side views, and auxiliary view in direction of arrow.

with mating parts. The draftsman's job is to make the drawing so that it conveys this information unmistakably to the shop. The workman is not expected to exercise judgment in engineering matters, but only to carry out instructions on the drawings.

Accuracy

The shops are equipped with precision machine tools and measuring devices, and the draftsman can specify extreme degrees of accuracy merely by lettering the desired numbers on the drawing. Since this affects both cost and function, it is obvious that a great

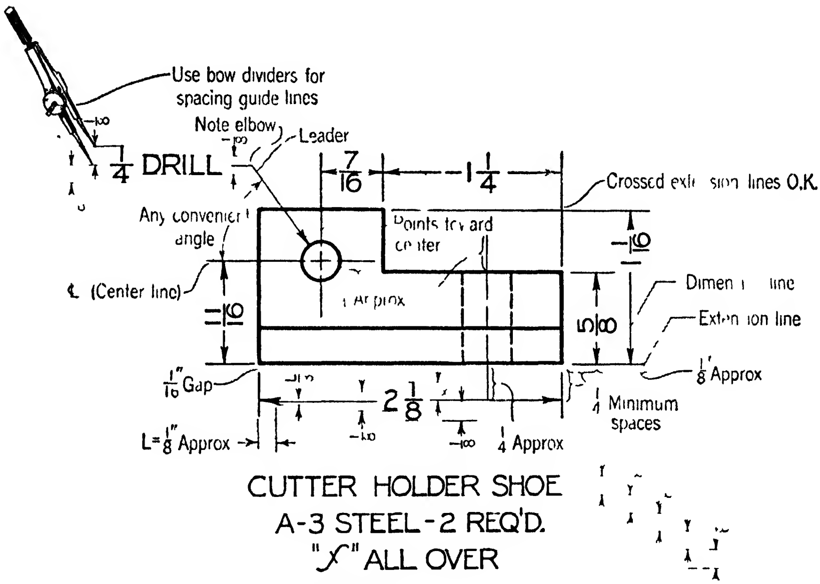


FIG. 164. APPLICATION OF DIMENSIONS

deal of thought must be given by the draftsman to dimensioning a drawing. Further, the dimensions must be such as to conform to the equipment in the shop and to the logical use of it by the workman. The draftsman, therefore, must be thoroughly familiar with the shop processes involved in the part to be dimensioned.

Terminology and spacing

The terminology and the spacing of dimensions are illustrated in Fig. 164. Rigid uniformity and care in spacing should be exercised to insure a pleasing and legible drawing. It is well to measure spacings at first and to use guide lines for all lettering, as shown. Later, dimensions may be lettered without guide lines, but notes should never be lettered without them. For the average dimension sufficient space is available (Fig. 165), but frequently limited spaces necessitate adjustments, as shown.

Arrowheads may be drawn with two strokes (Fig. 166 *a*) or filled in as shown at *b*, with preference for the latter.

Legibility of dimension figures requires special methods of spacing when dimensions are crowded

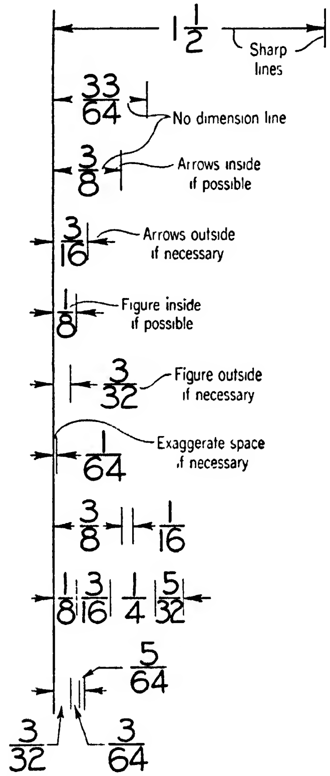


Fig. 165. Dimensions in limited spaces

Angles and radii

Angles are dimensioned in degrees, lettered horizontally, except for large arcs (Fig. 167). Radii are always designated by the letter R, as shown in Fig. 168, and small arcs require adjustments to assure clear, uncrowded numerals, as shown.

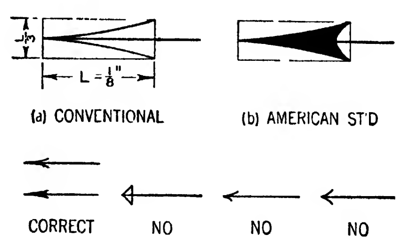


Fig. 166. Arrowheads

A surface on a casting or forging is said to be *rough* or *finished*, depending upon whether or not it has been machined (see Fig. 169).

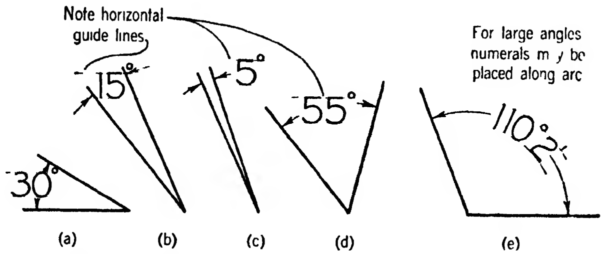


FIG. 167. ANGLES

If two rough surfaces intersect, the edge should be rounded, producing for an exterior edge a *round* and for an interior edge a

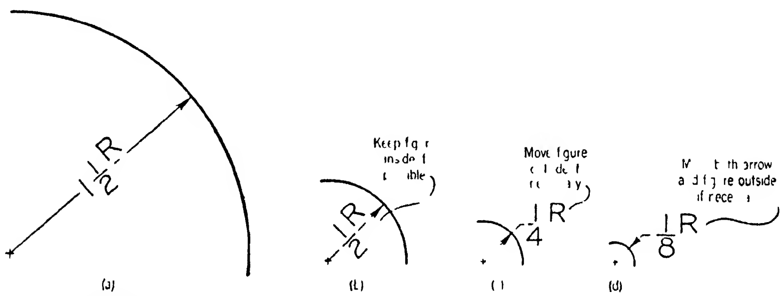


FIG. 168 RADII

fillet. These are intended by the designer to add strength to the edge or corner and to facilitate casting or forging. If either or both of two intersecting surfaces are finished, the edge will be sharp *Fillets and rounds*

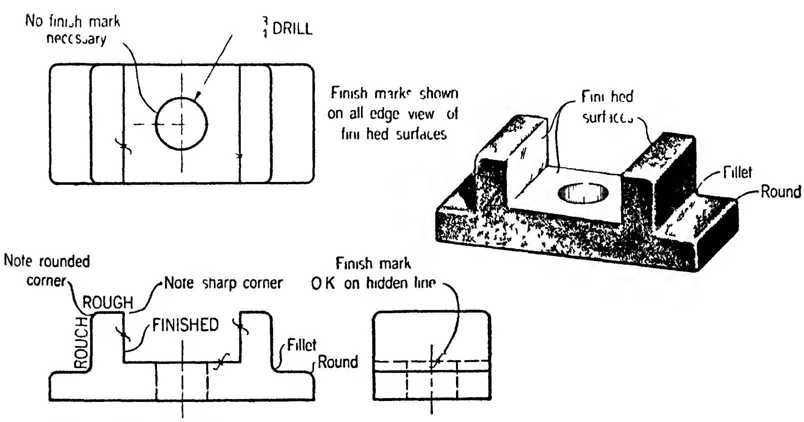


FIG. 169. FINISH MARKS

unless otherwise designated; if both are rough, the edge will be rounded or filleted.

Finish mark

The italic *f*, placed only on the edge views of finished surfaces (Fig. 169), is still the most nearly standard finish mark used in industry, despite all efforts to bring about an agreement for some form approaching the capital V.

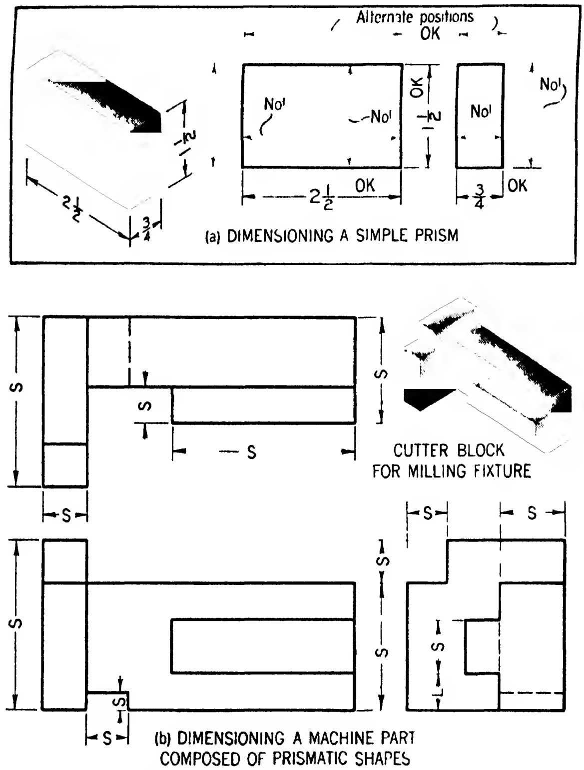


FIG. 170. DIMENSIONING PRISMS

Geometric shapes

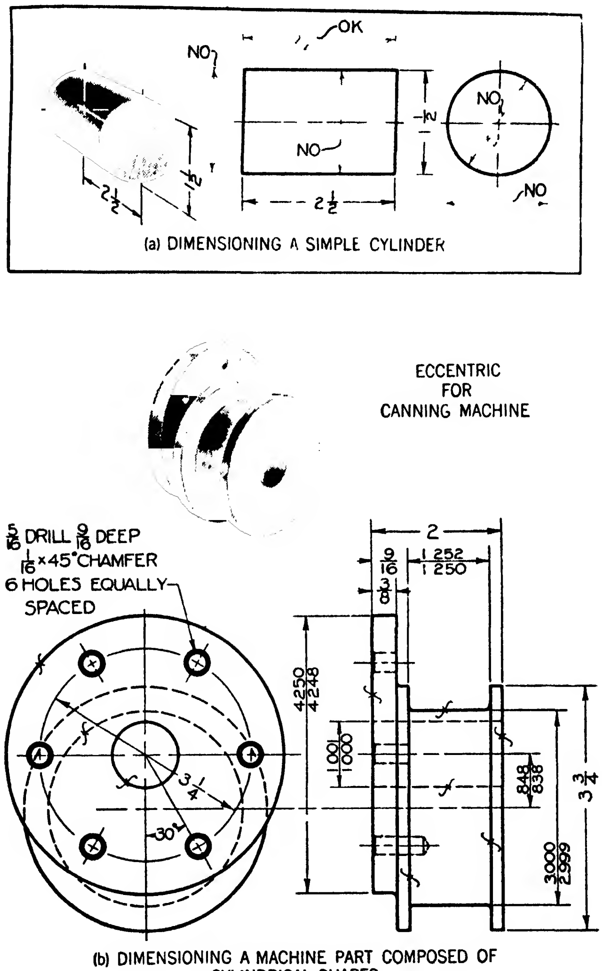
Any machine or engineering structure, when broken down into its elemental forms, is found to be composed largely of geometric shapes, such as the prism, cylinder, pyramid, cone, and sphere. In dimensioning a drawing we must first indicate the sizes of the elements with *size dimensions;* second, we must locate these elements with respect to each other with *location dimensions.*

Size dimensions

Three dimensions, for *width*, *height*, and *depth*, are necessary in dimensioning a prism (Fig. 170 *a*). An application to a machine part made up of prismatic shapes is shown at *b*. In general a dimension should be placed outside or between the views. Dimen-

-sion lines cutting across views are objectionable. Dimension numerals are lettered on guide lines parallel to the dimension lines and placed so that they are read either from the bottom or the right-hand side of the sheet.

Reading dimensions



(a) DIMENSIONING A SIMPLE CYLINDER

(b) DIMENSIONING A MACHINE PART COMPOSED OF CYLINDRICAL SHAPES

FIG. 171. DIMENSIONING CYLINDERS

The size-dimensioning of a cylinder and its application on a machine part are shown in Fig. 171. Cylindrical shapes are generally dimensioned with both the length and diameter in the rectangular view, as shown. Machined holes are usually designated by notes which specify the diameter and the operation required. Several of the most common types are illustrated in Fig. 172.

Machined holes

A cone is dimensioned by giving its altitude and diameter of base in the triangular view. A sphere is dimensioned by giving its diameter between two views.

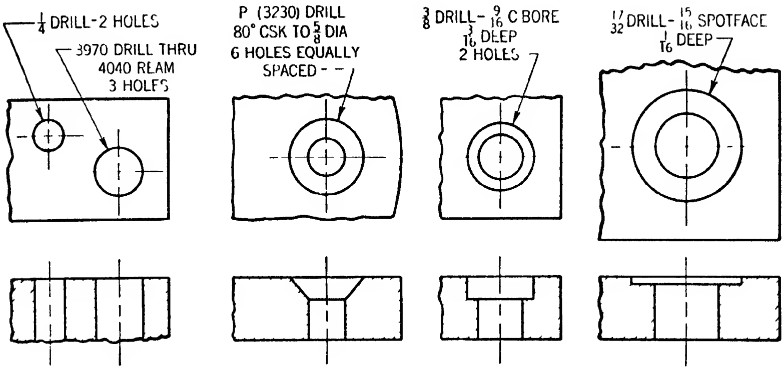
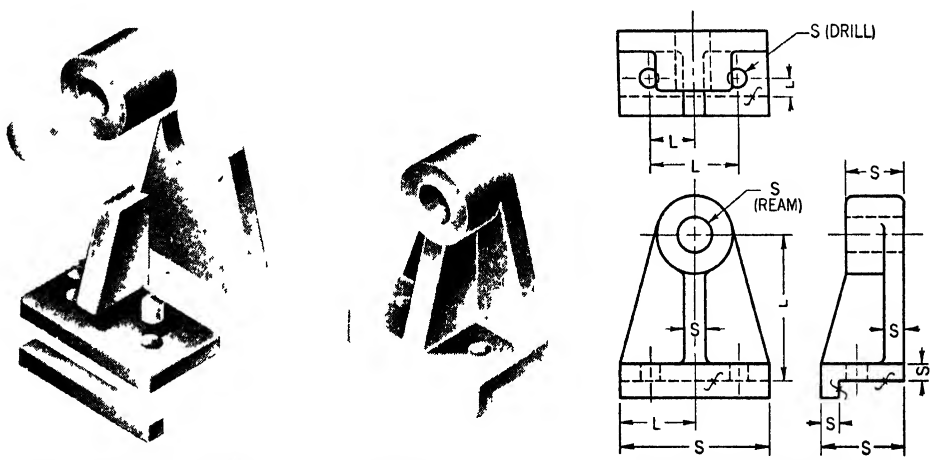


Fig. 172. Shop notes for machined holes

Location dimensions

A practical application of location and size dimensions on a drawing is shown in Fig. 173. Notice that all cylinders are located from their center lines and with respect to finished surfaces. Observe also that dimensions are attached to the view which most clearly identifies the form dimensioned; this is called *contour dimensioning*. Location dimensions for holes are accordingly given on the view where the holes show as circles. Dimensions should never be repeated on a drawing.



(a) THE GEOMETRICAL ELEMENTS (b) BEARING BRACKET (c) SIZE & LOCATION DIMENSIONS

Fig. 173. Size (s) and location (l) dimensions

Mating dimensions

A machine part cannot be intelligently dimensioned without consideration of its relation to mating parts. In Fig. 174 the dimension A must appear on both the drawings of the *bracket* and of the *frame*, and is therefore an important *mating dimension*. In Fig. 175, showing a redesign of this part, dimension A is not used as it is not necessary to control closely the distance between cap screws; but dimensions F are now important mating dimensions and must appear in both drawings. Other dimensions E, D, B, and C are not mating dimensions, since they do not directly affect the matching of parts (Figs. 174 and 175).

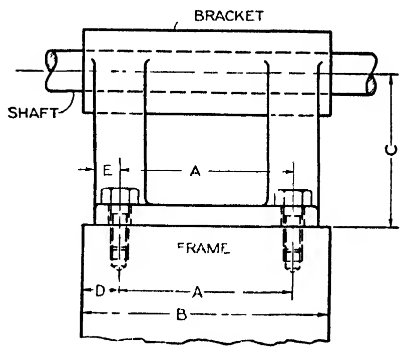


Fig. 174. Bracket assembly — single bracket

Fits and interchangeability

Where accurate fits are required, the old practice was to give the basic or nominal sizes of both mating parts in common whole numbers and fractions, and to specify in a note the types of *fits* desired, such as "running fit" or "drive fit." The workman in the shop was relied upon to make the parts so that they would fit properly. Today the workman is no longer expected to decide upon fits of mating parts. Mass production has introduced *interchangeable manufacturing* by which mating parts may be made in different factories and possibly assembled in a third factory. Thus it is necessary on drawings of closely fitting parts to specify sizes in such a manner that machine operators in widely separated shops can produce parts which are interchangeable.

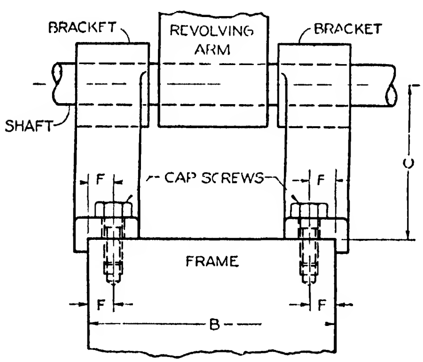


Fig. 175. Bracket assembly — double bracket

Problems

No formal problems in dimensioning are included here because you can apply the principles directly to the problems given at the end of the sections on shape description, page 198, and working drawings, page 243.

PICTORIAL DRAWING

The four kinds of projection are illustrated in Fig. 176, and all except the regular multiview projection (*a*) are pictorial types since they show several sides of the object at once.

Axonometric projection

In axonometric projection you consider yourself (as the observer) to be an infinite distance away from the object so that the visual rays from your eye to the object are theoretically parallel to each other and perpendicular to the plane of projection (*b*). The object may be tilted at any position with respect to the plane of projection, and any of these projections would be *axonometric projections*, provided the visual rays are kept perpendicular to the plane of projection.

Isometric projection

If the object happens to be placed so that lines *OA*, *OB*, and *OC* make equal angles with the plane of projection (Fig. 176 *b*), the

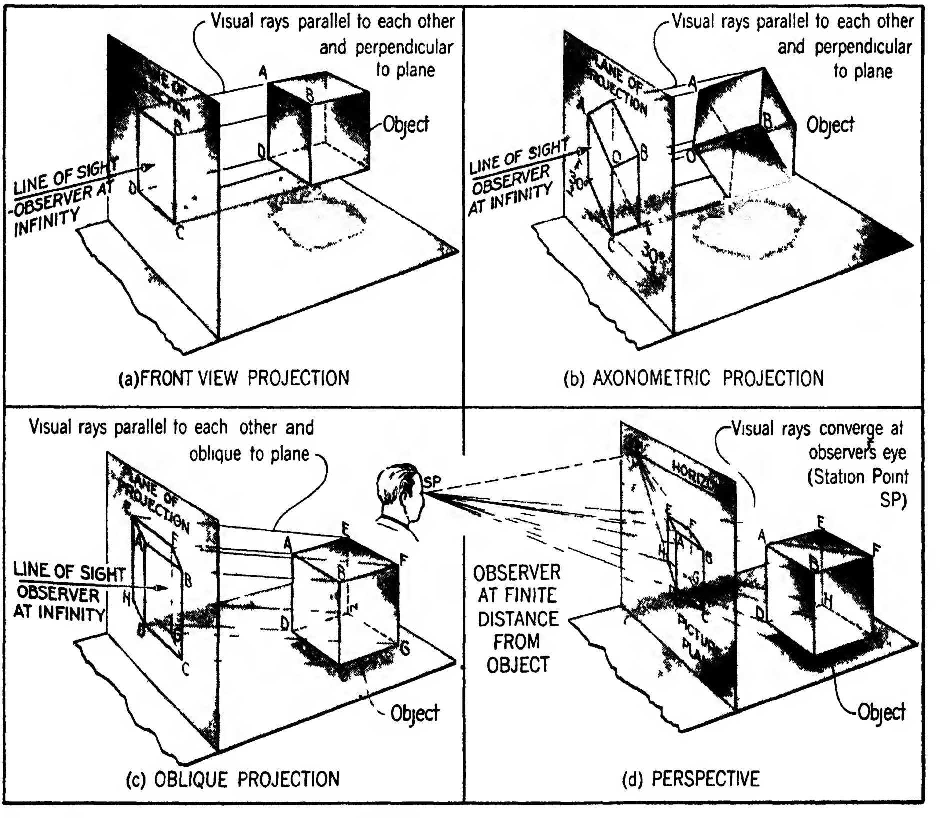


FIG 176 THE FOUR TYPES OF PROJECTION

result is an *isometric projection;* each of the lines is projected automatically to about 80 per cent of its actual length. An isometric projection may be obtained as in Fig. 177 *b* by revolving the object and then taking an auxiliary view. To accomplish this, the cube is revolved from its ordinary position in the top view (dotted)

Revolving the object

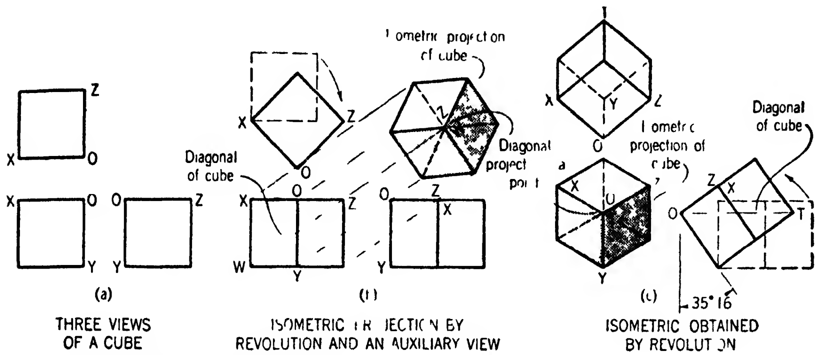


FIG. 177. ISOMETRIC PROJECTION

to the position shown. In this position the diagonal ZW through the cube would appear true length in the front view. An auxiliary view is then taken in which the observer looks in a direction parallel to the diagonal so that the diagonal is seen as a point in the auxiliary view. The cube then appears in true isometric in this auxiliary view. We have obtained an isometric projection by a combination of moving the object and moving the observer so that the two are properly related to produce an isometric projection.

Isometric projection

Instead of making the auxiliary view shown in Fig. 177 *b*, you could further revolve the object as shown in the side view at *c*. Here in the new side view position, the diagonal OT is horizontal and is shown in true length. The front view must show the diagonal as a point, and the front view is therefore a true isometric projection.

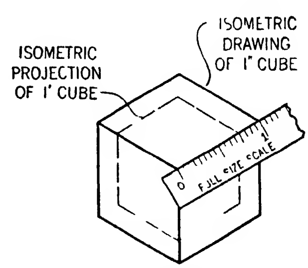


FIG. 178. COMPARISON OF ISOMETRIC DRAWING AND ISOMETRIC PROJECTION

Isometric drawing

To *draw* an *isometric projection* directly on paper, a special scale would have to be prepared to reduce the lengths of all lines about 20 per cent. However, in practice, the draftsman simply ignores this reduction, drawing all lines full length; and the result is called an *isometric drawing*, to distinguish it from the "true" *isometric pro-*

jection, which is about 20 per cent smaller. This difference between the "true," or theoretical, *projection* and the full-size isometric *drawing* is clearly shown in Fig. 178.

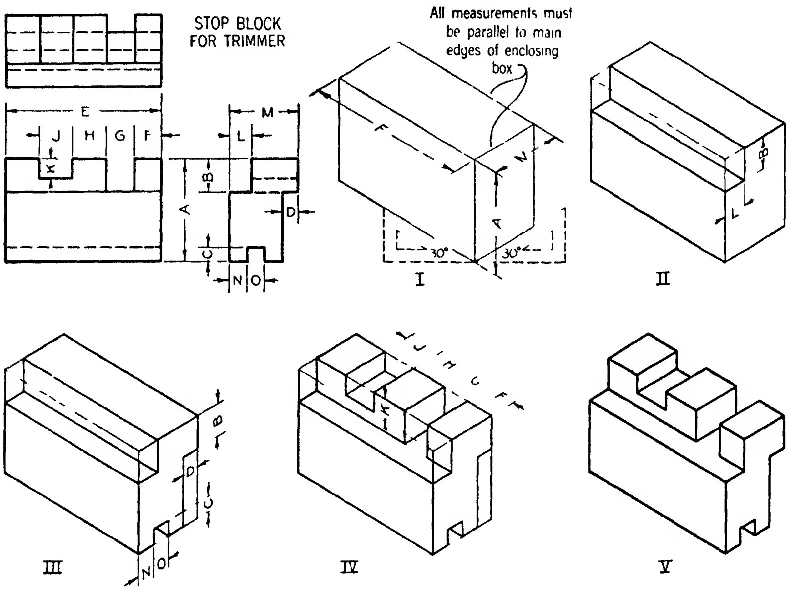


FIG. 179. ISOMETRIC DRAWING OF NORMAL SURFACES

The steps in constructing an isometric drawing of an object composed only of normal surfaces (see page 192) are illustrated in Fig. 179. Notice that all measurements are and must be made parallel to the main edges of the enclosing box. No measurement of a diagonal on any surface or through the object can be set off directly with the scale.

Inclined surfaces

The method of constructing an isometric drawing of an object composed partly of inclined surfaces (and oblique edges) is shown

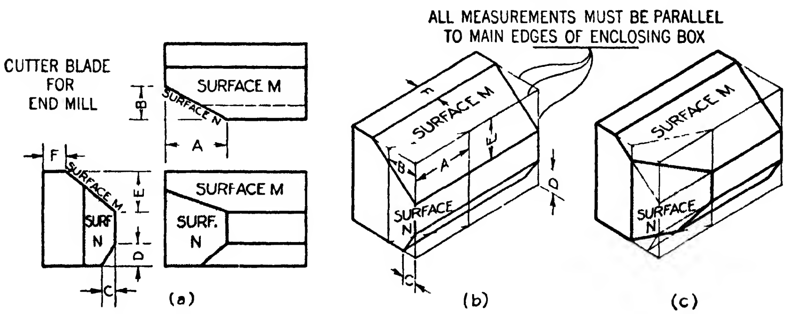


FIG. 180. INCLINED SURFACES IN ISOMETRIC

in Fig. 180. Notice that inclined surfaces are located by *offset measurements* along the main edges of the enclosing box. For example, dimensions E and F are set off to locate the inclined surface M, and dimensions A and B are used to fix the position of the surface N.

Circles in isometric A circle in isometric is an ellipse, due to the fact that the observer is not looking perpendicularly toward the plane containing the

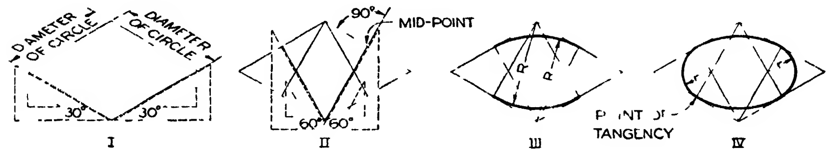


FIG. 181. STEPS IN DRAWING FOUR-CENTER OVAL

circle. The construction of the true projection of the circle is relatively difficult and time-consuming; therefore it is seldom used. The accepted approximate construction of the isometric ellipse is that of the *four-center oval* (Fig. 181). A typical example of the isometric ellipse is shown in Fig. 182. Notice that the centers for the large ellipses cannot be used for the smaller ellipses though the ellipses are concentric, and that the centers for the lower ellipses are obtained by projecting the upper centers down a distance C for each center.

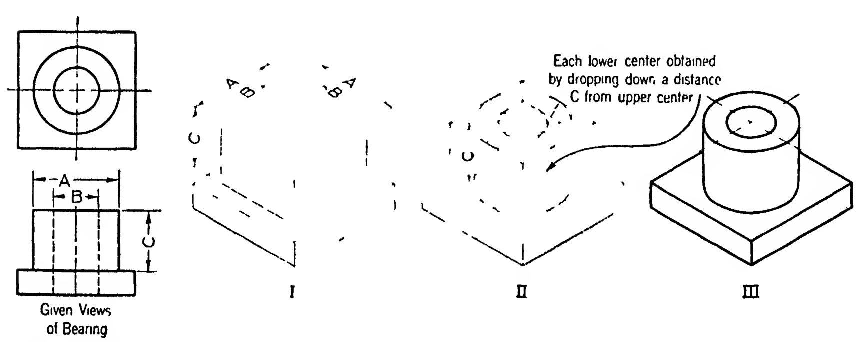


FIG. 182. ISOMETRIC DRAWING OF A BEARING

OBLIQUE PROJECTION

Oblique projection In oblique projection (Fig. 176 *c*) the observer is stationed at infinity, so that the visual rays are parallel to each other but oblique to the plane of projection. All surfaces of the object which are parallel to the plane of projection will be projected in their true

sizes and shapes on that plane. Thus the front surfaces *A–B–C–D* will be projected to true size and shape. The chief advantage of oblique projection over isometric projection is that any object having circular arcs can easily be drawn if the object is placed so that these arcs are parallel to the plane of projection (Fig. 183).

Steps in construction

The steps in constructing an oblique drawing are illustrated in this figure. *The receding lines may be drawn at any angle*, but 30 degrees or 45 degrees is usually used for convenience.

Not only may you draw the receding lines at any angle with the horizontal but they may theoretically be drawn any length as well. Usually the receding distances are drawn full size, as it is more convenient to make the entire drawing to agree with the actual dimensions of the given object.

Note in Fig. 183 that the position of the object is such that all circles and circular arcs are in planes parallel to the front plane

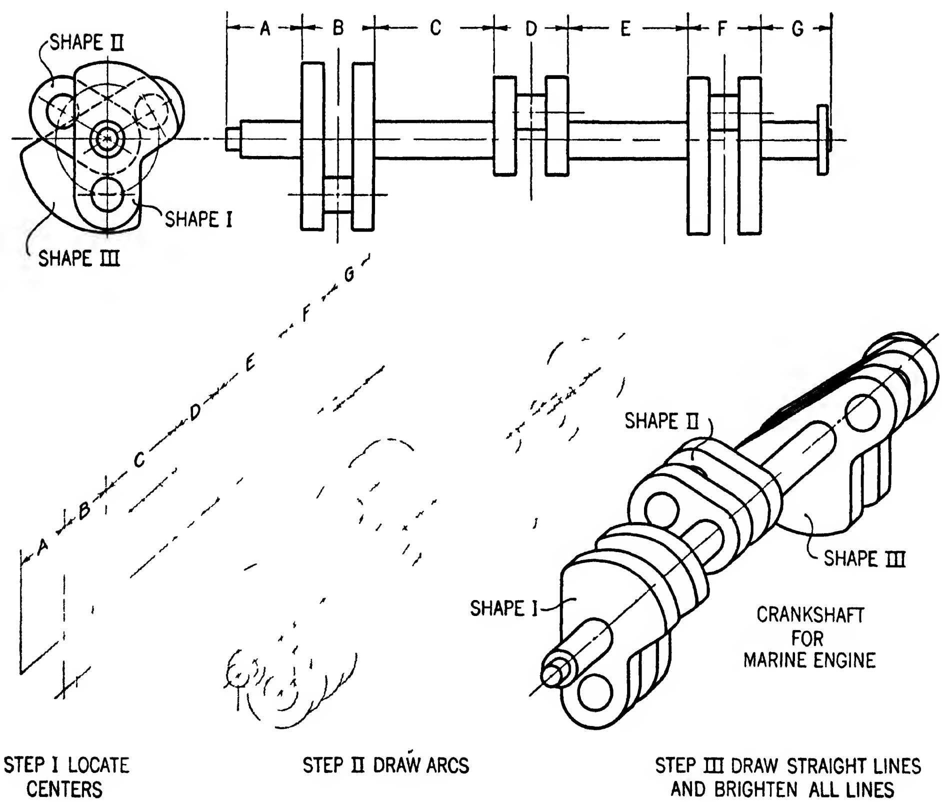


FIG. 183. OBLIQUE PROJECTION OF A CRANKSHAFT

Circles in oblique

of projection. This is purposely so in order that such curves could be drawn directly with the compass. If circular curves lie in receding planes of the object, they will appear as ellipses, and the construction would be made much more difficult and the result less pleasing. Notice that this drawing is built upon an important center line and that all receding measurements are set off along this line.

PERSPECTIVE

In *perspective projection* the observer is relatively near the object being viewed, this position resulting in a convergence of the visual rays from all points on the object toward the observer's eye or *SP* (Station Point). See Fig. 176 *d*.

Simple perspective

In Fig. 176 *d* the perspective is the picture on the *picture plane* (*PP*), formed by all the piercing points of the visual rays from the object to the eye, or station point (*SP*). The problem to be solved is how to *construct* the perspective from the ordinary top and side views of the object. In Fig. 184, we have the top and right side views of the object, of *PP*, and *SP*, and the perspective is shown in the center where the front view is usually placed. In the top and side views we have the edge views of *PP* and can see where the visual rays pierce *PP*. In the perspective, points *A–B–C*, and so on, are found by projecting down from these piercing points in the top view and across from the corresponding piercing points in the

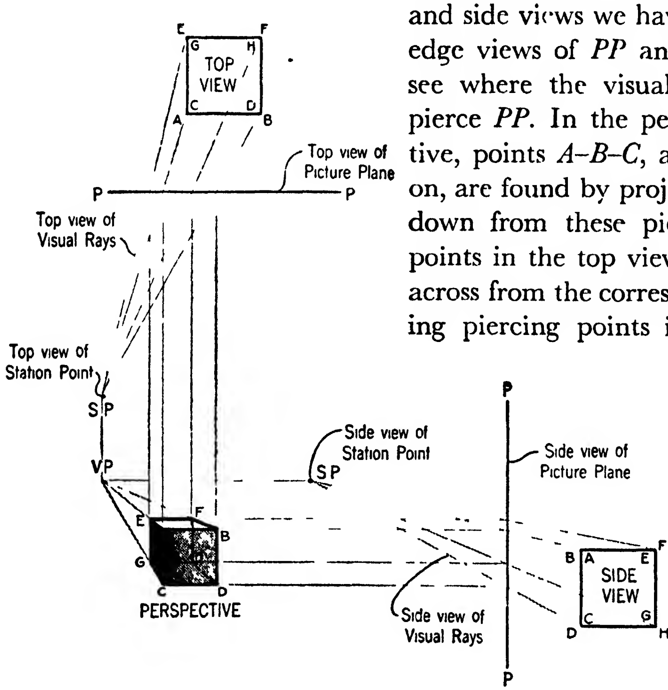


Fig. 184. Perspective projection

side view — for example (top view) the piercing point of visual ray *SP–B* is projected down to *B* in the perspective; and (side view) the piercing point of *SP–B* is projected across to *B* in the perspective. All other points are projected in the same manner.

Notice that the receding lines *A–E*, *B–F*, *D–H*, and *C–G* of the perspective meet at a vanishing point (*VP*), and that *VP* lines up with the top and side views of *SP*. Such a perspective, having only one vanishing point, is called a *one-point* perspective.

Two-point perspective

If, on the other hand, the top view is turned at an angle to *PP* (Fig. 185), there would be two vanishing points, *VPL* and *VPR*. No side view of *PP*, *SP*, and the visual rays is necessary or convenient, but the side view of the object is needed to obtain needed *height measurements*. Here, much depends upon the vanishing points, of which there are two in this case; hence, this is a *two-point perspective*.

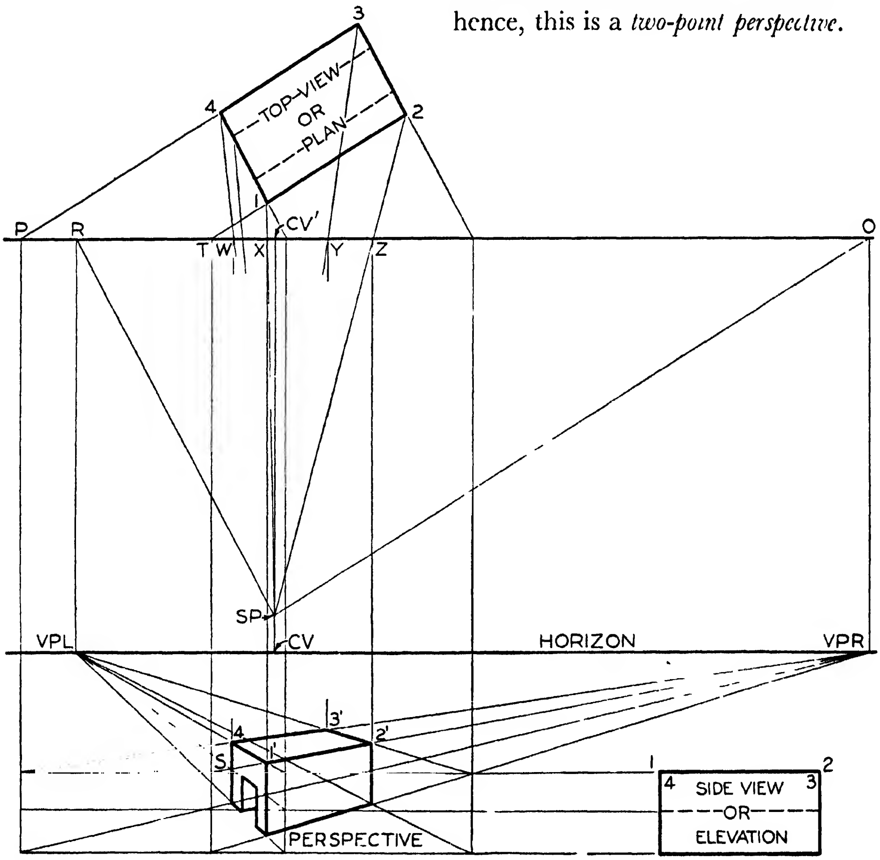


Fig. 185. Perspective of a single object

The top portion of the drawing in Fig. 185 (above the *horizon*) is a regular top view of the object, picture plane, and the observer as in Fig. 184. To draw the perspective you must simply learn how to draw the perspective of any line, and repeat this procedure until the perspective drawing is completed. *The setup*

In Fig. 185, to find the perspective of line 1–2 (top view), draw a line from *SP* parallel to 1–2, striking the picture plane *PP* at *O*. Then project downward to the horizon from *O* to obtain the vanishing point *VPR* of line 1–2 (and all lines of the object which are parallel to 1–2). Next (top view) extend the line 1-2 to the picture plane at *T* and project downward from *T* to *S*; then *S* is the *piercing point* of the line 1–2 in the picture plane, the height of *S* being projected across from 1–2 in the side view. *Steps in drawing perspective*

Then join *S* to *VPR* with a straight line. The visual rays (top view) from *SP* to the points 1 and 2 intersect the picture plane at *X* and *Z*. Project downward from these points to the line *S–VPR* to obtain the ends 1′ and 2′ of the perspective of the line 1–2. Follow this same procedure with all other lines of the object to complete the perspective. Always follow this procedure in finding the perspective of a given line:

Principles

1. Find the vanishing point *VP* of the line.
2. Find the piercing point of line in the picture plane *PP*.
3. Connect these two points with a straight line.
4. Find the end points of the perspective of line 1–2 by projecting downward from the piercing points of visual rays with the picture plane.

Remember the following:

1. Vanishing points of all horizontal lines are on the horizon.
2. Parallel lines have the same vanishing point.
3. Lines which are parallel to the picture plane have no vanishing points.

A perspective is geometrically the same as a photograph; and except for the fact that the retina of the eye is spherical, instead of flat like a sheet of drawing paper, it corresponds exactly to the impression obtained by the eye. By means of perspective the most complicated objects can be represented accurately and naturally. Architects use perspective extensively in the process of design to study the appearance of buildings before they are built. They also prepare perspectives to show clients how the building will look upon completion. As a matter of fact, perspectives are used in all kinds of illustration where natural appearance is desired.

Production illustrations

Modern industry, particularly aircraft manufacturing, has found a new and valuable use for perspective. Because we find that many thousands of workers cannot "read" blueprints, especially the complex drawings of airplanes, perspectives have come into extensive use to supplement the blueprints by providing the extra visualization needed for a clear understanding. It was soon found that engineers, executives, and all other persons connected with the production of these intricate mechanisms came to regard these perspectives as a great help in clearly visualizing the separate parts and the assemblies thereof.

These drawings are known as "production illustrations," and are defined roughly as pictorial drawings used in the shop or the production line to show clearly how all the parts are fabricated and assembled. A pen-and-ink perspective drawing of a complete airplane showing the details of construction is shown in Fig. 186. Obviously a considerably more extensive study of perspective is necessary to produce a drawing of this character than can be provided in these pages.

A more natural gradation in rendering light and shade on a perspective is afforded through the use of the *air-brush.* An excellent example of this type of drawing is shown in Fig. 187. By means of this ingenious tool the paint is sprayed on the drawing while certain parts are "masked," or covered, to protect them from the spray.

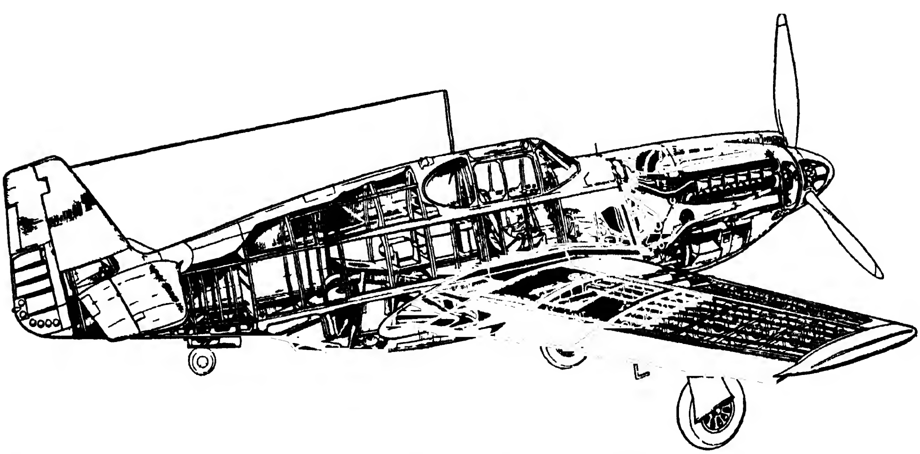
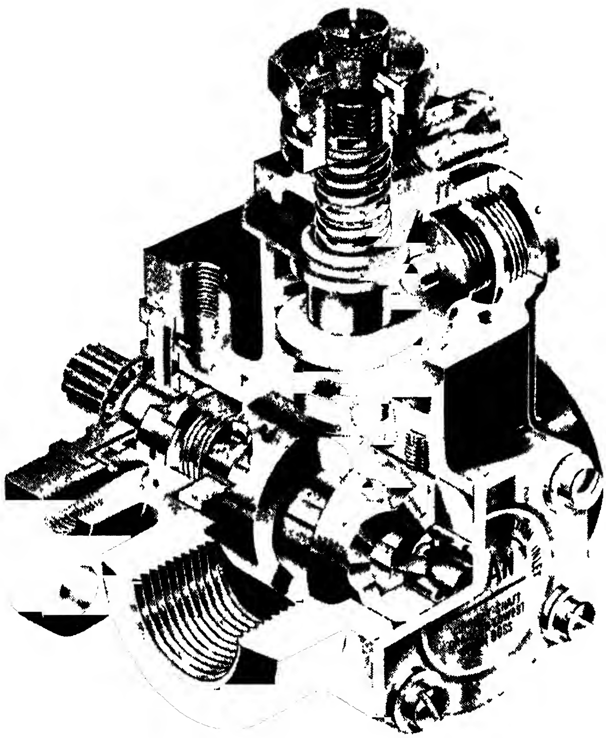


FIG. 186. PRODUCTION ILLUSTRATION OF THE P–51 MUSTANG FIGHTER
An example of complex technical drawing — a true perspective.

Problems

It is not expected that you can, as a result of the brief treatment of perspective afforded here, produce pictorial drawings of complicated objects. However, a large selection of elementary problems is provided in Fig. 188 and you are urged to do as many of these as you can before attempting anything more difficult. The isometric and oblique problems can be drawn on the sheet layout of Fig. 162, page 214. For perspective drawings use larger paper sizes of your own choosing.



Courtesy Candler-Hill Corporation

FIG. 187. PICTORIAL DRAWING SHADED WITH THE AIR-BRUSH -- TITAN AIRCRAFT FUEL PUMP

TECHNICAL SKETCHING

Importance of sketching

The value of freehand technical sketching to the engineer cannot be overstated. It enables him to convey technical ideas to others quickly and unmistakably. He can employ all of his knowledge of graphic expression and can record his ideas easily and completely. "A picture is worth a thousand words," according to an old Chinese proverb.

Most original mechanical ideas find their first expression through a freehand sketch. Executives resort to it daily to explain their ideas to subordinates. Engineers usually prepare their designs and

Straight lines

In general, before drawing a straight freehand line, decide upon your starting and stopping points; then swing your pencil back and forth between these points until you feel that you are "set" to make the line. First make a very light sketch line between the points; then, more slowly, go over the line and make it the proper weight. Much can be learned by observation of good sketches, and

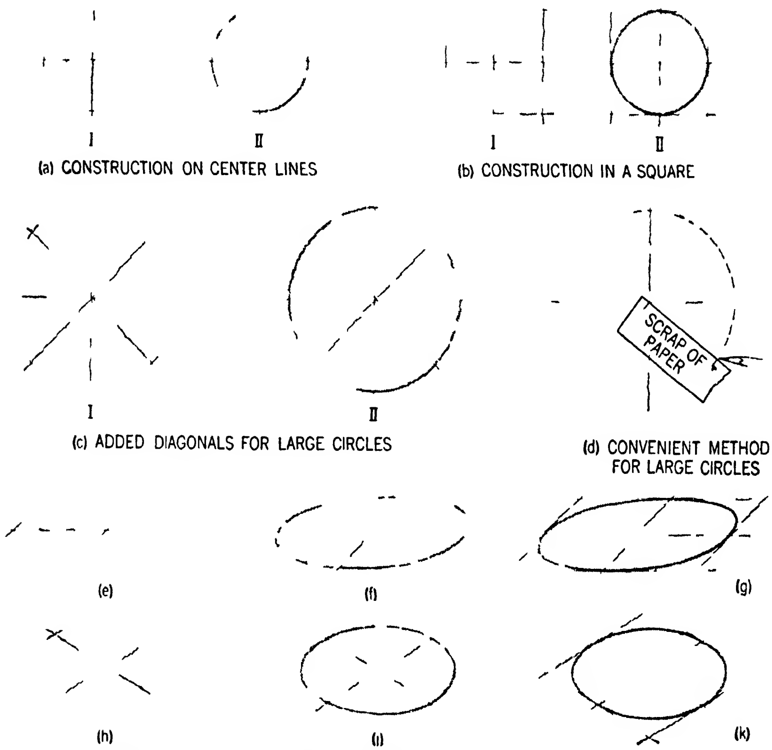


Fig 190 Sketching circles and ellipses

by practice. The draftsman who perseveres will eventually evolve a satisfactory technique of his own

Some of the various methods used in constructing circles and ellipses are shown in Fig. 190. In general, construction lines should be so light that they will not need to be erased.

Views of objects

Sketches such as Fig. 191 are frequently made showing the views of objects carrying complete dimensions and shop notes. First, block in the main geometric elements in the views chosen; second, sketch in the details; and third, heavy-in the required lines, and add dimensions and notes. The final drawing should

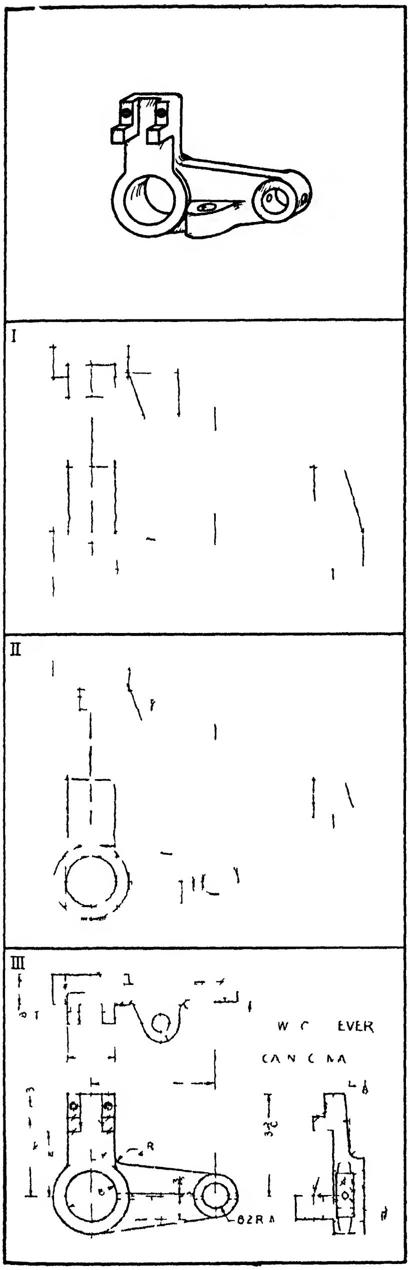


FIG 191 SKETCHING A WORKING DRAWING

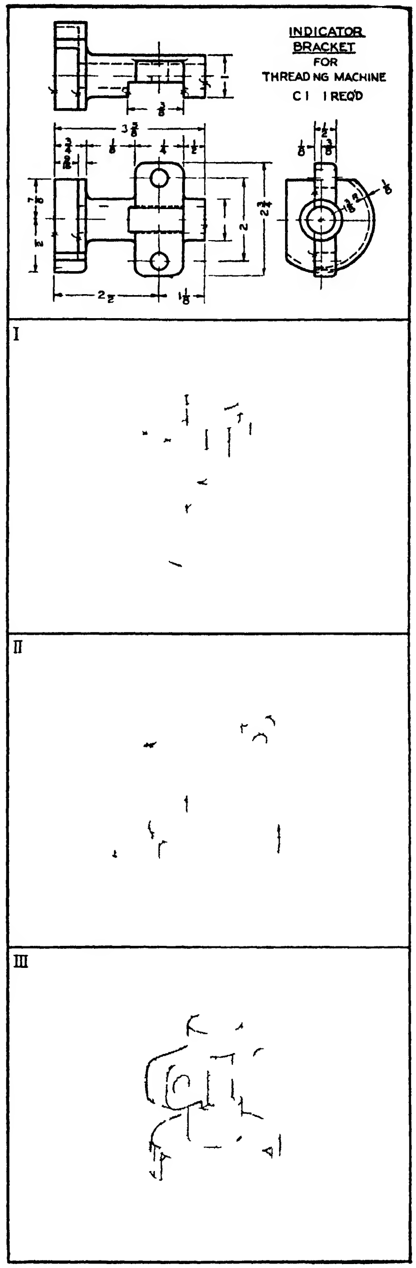


FIG 192 SKETCHING A PICTORIAL DRAWING

have the contrast in lines of a mechanical drawing, but the character of the lines should be strictly freehand. In other words, do not try to make every line "perfect," but be very sure that your general proportions and estimates of distances and areas are well considered.

Pictorial sketches

Pictorial sketches may be made to approximate isometric, oblique, or perspective projection. (See Fig. 192.) Considerable liberty can be taken in a pictorial sketch if the draftsman has a natural aptitude for sketching, but most persons should rely entirely upon the fixed methods of projection used in mechanical drawing. The pictorial drawing, unlike the working drawing, can be shaded if desired. Shading, however, is an art in itself and beginners are cautioned to keep shading to a minimum. Many a sketch is ruined by "overdoing" this part of the drawing. The best rule to follow is to use restrained shading, and to use it only when the shape is not clear without it.

Problems

Any of the problems given heretofore for mechanical drawing will make equally suitable problems for technical sketching, and you are therefore referred to pages 198, 209, 213 for problem material. Ordinary buff detail drawing paper of the same kind used for mechanical drawings will be found excellent for these sketches.

MACHINE FASTENERS

Some machine elements occur so often in engineering that the draftsman must be familiar with accepted methods of representation of such parts on drawings. These are chiefly permanent fasteners such as rivets, or movable fasteners such as bolts, screws, and keys.

Threads

Threads are of course the most common fastener. On the outside of a member they are called *external threads* and on the inside,

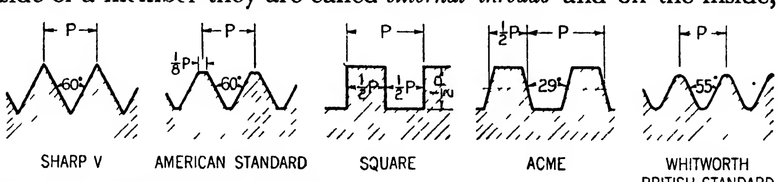


FIG. 193. SCREW THREAD FORMS

internal threads. The diameter is always understood to be the major diameter, or outside diameter. A number of thread forms, or profiles, are in use today, the most common of which are shown in Fig. 193. The American Standard is the general all-purpose thread, but either the *square thread* or the *Acme thread* is widely used to transmit power.

Screw threads are heliccidal in shape but the curved lines are never drawn because of the considerable labor and time involved.

Threads under approximately 1-in. diameter on the drawing are drawn by the conventional method (Figs. 194 and 195). Here the visible thread is represented by alternating long thin lines and short thick lines at right angles to the shaft. The long lines are spaced by eye to approximate the spacing between the ridges.

Conventional threads

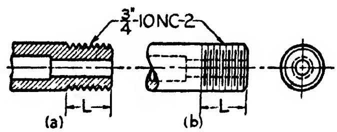


Fig. 194. External thread symbols — regular

Complete tables covering the various threads giving the number of threads per inch and other data for various diameters are given in the handbooks. A few of the most important tables are included here.

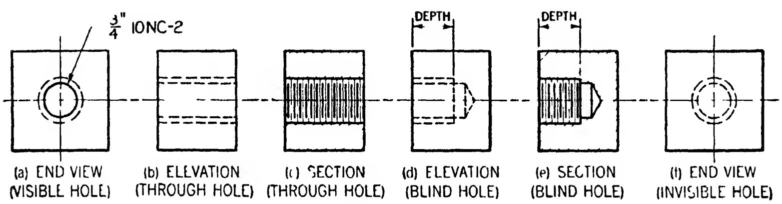


Fig. 195. Internal thread symbols — regular

AMERICAN STANDARD SCREW THREADS

Thread spacing

Sizes	*Threads per Inch*			*Sizes*	*Threads per Inch*		
	NC National Coarse	*NF National Fine*	*EF S.A.E. Extra Fine*		*NC National Coarse*	*NF National Fine*	*EF S.A.E. Extra Fine*
0		80		9/16	12	18	24
1	64	72		5/8	11	18	24
2	56	64		3/4	10	16	20
3	48	56		7/8	9	14	20
4	40	48		1	8	14	20
5	40	44		1 1/8	7	12	18
6	32	40		1 1/4	7	12	18
8	32	36		1 3/8	6	12	18
10	24	32		1 1/2	6	12	18
12	24	28		1 3/4	5		16
1/4	20	28	36	2	4 1/2		16
5/16	18	24	32	2 1/4	4 1/2		16
3/8	16	24	32	2 1/2	4		16
7/16	14	20	28	2 3/4	4		16
1/2	13	20	28	3	4		16
				Over 3			16

Uses and fits

The *form* of the American Standard thread is shown in Fig. 193, and in addition several *series* have been standardized, the most important of which are shown in the preceding table. National Coarse threads are for general use, National Fine threads are for special uses requiring a larger number of threads per inch, and the S.A.E.

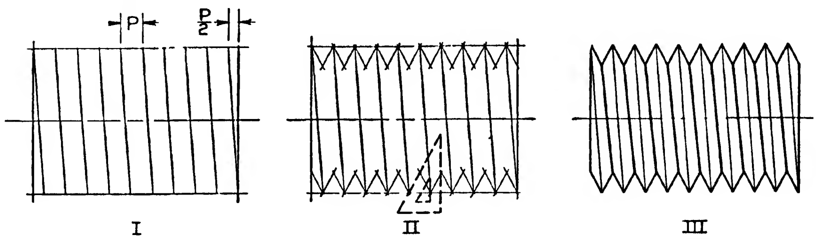


FIG. 196. STEPS IN DRAWING SEMICONVENTIONAL SHARP-V, OR AMERICAN STANDARD THREADS

Extra Fine threads are intended for airplane construction and other uses where extremely fine threads are needed.

Tables of fits for American Standard threads are also available, specifying the tightness of fit between internal and external mating threads. In Figs. 194 and 195 the first figure in the thread note designates the major diameter, followed by the number of threads per inch, the symbol for the series, and finally the class of fit.

Pitch of a thread

Threads of over 1-in. diameter on the drawing are drawn by the *semiconventional* method (Figs. 196 and 197), which more closely resembles the actual thread. In all cases the pitch, $P = \dfrac{1''}{\text{no. threads per inch}}$, and the number of threads per inch may be found in the tables given here.

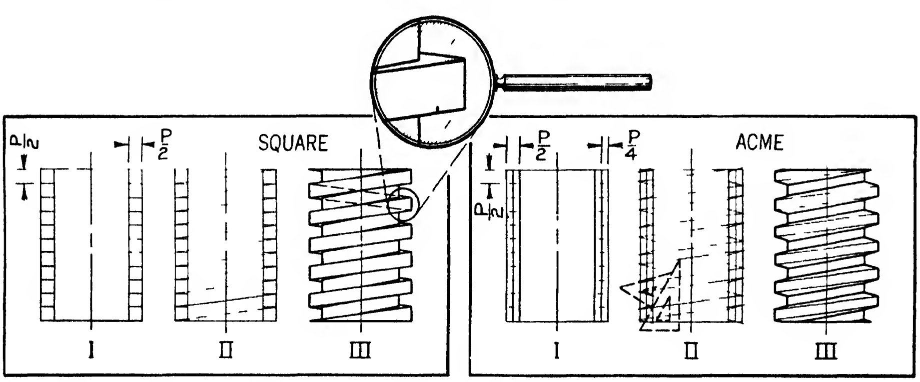


FIG. 197. DRAWING SEMICONVENTIONAL SQUARE AND ACME THREADS

SQUARE AND ACME * THREADS

Thread spacing

Size	*Threads per Inch*		*Size*	*Threads per Inch*		*Size*	*Threads per Inch*	
	Square	*Acme*		*Square*	*Acme*		*Square*	*Acme*
$\frac{1}{4}$		16	$\frac{7}{8}$	5	6	2	$2\frac{1}{2}$	4
$\frac{3}{8}$	12		1	5	5	$2\frac{1}{4}$	2	3
$\frac{5}{16}$		14	$1\frac{1}{8}$	4	5	$2\frac{1}{2}$	2	3
$\frac{3}{8}$		12	$1\frac{1}{4}$	4	5	$2\frac{3}{4}$	2	3
$\frac{7}{16}$	10	12	$1\frac{1}{2}$	3		3	$1\frac{1}{2}$	2
$\frac{1}{2}$	10	10	$1\frac{3}{8}$		4	$3\frac{1}{4}$	$1\frac{1}{2}$	
$\frac{9}{16}$	8		$1\frac{1}{2}$		4	$3\frac{1}{2}$	$1\frac{1}{3}$	
$\frac{5}{8}$	8	8	$1\frac{3}{4}$	$2\frac{1}{2}$	4	$3\frac{3}{4}$	$1\frac{1}{3}$	
$\frac{3}{4}$	6	6				4	$1\frac{1}{3}$	2
						$4\frac{1}{4}$	$1\frac{1}{3}$	
						$4\frac{1}{2}$	1	
						5		2

* Acme threads are American Standard (ASA B1.3–1941)

Threads in section are shown in Fig. 198. Note that the stud and nut are not sectioned, as there is no need for showing internal construction. Note also that at the bottom end of the stud the back side of the threaded hole and a portion of the tap drill hole are shown. The angle of the point of the drill is 60° with the center line.

R.H. and L.H. threads

A thread is understood to be R.H. (right hand) unless specified L.H. (left hand) at the end of the thread note. A right-hand thread advances when the screw is turned clockwise as illustrated by Fig. 199 *b;* a left-hand thread advances when the screw is turned counterclockwise (Fig. 199 *a*). The shop man remembers this by the position of his thumb around a screw thread, as shown.

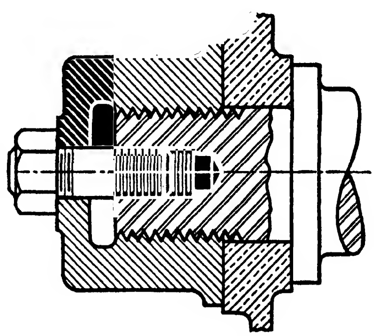


Fig. 198. Threads in section

Bolts

Regular American Standard bolts are illustrated in Fig. 200, showing convenient draftsman's proportions based on diameter *D*. These bolts have National Coarse threads.

A number of other miscellaneous machine fasteners are shown in Fig. 201. These are generally shown only in assembly drawings and their dimensions are obtained from handbooks.

WORKING DRAWINGS

Originality in design

When you look at some complicated structure, such as an airplane, a full-color printing press, or a late-model milling machine, don't be discouraged. Not one but hundreds of persons have contributed their brains and originality to these modern wonders, and you may be sure that the first design was relatively crude by our standards, revolutionary as it might have been at the time.

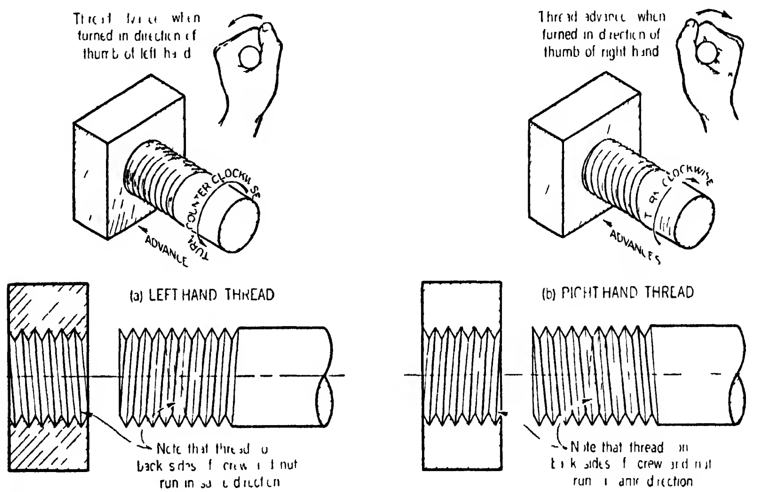


FIG. 199. RIGHT-HAND AND LEFT-HAND THREADS

Leonardo's inventions

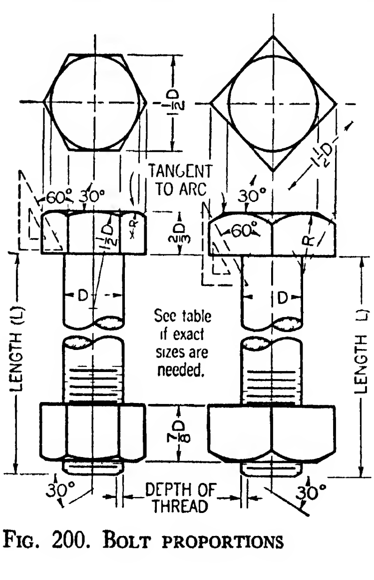


FIG. 200. BOLT PROPORTIONS

Leonardo da Vinci, perhaps the greatest scientific and artistic genius of all time, is credited as being the first to tackle seriously the problem of human flight. One of his many strange sketches, with left-handed mirror-written notes, is shown in Fig. 214.

Since Leonardo, thousands of other brilliant and daring minds, notably the Wright brothers, have added their ideas. Each designer has contributed small improvements through the years to produce the intricate mechanisms of today (such as Fig. 186), and

often bitter tragedy taught men to abandon one idea and try another. The point is that engineering design is cumulative, and each designer hopes only to improve a little upon existing mechanisms.

DRAFTING PROCEDURE

Usually a designing engineer conceives the basic idea, but often the ideas come from draftsmen, shopworkers, or salesmen. In any case this fortunate individual visualizes the mechanism in three-dimensional space and not in terms of a word description. It is *1. The idea sketch*

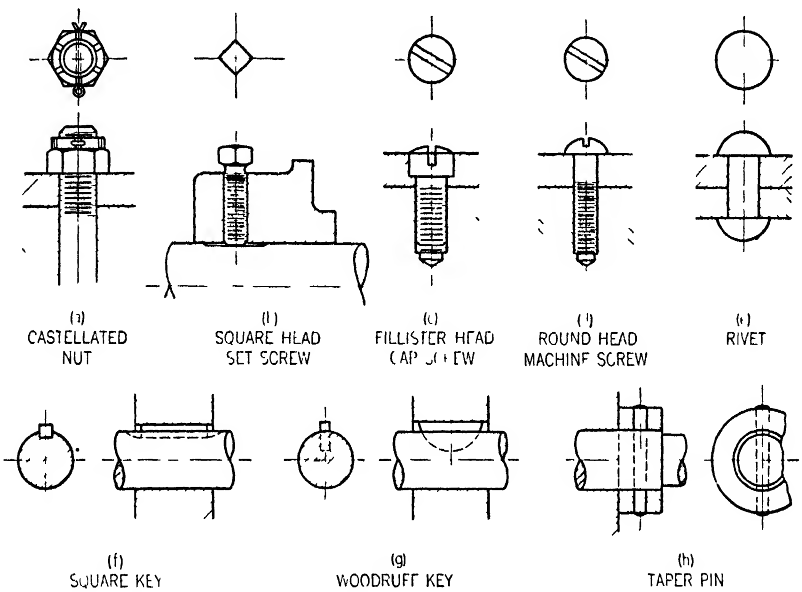


Fig. 201. Miscellaneous fasteners

only natural that at this moment he would seize a pencil and any scrap of paper to record the new idea at once in the form of a freehand technical sketch (page 235). Such a sketch is shown in Fig. 202, of an "Arm Adjusting Mechanism" for a milling machine.* See Fig. 203 (handle in upper left portion of milling machine).

Then the originator, or another designer-draftsman, uses the idea sketch as a basis for the *layout drawing* of Fig. 204, drawn in pencil with instruments to exact scale but without dimensions. This is a *design assembly drawing*, showing the part in its natural *2. The layout drawing*

* Figures 202 to 207 are reproduced through courtesy of the Brown & Sharpe Mfg. Co.

position and in relation to all adjacent parts. The layout is usually full size to enable the designer to visualize more clearly the actual sizes of the parts. When working with drawings to half-size or less, a designer has a tendency to design large, heavy, or clumsy parts, while in working double size or more he is apt to get an exaggerated idea of strength and rigidity.

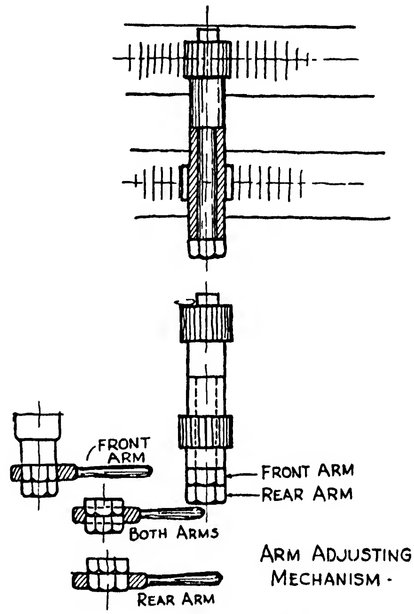


FIG. 202. IDEA SKETCH FOR IMPROVEMENT ON MILLING MACHINE

The designer-draftsman makes use of all the sources of information available to him, including physics, mathematics, chemistry, and mechanics. He adopts the conclusions and recommendations obtained from experimental tests and laboratory studies and he uses the sound experience that he and his company have obtained from success and failure of past machines, following carefully the performance and maintenance records of machines in use. Much design information is compiled in handbooks and designers use these so frequently that they usually own their own copies.

Machine design

Some mechanisms are relatively simple in form and operation and little thought is required to design the shapes and arrangement. They may be, however, subject to great loads, so that careful computation of strength is necessary to design the parts to withstand the stresses and strains. A good example of this type of

design problem would be a simple hoist. Other mechanisms, such as typewriters and adding machines, are not subject to large stresses, and the chief problem is one of arrangement and shape of parts for effective operation and low production cost.

3. The detail drawing

When the designer's layout has finally been approved, it is turned over to the detailer to make *detail working drawings* (Fig. 205).

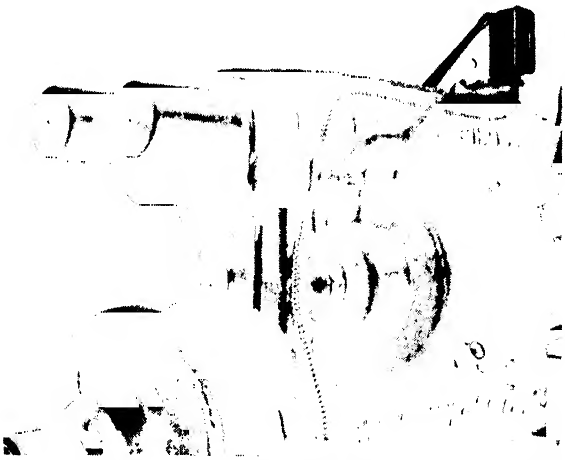


Fig. 203. "Arm Adjusting Mechanism" on the milling machine

These are separate drawings of the individual parts, completely dimensioned and including all shop notes and other data required in the manufacture of the parts. The detailer takes his dimensions directly from the layout with scale and dividers.

4. The assembly drawing

The shop also needs an assembly drawing to direct the workers in assembling the parts properly. Since the original layout is an assembly, it usually can be traced, omitting unnecessary adjacent parts, and used as the assembly drawing (Fig. 206).

5. The patent drawing

Finally, the patent drawing is made and sent to the United States Patent Office (Fig. 207). Patent drawings are shaded, lettered in script, and otherwise follow rigid Patent Office specifications.

Problems

Three detail drawing problems are given in Figs. 208–210. These will test your ability to read a drawing and to visualize fairly complicated details. Use a 17″ × 22″ sheet, design your own title block, and follow the instructions given with each problem. Dimension the problems fully and letter all notes. Place the dimensions where they should be in the new drawing, remembering that in many cases they could not be placed properly on the limited views given. This drawing can be drawn directly on vellum in pencil, if desired, or the pencil drawing on detail paper can be traced on cloth in ink. (Continued on p. 248.)

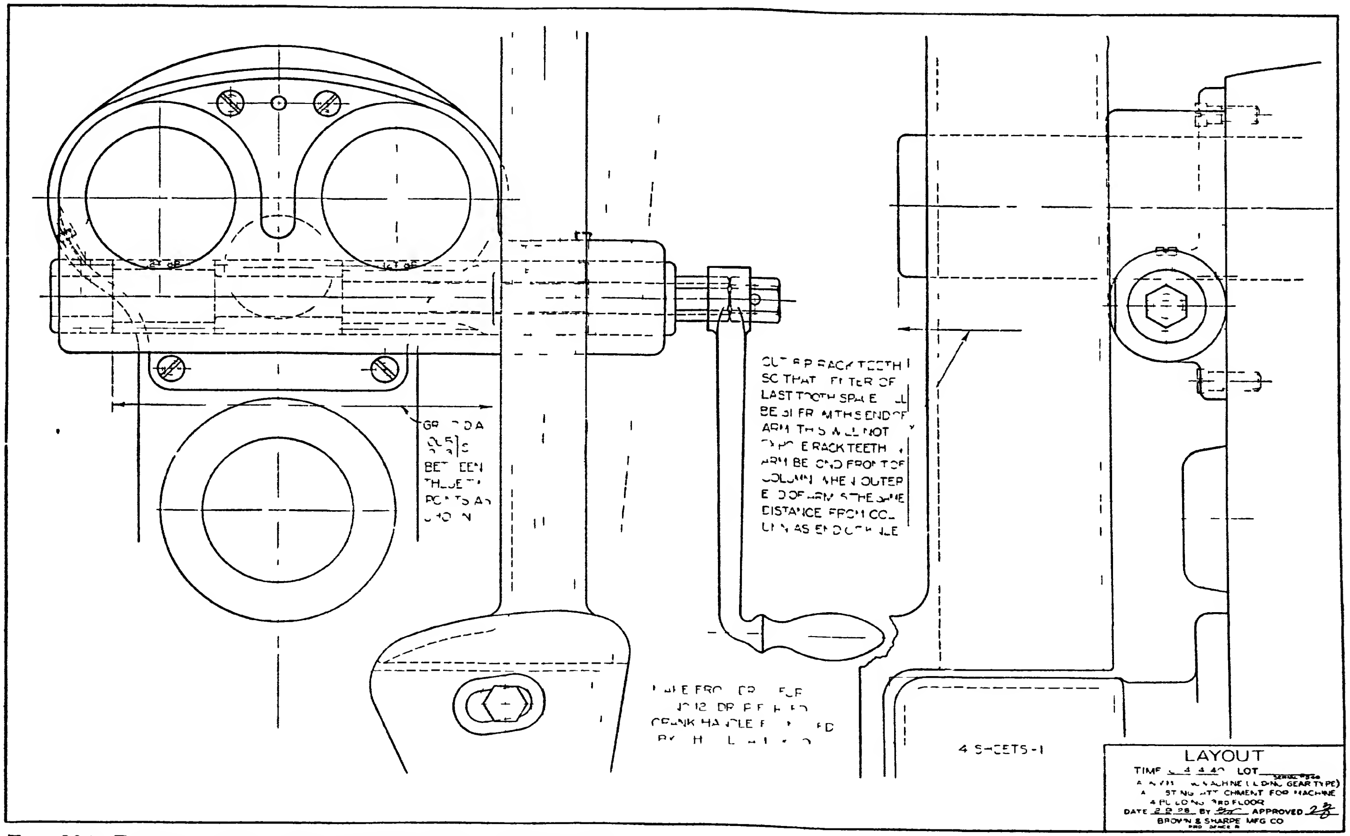


FIG. 204. DESIGN LAYOUT OF "ARM ADJUSTING MECHANISM"

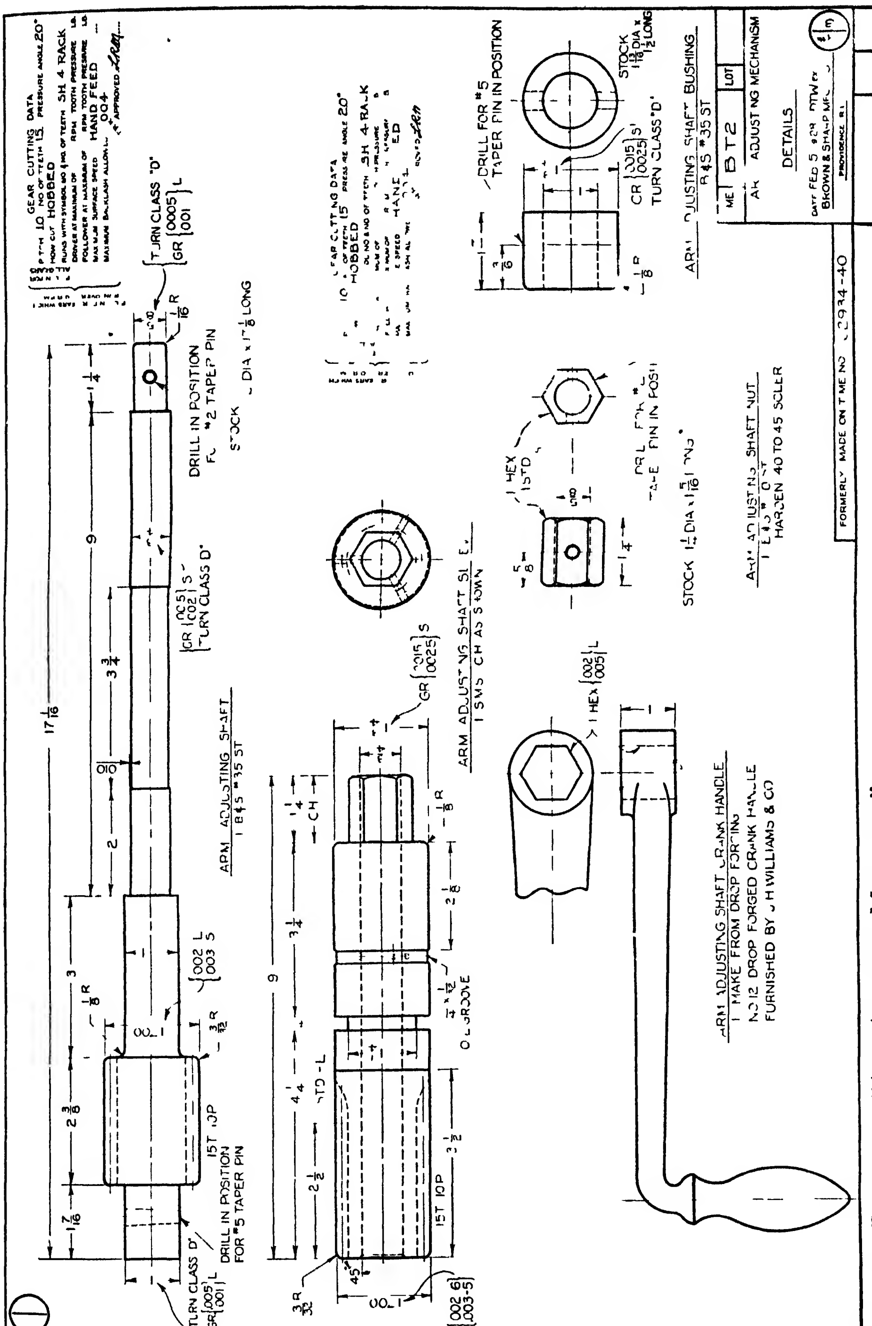


Fig. 205. Details of "Arm Adjusting Mechanism"

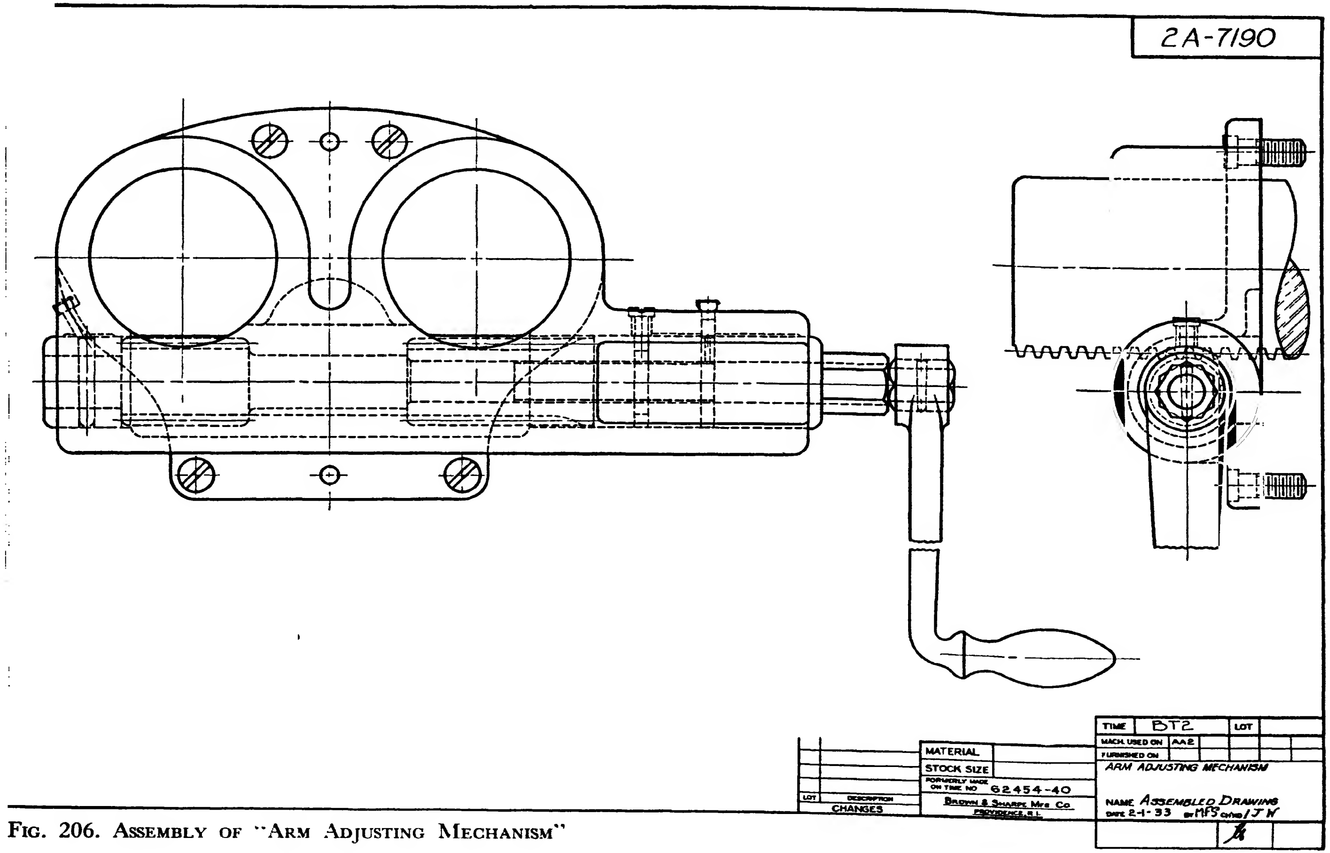


FIG. 206. ASSEMBLY OF "ARM ADJUSTING MECHANISM"

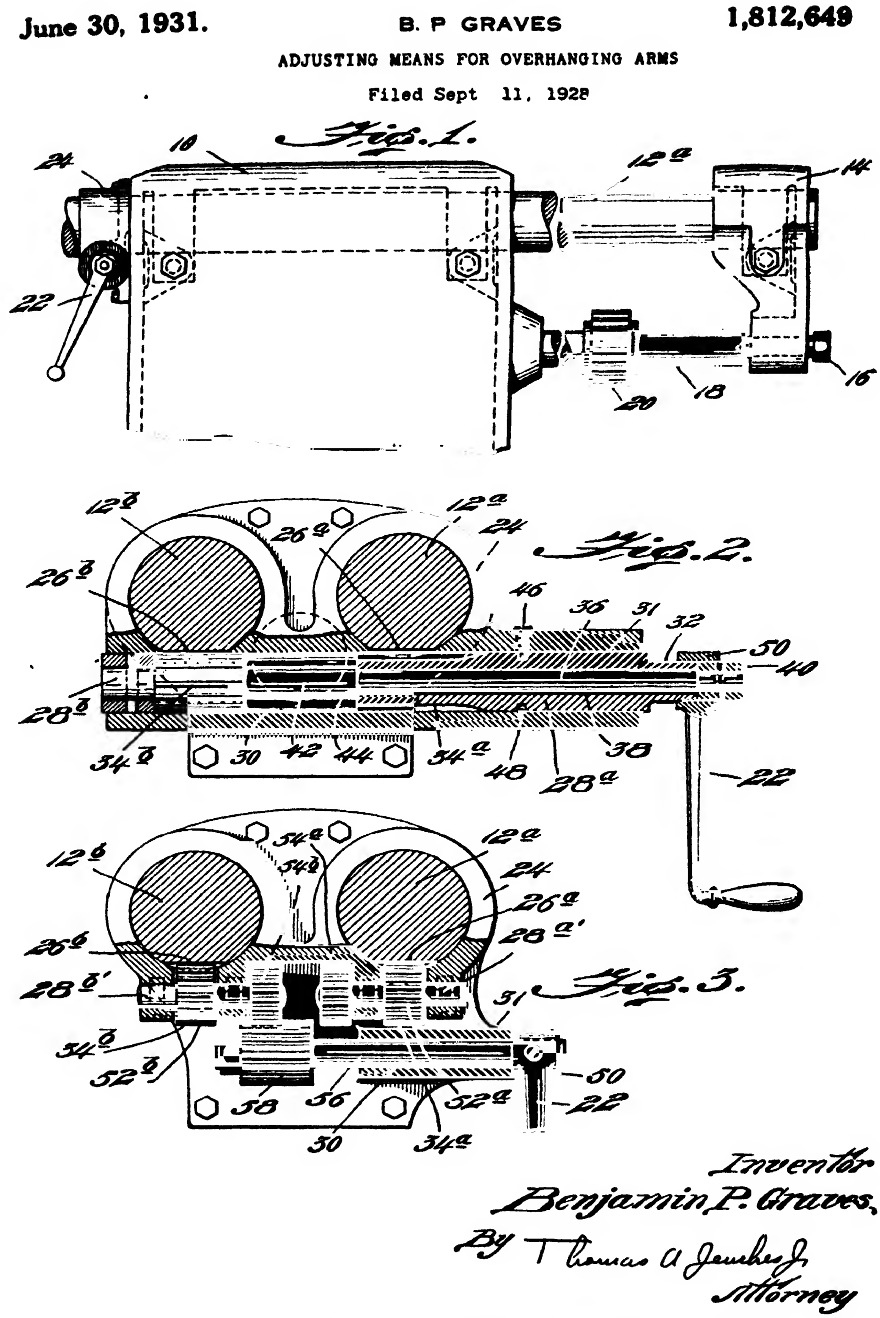


FIG. 207. PATENT OFFICE DRAWING FOR "ARM ADJUSTMENT MECHANISM"

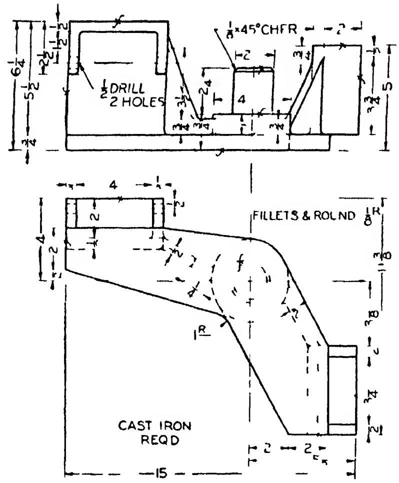


Fig. 208 Skip drum bracket for a concrete paver

Given Front and bottom views
Required: To draw front, top and right side views.

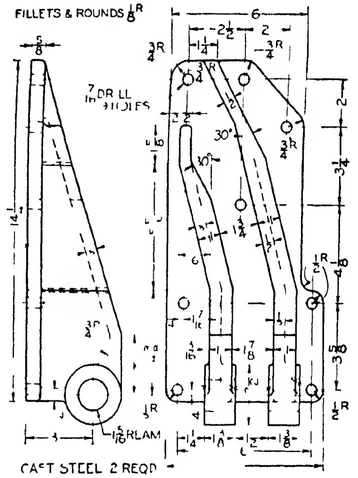


Fig. 209. Bracket for a Diesel engine

Given: Front and left side views
Required Revolve present front view 90° clockwise, and let it become a top view Then add front and right side views.

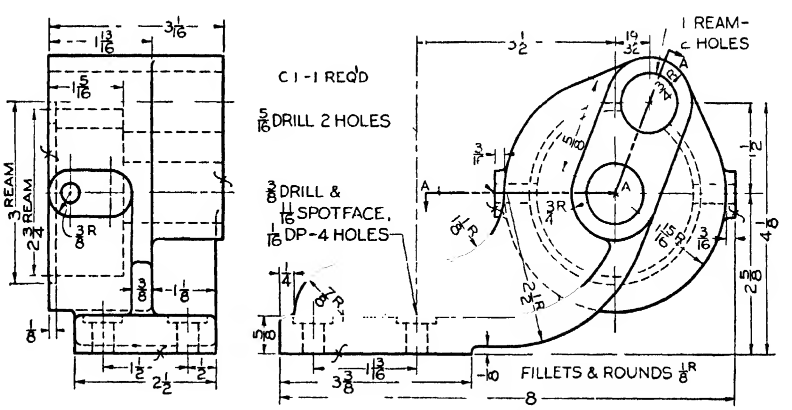


Fig. 210 Pump bracket for a thread milling machine

Given: Front and L. S. views. Required: Draw front view, R. S. view, and top view sectioned on A. A.

Problems continued

All of the details, together with a pictorial assembly drawing of a "Fixture for Centering Connecting Rod," are given in Figs 211–213. Draw complete detail and assembly working drawings Use 17″ × 22″ sheets, placing several details per sheet. Include a complete parts list on the assembly drawing

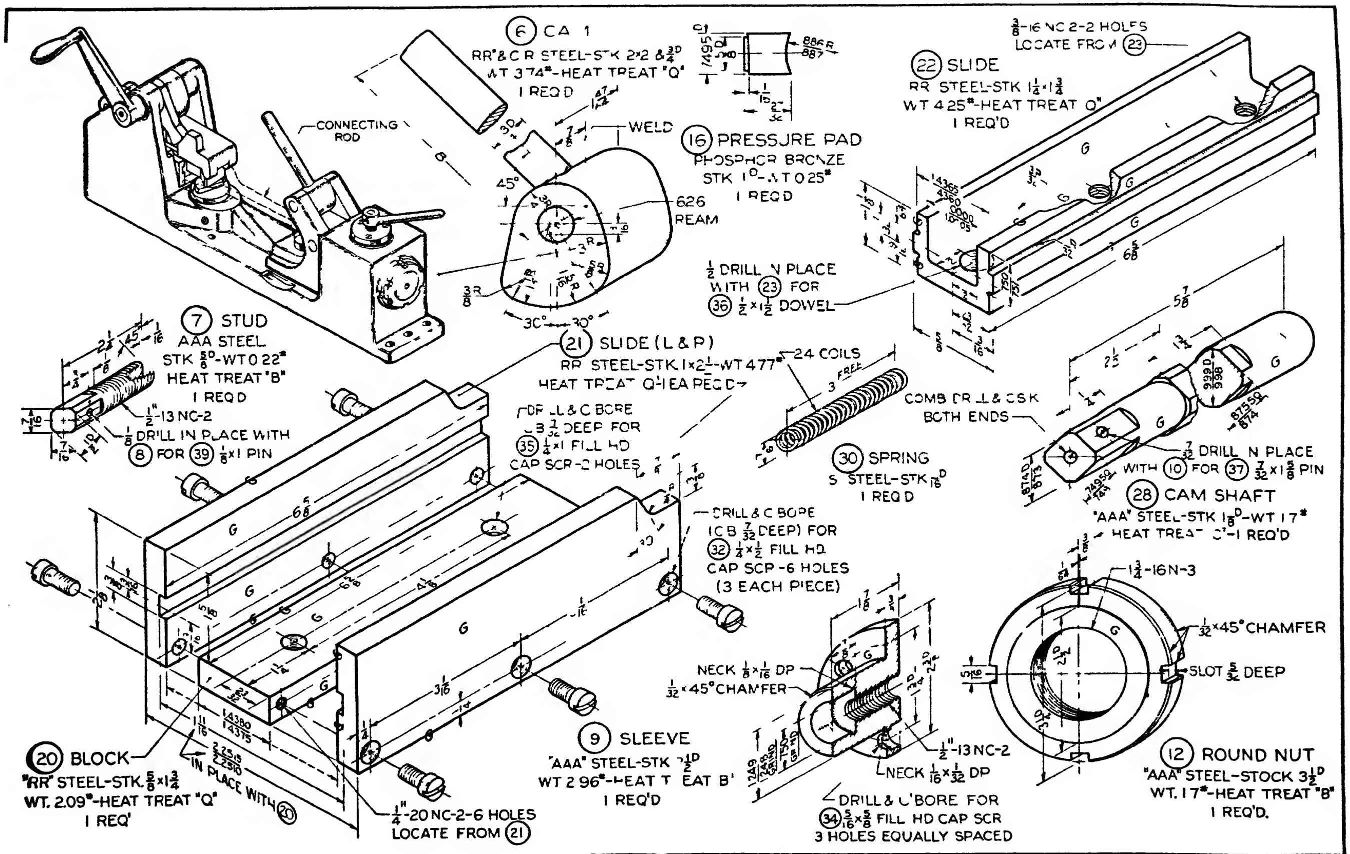


Fig. 211. Fixture for centering connecting rod

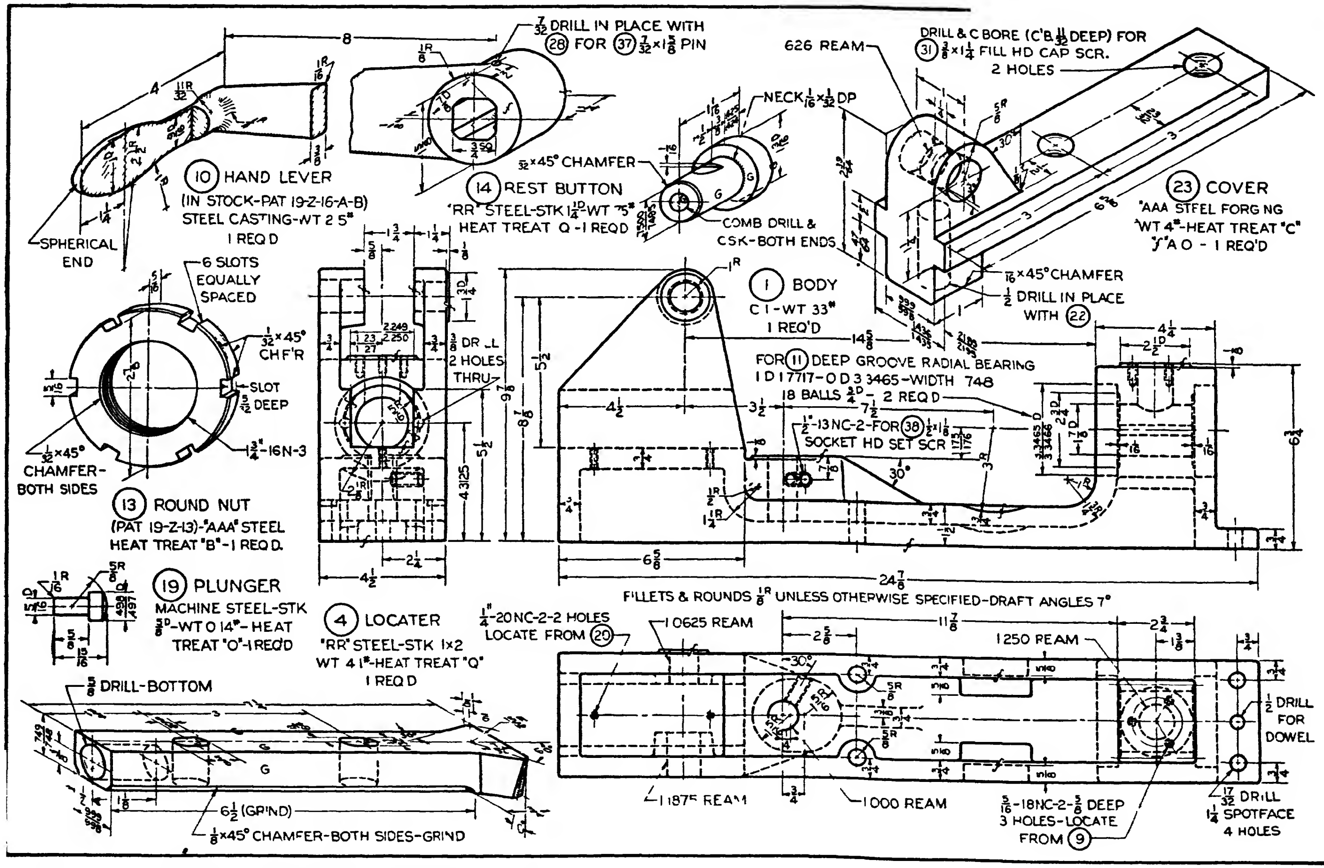


FIG. 212. FIXTURE FOR CENTERING CONNECTING ROD

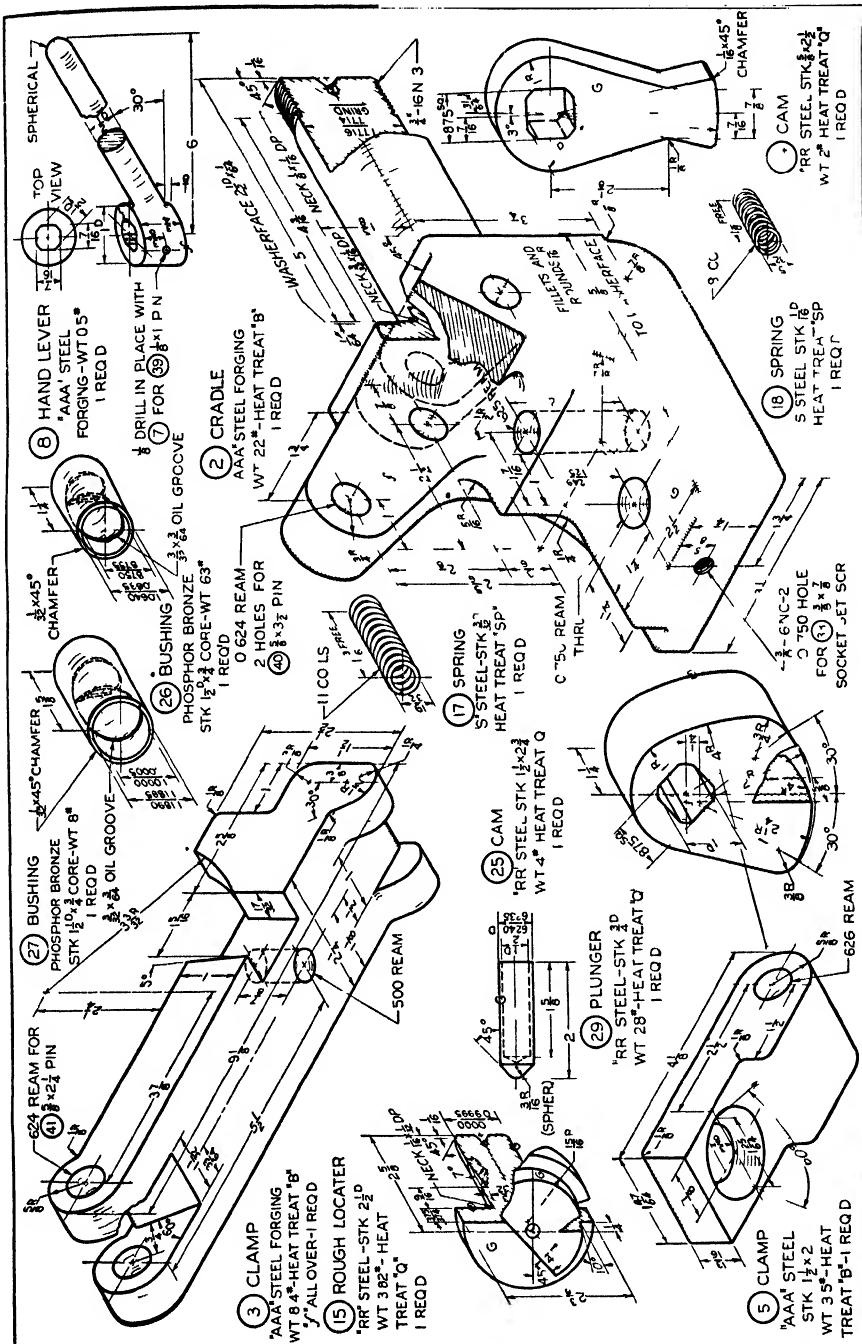
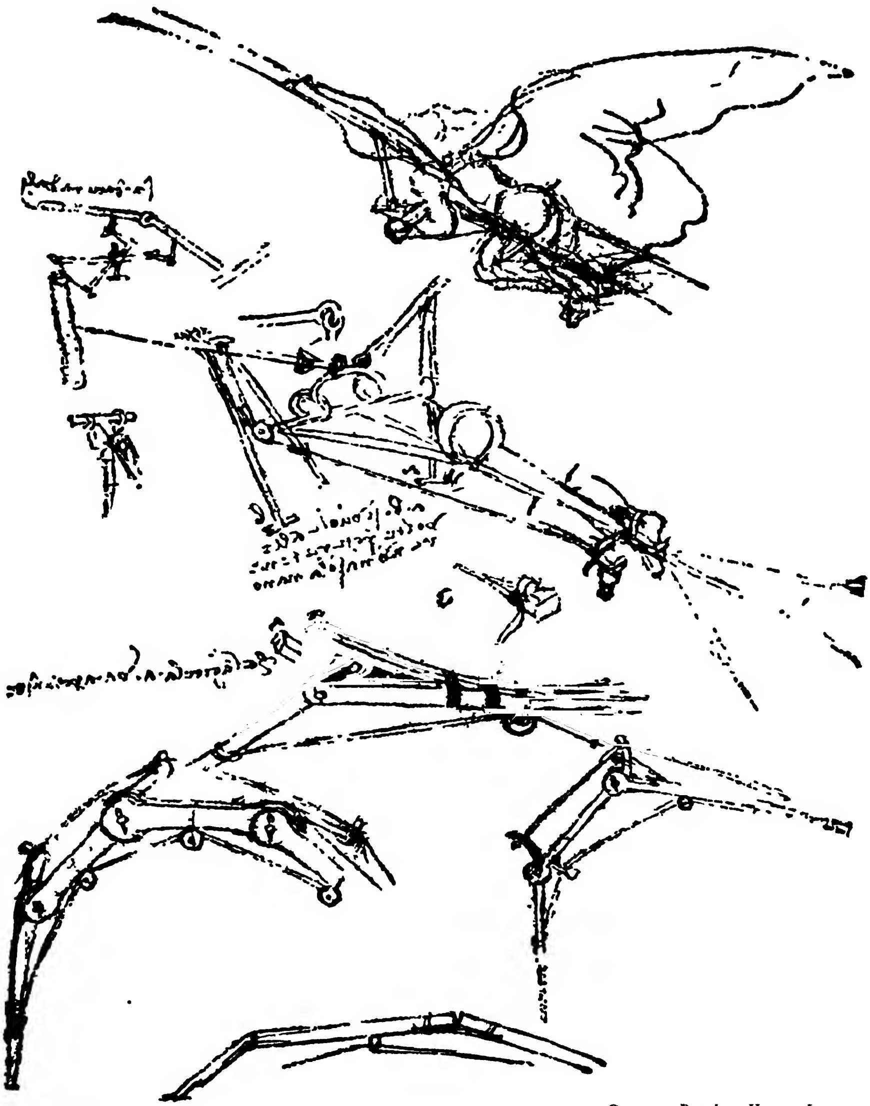


FIG. 213. FIXTURE FOR CENTERING CONNECTING ROD



Courtesy Random House, Inc

Fig. 214. Leonardo da Vinci's idea sketch of an airplane

In conclusion, from Leonardo da Vinci's sketches (Fig. 214), to the complicated drawings used in modern industry, it is evident that technical ideas have always been expressed graphically. Hence, your objective should be to produce drawings which are neat and simple, yet clear and complete.

Mathematics

A UNIVERSAL TOOL OF ENGINEERING

Basic to all fields of engineering is the prediction of how some thing designed on paper will work. To make such predictions an engineer must know various laws of nature under which physical systems operate. These laws are usually expressed in mathematical form because they are relations between the measures of quantities, such as the mass and acceleration of a projectile and the force acting on it. From such laws the engineer can determine how to increase the efficiency of a given device. He can also diagnose failures in equipment and suggest improvements. When the engineer uses numbers, and hence mathematics, his work becomes more exact, and his field more scientific. *Mathematical laws*

Certain outstanding inventions like the radio were discovered through mathematics, whereas others such as the dynamo were found experimentally, and subsequently improved by mathematics. If, therefore, you wish to build or maintain some engineering device such as a bridge, electrical generator, airplane, or oil-cracking plant, or if you wish to add to the comforts of man through original scientific inventions, you will need mathematics.

Mathematics had its origin in the need of our ancestors to manipulate numbers rapidly. Prehistoric man probably used the whole numbers 1, 2, 3, and so on, and excavations show that even before 2000 B.C. the Babylonians divided one of these whole numbers by another. Mathematicians speak of whole numbers as *positive integers*. *Positive integers*

The result of such division is called a *quotient*. The quotient of 4 by 3 is represented by $\frac{4}{3}$. In a quotient the first, or top, number is called the *numerator* and the second, or bottom, is the *denominator*. In the fraction $\frac{4}{3}$ the numerator is 4 and the denominator is 3.

If the denominators of two fractions are identical, they are said to have a *common denominator*. In this case the fractions may be added to yield a fraction whose numerator is the sum of the numerators of the given fractions and whose denominator is the *Addition of fractions*

same as the common denominator of the given fractions. For example,

$$\tfrac{4}{3} + \tfrac{7}{3} = \tfrac{11}{3}.$$

This rule for the addition of fractions probably arose from a consideration of parts that may be added up to make a whole, as in the following summations:

$$\tfrac{1}{2} + \tfrac{1}{2} = \tfrac{2}{2} = 1, \quad \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{3}{3} = 1, \quad \tfrac{1}{3} + \tfrac{2}{3} = \tfrac{3}{3} = 1.$$

Symbols representing numbers

The rule for the addition of two fractions may be stated very briefly by the use of symbols to represent numbers, whose values may be anything we please to make them. If a, b, and c represent numbers, the relation

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \tag{1}$$

says that the sum of the fractions a/b and c/b is the sum of the numerators over the common denominator, or $(a + c)/b$.

If we let $a = 4$, $b = 3$, and $c = 7$, equation 1 reduces to the sum of the fractions $\frac{4}{3}$ and $\frac{7}{3}$ above, whereas if $a = 1$, $b = 3$, and $c = 2$ we have the last of the above sums. The formula 1 thus represents an unlimited number of equations, because an unlimited number of equations can be obtained from it by assigning different values to a, b, and c.

The result of multiplying numbers together is called the *product* of these numbers. The given numbers are termed the *factors* of the product. Thus the product of the numbers 2, 3, and 7 is $2 \times 3 \times 7$, or 42. The numbers 2, 3, and 7 are factors of 42, that is, of the product $2 \times 3 \times 7$.

Notation for product

To avoid confusion between the symbol $\times$ meaning "times" and the symbol x standing for a number, and to simplify notations, it will be convenient to omit the "times" sign when no confusion will result. For example, $2\,a$ means 2 times a, and $12\,abc$ represents 12 times a times b times c. If, however, you wish to write 3 times 4, you will have to use a times sign because 34 stands for thirty-four.

Reducing fractions

Since $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ have the same value, it is obvious that fractions with different numerators and denominators may be equal to one another. Because it is easier to work with small numbers, the fraction $\frac{1}{2}$ is to be preferred to $\frac{2}{4}$ or $\frac{3}{6}$. In the fraction $\frac{2}{4}$ the numerator and denominator have the common factor 2; and, similarly, for $\frac{3}{6}$ the common factor is 3. Removal of the common

factor by dividing it into numerator and denominator yields $\frac{1}{2}$ in each instance. The process of removing common factors of the numerator and denominator of a fraction is said to be *reducing the fraction.*

Factoring

The process of splitting a number into a product of factors is called *factoring the number.* It is customary in factoring a whole number to use only whole numbers as factors. Thus 462 can be factored as shown:

$$462 = 2 \times 3 \times 7 \times 11.$$

Likewise, the number 75 is a product of the factors 3, 5, and 5.

You can readily see that any whole number can be split up uniquely into a product of whole numbers, each of whose only factors are that number and 1.

Rules for reducing fractions

To reduce a fraction you should follow these steps:

1. *Factor the numerator and denominator* completely so that further factoring with whole numbers other than 1 is not possible.

2. *Remove common factors from numerator and denominator.*

To reduce the fraction $\frac{462}{308}$, we factor numerator and denominator to obtain

$$\frac{462}{308} = \frac{\cancel{2} \times 3 \times \cancel{7} \times \cancel{11}}{2 \times \cancel{2} \times \cancel{7} \times \cancel{11}} = \frac{3}{2},$$

removing common factors as shown. By using algebraic terms we may write

$$\frac{ab}{cb} = \frac{a}{c}, \tag{2}$$

where a, b, and c stand for whole numbers. Thus you can multiply or divide the numerator and denominator of a fraction by the same positive integer (whole number) without changing the value of the fraction. The right side of formula 2 is read "*a over c*."

Lowest common denominator

The rule given in equation 2 may be used to add fractions with different denominators. If we wish to add $\frac{4}{10}$ and $\frac{6}{100}$, we first multiply numerator and denominator of the first fraction $\frac{4}{10}$ by 10 to get $\frac{40}{100}$ because we know (from equation 1) that the denominators of both fractions to be added must be alike. Addition now yields

$$\frac{40}{100} + \frac{6}{100} = \frac{40+6}{100} = \frac{46}{100}.$$

To add $\frac{1}{2}$ and $\frac{1}{3}$ we perform the steps indicated:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}.$$

Naturally, in changing fractions to a common denominator we should choose the smallest possible denominator, since $\frac{1}{5}$ is a simpler result than $\frac{3}{15}$, for example. This smallest possible denominator is called the *lowest common denominator.* For the fractions $\frac{1}{2}$ and $\frac{1}{3}$, the lowest common denominator is 6. Of course, we can add $\frac{1}{2}$ and $\frac{1}{3}$ by writing

$$\frac{1}{2}+\frac{1}{3}=\frac{6}{12}+\frac{4}{12}=\frac{6+4}{12}=\frac{10}{12}=\frac{5}{6},$$

but this involves a final unnecessary operation. The use of the lowest common denominator 6 is simpler. For $\frac{3}{10}$, $\frac{5}{7}$, and $\frac{11}{4}$, the lowest common denominator is $5 \times 7 \times 4$, namely, 140. That 140 is the lowest common denominator of these fractions can be seen from writing the denominators in the factored forms $10=2\times 5$; $7 = 7$; $4 = 2 \times 2$ respectively. Hence, the lowest common denominator is $5 \times 7 \times 2 \times 2$, or 140.

Decimals

The numbers 0, 1, 2, $\cdots$ 9 are called *digits.* Just remember that there can be only ten digits, the same as the number of fingers. To facilitate operations with the quotients of whole numbers, the Babylonians invented the *decimal system.* This system is the process of writing such numbers as 5.2 instead of $5\frac{1}{5}$. In decimals, the position and value of the digit designates a given multiple of one of the following numbers: one, ten, one hundred, and so forth; or one tenth, one hundredth, one thousandth, and so on. The use of several digits in a number indicates that corresponding multiples are to be added together. For example, the decimal 23.784 stands for the sum

$$2 \times 10 + 3 \times 1 + 7 \times \tfrac{1}{10} + 8 \times \tfrac{1}{100} + 4 \times \tfrac{1}{1000}.$$

This sum is the quotient of two positive integers, namely,

$$\frac{23{,}784}{1000}.$$

Decimal numbers

A number written in decimal notation is referred to as a *decimal number.* The ordinary decimal numbers are quotients of whole numbers. We use the word *ordinary* to distinguish the common decimal number from unending decimals such as $0.333 \cdots$ which is understood to be $\frac{1}{3}$. Although this fact will not be proved here, each decimal number, ending, or formed from some point on by the eternal repetition of a block of digits as in 1.23,756,756,756, . . . ,

can be written as an ordinary fraction. Accordingly, many decimal numbers are not new numbers, but rather equivalents of ordinary fractions. The representation 23.784 is easier to write than the long sum given above, and has the advantage over

$$\frac{23{,}784}{1000}$$

of omitting the denominator.

An ordinary fraction can be changed to its *decimal equivalent* by dividing the numerator of the fraction by its denominator. For example, the quotient of 7 by 8 is 0.875 as shown.

$$\begin{array}{r@{}l} & 0.875 \\ 8 \,\big) & \overline{7.0} \\ & \underline{6\,4} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{40} \end{array}$$

Hence 0.875 is the decimal equivalent of the fraction $\frac{7}{8}$.

Subtraction of fractions

The rule for the subtraction of one fraction from another is similar to rule 1 for addition, namely,

(3) $$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Thus, we obtain

$$\frac{12}{5} - \frac{4}{5} = \frac{12-4}{5} = \frac{8}{5}.$$

Fractions with denominators not necessarily the same can be combined according to the following laws:

Cross multiplication

(4) $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}; \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

The denominator bd will often not be the lowest common denominator of the given fractions. Such an example is the following:

$$\frac{7}{10} + \frac{4}{15} = \frac{7 \times 15 + 10 \times 4}{10 \times 15} = \frac{105+40}{150} = \frac{145}{150} = \frac{29}{30}.$$

Use of the lowest common denominator yields

$$\frac{7}{10} + \frac{4}{15} = \frac{7}{2 \times 5} + \frac{4}{3 \times 5} = \frac{7 \times 3}{2 \times 5 \times 3} + \frac{4 \times 2}{3 \times 5 \times 2} = \frac{21+8}{30} = \frac{29}{30}.$$

Combining operations

Operations on several fractions can be performed at once as shown in the example:

$$\frac{5}{8} - \frac{1}{12} + \frac{3}{10} = \frac{75}{120} - \frac{10}{120} + \frac{36}{120} = \frac{75 - 10 + 36}{120} = \frac{101}{120}.$$

In this example the denominators can be factored into

$$2 \times 2 \times 2, \quad 3 \times 2 \times 2, \quad \text{and} \quad 2 \times 5,$$

whence the lowest common denominator is $2 \times 2 \times 2 \times 3 \times 5$, which is 120. Each fraction is changed to one with this denominator by multiplying above and below by the proper factor. The numerators are then combined according to the operations indicated. A method analogous to equation 4 can be used for working the same example:

$$\frac{5}{8} - \frac{1}{12} + \frac{3}{10} = \frac{5 \times 12 \times 10 - 1 \times 8 \times 10 + 3 \times 8 \times 12}{8 \times 12 \times 10}$$

$$= \frac{600 - 80 + 288}{960} = \frac{808}{960} = \frac{101}{120}.$$

Multiplication of fractions

From such examples as

$$2 \times \frac{3}{4} = \frac{6}{4}, \quad \frac{\frac{3}{4}}{3} = \frac{3}{12} = \frac{1}{4},$$

we are led to the rule that the product of two fractions is *the fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of these fractions.* In brief, using numbers, a, b, c, and d, we may write

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}. \tag{5}$$

As an example we obtain the product of the two fractions

$$\tfrac{3}{4} \times \tfrac{7}{8} = \tfrac{21}{32}.$$

Also, since 11 is $\frac{11}{1}$, we have

$$11 \times \frac{7}{8} = \frac{11}{1} \times \frac{7}{8} = \frac{11 \times 7}{1 \times 8} = \frac{77}{8}.$$

Division of fractions

Since division by 8 is the same as multiplication by $\frac{1}{8}$, the fraction $\frac{3}{4}$ divided by 8 is

$$\frac{\frac{3}{4}}{8} = \frac{3}{4} \times \frac{1}{8} = \frac{3 \times 1}{4 \times 8} = \frac{3}{32}.$$

Division of one fraction by another can always be reduced to multiplication by use of the following relations

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{cd}{dc}} = \frac{a}{b} \times \frac{d}{c}. \tag{6}$$

When we change from a fraction a/b to its *reciprocal* b/a, we say that we *invert* a/b. The formulas in equation 6 imply that to divide one fraction by another, we invert the second fraction and multiply the result by the first. In particular,

$$\frac{\frac{3}{4}}{\frac{7}{8}} = \frac{3}{4} \times \frac{8}{7} = \frac{24}{28} = \frac{6}{7}.$$

Per cent

Fractions with denominator 100 occur so frequently that it is desirable to write such fractions in a shorter way than usual. We use the symbol % to mean "divided by 100." Thus $\frac{4}{100}$ is 4%, and $\frac{5}{200}$ is $2\frac{1}{2}$%. The expression 4% is read 4 per cent, an expression derived from the latin "per centum."

The concept of per cent was introduced into mathematics to enable us to compare one number with another more easily. It is often difficult to decide which of two numbers is the larger when these numbers are fractions with different denominators. When the denominators are the same, the comparison is easy, because the larger numerator goes with the larger number. It may be difficult at a glance for you to pick the larger of the numbers $\frac{7}{11}$ and $\frac{8}{12}$. When, however, we write $\frac{7}{11}$ and $\frac{8}{12}$ respectively as

$$\frac{84}{132} \quad \text{and} \quad \frac{88}{132},$$

we can see at once that the second fraction $\frac{8}{12}$ is the larger. The number $\frac{7}{11}$ is about 63.6% and $\frac{8}{12}$ is about 66.7%, from which you can verify that $\frac{7}{11}$ is smaller than $\frac{8}{12}$. Because of the common use of the decimal system it is particularly convenient to employ the denominator 100 as the basis for comparing these numbers.

How to use per cent

To write an ordinary fraction in terms of "per cent," simply divide the denominator into the numerator; then move the decimal point two units to the right in the answer. To change $\frac{7}{11}$ to a decimal we divide as indicated,

$$\begin{array}{r} 0.636 \\ 11\overline{)7.0} \\ \underline{6\,6} \\ 40 \\ \underline{33} \\ 70 \\ \underline{66} \\ 4 \end{array}$$

Moving the decimal in 0.636 two places to the right we have 63.6. Thus $\frac{7}{11}$ is 63.6% approximately. The number 1 is 100%, the

number 2 is 200%, and so on. By 5% of 10 we mean $\frac{5}{100}$ths of 10, or $\frac{1}{2}$. Thus we can operate with fractions written as per cents just as with ordinary fractions. For example, 10% of 50% is 5%, or

$$\frac{10}{100} \times \frac{50}{100} = \frac{500}{10{,}000} = \frac{1}{20}.$$

Ratio

A fraction may be thought of as a way of comparing the sizes of two numbers. This comparison is emphasized in mathematics by writing the given fraction, say a/b, as $a : b$, read as the *ratio of a to b*. Hence $3 : 4$ is the ratio of 3 to 4, that is, the fraction $\frac{3}{4}$.

The splitting of 28 into the unequal parts 12 and 16 is a divi-

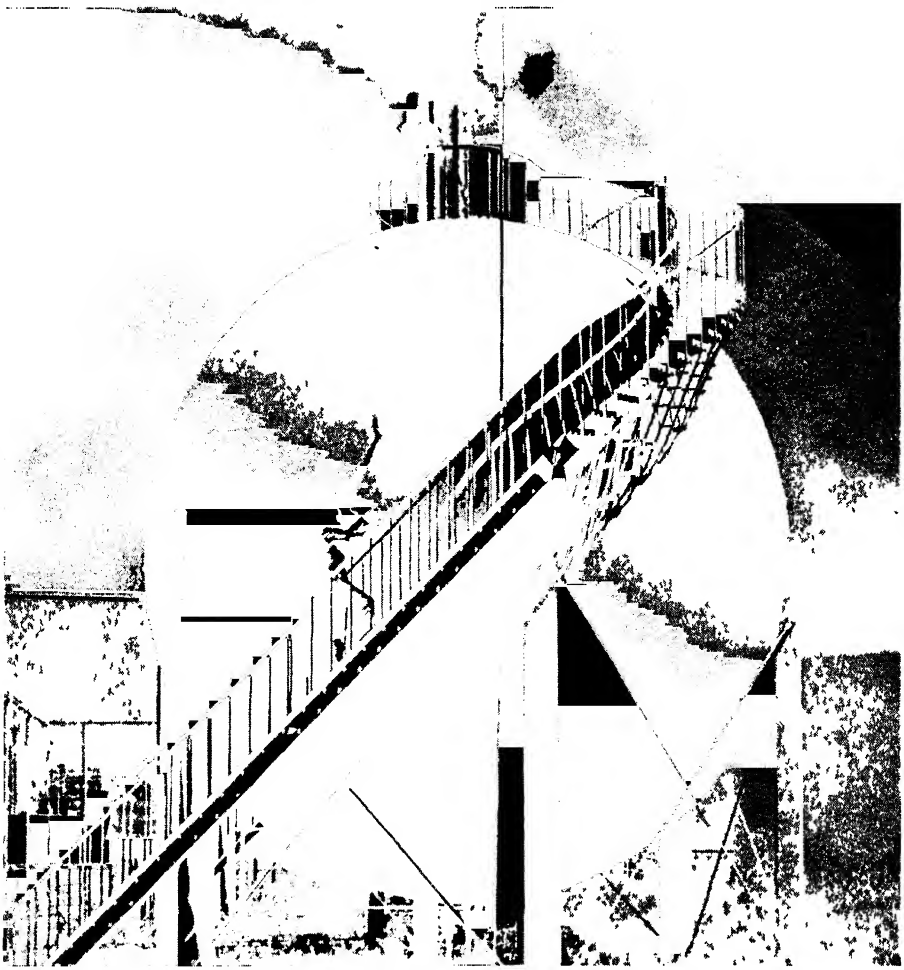


FIG. 215. GEOMETRICAL SHAPES ARE FOUND IN INDUSTRY

sion of the number 28 into two parts whose ratio is 3 : 4. If one gear weighs 2 lb. and another $5\frac{1}{2}$ lb., the ratio of their weights is $2 : 5\frac{1}{2}$, or 4 : 11.

The equality of two ratios is a *proportion.* It follows that

Proportion

$$\tfrac{2}{3} = \tfrac{4}{6}$$

is a proportion. This equality is read "2 is to 3 as 4 is to 6." If the values which x and y can take on in a given problem are such that the ratio $x : y$ is constant, y is said to be *directly proportional* to x.

Since the circumference c of a circle divided by the diameter d is a constant π, the ratio $c : d$ is π, and d is directly proportional to c. Also, c is directly proportional to d.

If y is directly proportional to x, we also say that *y varies directly as x.* If y varies directly as $1/x$, we say that *y varies inversely as x.*

Variation

We will let p denote the pressure of a gas, and v the volume, keeping the temperature of the gas fixed. The gas then satisfies Boyle's law (discussed on pp. 67–68), that is, for a constant k whose value depends only on the kind and amount of gas used,

$$p = \frac{k}{v}, \quad \text{or} \quad pv = k. \tag{7}$$

The constant k is called the *constant of proportionality.* For example, if the pressure of a given weight of gas is known to be 20 lb. per square inch when the volume is reduced to 1000 cu. in., substitution of 20 for p and 1000 for v gives

Constant of proportionality

$$k = 20 \times 1000 = 20{,}000 \text{ (units being inch pounds).}$$

Knowing k, we can now find the pressure of the gas for any volume. Such a problem arises when we change the volume of gas in a cylinder by moving a piston in the cylinder. With $v = 2000$, the formula

$$p = \frac{20{,}000}{v} = \frac{20.000}{2000} = 10$$

implies that the pressure p is 10 lb. per square inch, whereas with a volume v of 500 cu. in., the pressure p is increased to 40 lb. per square inch.

Boyle's law is a special case of a general gas law,

General gas law

$$pv = RT, \tag{8}$$

where T is the *absolute temperature.* (See p. 69.) Here T varies directly as p and directly as v. On the other hand, p varies directly as T and inversely as v.

If T is 492° absolute temperature on the Fahrenheit scale when p is 14.7×144 lb. per square foot and v is 12.39 cu. ft., the value of R in the general gas law is

$$R = \frac{14.7 \times 144 \times 12.39}{492} = 53.3,$$

where R is in foot-pounds per degree absolute on the Fahrenheit scale. This value of R applies to air.

From the formula

$$pv = 53.3\ T$$

for the given gas, all possible sets of values of p, v, and T can be found by substitution. If the pressure p is 30 lb. per square foot and the absolute temperature T is 200°, the volume v in cubic feet is

$$v = \frac{53.3 \times 200}{30} = \frac{10{,}660}{30} = 355\tfrac{1}{3}.$$

Unknowns

It is often desirable to solve a proportion for one of the terms. For example, for what value of the *unknown* x is

$$\frac{2}{x} = \frac{4}{7}?$$

Multiplying both sides by the lowest common denominator $7x$, and dividing by 4, we obtain the solution for x, as shown:

$$x = \frac{7 \times 2}{4} = \frac{7}{2}.$$

ZERO AND THE NEGATIVE NUMBERS

Zero

You are so accustomed to working with the number zero that it probably seems to be the easiest number to employ, and you might assume therefore that it was one of the first numbers used in arithmetic. On the contrary, the Hindus introduced a notation for it around 500 A.D. They treated zero like the other numbers, multiplying, adding, and subtracting with it as with the whole numbers. *The number zero is, by definition, an integer.*

Negative numbers

The Hindus found that relations such as

$$x + 1 = 0, \quad x + 2 = 0$$

cannot be solved for x by the use of those numbers described up to this point. Wishing to assign a meaning to such equations, and also desiring to think of such expressions as

$$\tfrac{1}{2} - \tfrac{3}{2}$$

as numbers, the Hindu mathematicians were led to devise negative quantities. The whole numbers used above are positive. The quotient of any two of these is also said to be positive. These are examples of positive numbers which will be defined later.

Given a positive number a, the *negative number* $-a$ is by definition the solution for x from the equation

$$x + a = 0. \tag{9}$$

When we say "solution," we mean that whenever we come across the sum $(-a) + a$, we can replace this sum by zero. We have put parentheses around the number $(-a)$ to separate this number from the rest of the text. It has no other significance. The sum $(-4) + 4$ is thus zero.

Negative integers

The negative integers are the numbers $-1, -2, -3, \cdots$, and so on; the whole numbers 1, 2, 3, etc. used above are the positive integers. The negative and positive integers and the number zero comprise the class of numbers ordinarily called *integers*.

Negative numbers enable us to subtract a number from a smaller number conveniently. Thus you can write $8 - 12$ as -4. If the temperature is 8° F. and drops 12° F., the resulting temperature is $(8 - 12)$° F., which is -4° F. The Hindu mathematicians thought of positive numbers as *possessed* and negative numbers as *owed*. In this sense if we have \$200 on hand but owe \$400 our "assets" amount to $-$ \$200.

Meaning of zero

The number zero has caused considerable difficulty because it cannot always be employed like the other integers. It can be shown that if we say that $\frac{0}{0}$ is 1 we soon arrive at a contradiction in our mathematics. To give some meaning to $\frac{0}{0}$, Isaac Newton (1642–1727), Gottfried Wilhelm Leibniz (1646–1716), and others introduced the branch of mathematics called the *calculus*.

Multiplication by zero

We shall illustrate with numbers a method whereby we can show that *zero times any number is zero*. Let us consider 0×5. We can write 0 as $1 - 1$. You can easily verify that the law which follows applies for all positive integers a, b, and c, and in fact holds for all numbers.

$$(a - b) \times c = ac - bc. \tag{10}$$

Therefore, we may write

$$(5 - 2) \times 3 = 5 \times 3 - 2 \times 3,$$

or, in words, the quantity (5 − 2) times 3 is the same as 5 times 3 minus 2 times 3. Applying this law to the product of 0 and 5 by writing zero as (1 − 1), we get

$$(1-1)\times 5 = 1\times 5 - 1\times 5 = 0.$$

Therefore, $0 \times 5 = 0$ and we conclude that the product of zero and any other number is also zero.

Division by zero

We shall show why *division by zero is not allowed* in mathematics. That a number not zero cannot be divided by zero can be illustrated with the fraction $\frac{5}{0}$. We recall that c is the quotient of a number a by a number b if $a = bc$. It follows that if there is a number m such that

$$\frac{5}{0} = m,$$

then we may write

$$5 = 0 \times m.$$

Contradiction

But since $0 \times m = 0$ instead of 5, we have arrived at a contradiction. Thus we conclude that there is no number m equal to $\frac{5}{0}$.

Suppose now that there is a number m such that

$$\frac{0}{0} = m.$$

From the meaning of *quotient* this expression may be rewritten as

$$0 = 0 \times m.$$

The last relation is satisfied by *any number m*. It would be absurd to say that $\frac{0}{0}$ is any number, such as 10, − 15, and 100, all at the same time. Therefore, $\frac{0}{0}$ is not a number.

Addition and subtraction with zero

We have treated multiplication and division by zero. To complete the picture you will note that if you add zero to or subtract zero from a given number, you again obtain this same number.

Signs

From the rules $(-1) \times a = -a$, $(-1) \times (-1) = +1$, and $(-1) \times (+1) = -1$, the *product of two negative numbers is positive as is the product of two positive numbers*, whereas the *product of a positive number and a negative number is negative*. The same remarks hold for division since the division of a by b can be thought of as the product of a by the reciprocal $(1/b)$ of b. If no plus or minus sign is written before a number, the sign before the number is understood to be plus.

Complex fractions

The sign conventions and procedures for handling fractions given above enable us to simplify fractions whose numerators and denominators may themselves contain fractions. Thus

$$\frac{\frac{2}{3}\times\frac{9}{8}-\frac{5}{6}}{\frac{4}{9}-\frac{1}{3}\times 2}=\frac{\frac{3}{4}-\frac{5}{6}}{\frac{4}{9}-\frac{2}{3}}=\frac{\frac{9}{12}-\frac{10}{12}}{\frac{4}{9}-\frac{6}{9}}=\frac{-\frac{1}{12}}{-\frac{2}{9}}=\frac{1}{12}\times\frac{9}{2}=\frac{3}{4\times 2}=\frac{3}{8}.$$

Similarly,

$$\frac{2+\cfrac{1}{3-\cfrac{4}{1+6}}}{\frac{1}{8}}=\frac{2+\cfrac{1}{3-\frac{4}{7}}}{\frac{1}{8}}=\frac{2+\cfrac{1}{\cfrac{21-4}{7}}}{\frac{1}{8}}=\frac{2+\frac{7}{17}}{\frac{1}{8}}=\frac{\cfrac{34+7}{17}}{\frac{1}{8}}$$

$$=\frac{\frac{41}{17}}{\frac{1}{8}}=\frac{41}{17}\times\frac{8}{1}=\frac{328}{17}.$$

Rational numbers

Any number that can be written as the quotient of two integers (division by zero excluded) is called a *rational number*. The numbers

$$0,\ 2,\ -3,\ \tfrac{4}{3},\ 1.741,\ 0.333\cdots,$$

are rational numbers. The number zero is rational because it can be written as $\frac{0}{1}$. Examples of numbers which are not rational will be given presently.

Number systems

Algebra is primarily concerned with the addition, subtraction, multiplication, and division of numbers (division by zero excluded). Under all of these operations, rational numbers always yield rational numbers. The rational numbers are the ordinary fractions of arithmetic, negative integers being included in the numerators and denominators. Because rational numbers with the above operations go into rational numbers, the rational numbers are said to form a *system of numbers*. When we work with them and limit our mathematical operations to those already mentioned, we are in a sort of prison from which we cannot escape.

REAL NUMBERS

Greek mathematics

The quotients of positive integers met the needs of the Egyptians and Babylonians who made use of their knowledge of mathematics in the construction of the pyramids, the evolution of the calendar, and other advances in astronomy and engineering. Much later the Greeks, who were greatly interested in mathematics for its own sake, continued investigations into the properties of numbers and added much to the knowledge of them.

Through its interest in the Pythagorean theorem, one of the most significant accomplishments in mathematics, the school of

Pythagoras about 500 B.C. was led to devise "irrational numbers," which will be discussed later. The importance of the Pythagorean theorem is evident when we consider that it is used in marine and aeronautical navigation, land surveying, airplane design, bridge construction, and indirectly to carry out innumerable functions of modern life.

Geometrical concepts

To state the Pythagorean theorem we shall need some concepts from plane geometry. If you remove from a straight line everything to one side of a fixed point P on the line, you obtain a *ray* as shown in Fig. 216. The point P is the initial point of the ray.

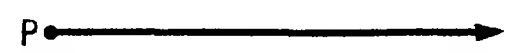


FIG. 216. RAY

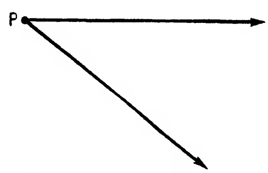


FIG. 217. ANGLE

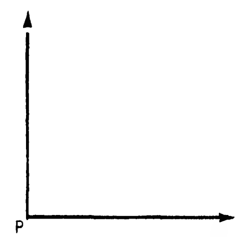


FIG. 218. RIGHT ANGLE

Angles and triangles

Two rays with the same initial point form an *angle* as shown in Fig. 217. Two perpendicular rays form a *right angle* as in Fig. 218. Thus the hands of a clock at 12 : 15 form a right angle. A geometric figure bounded by three lines is said to be a *triangle* because the sides form three angles. Two triangles are shown in Fig. 219.

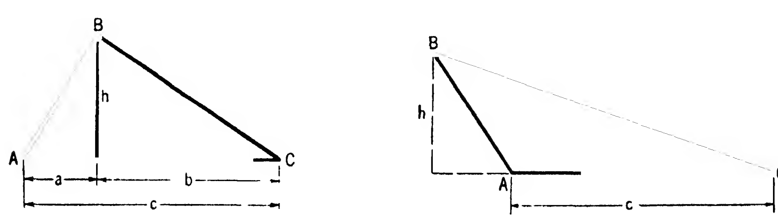


FIG. 219. TRIANGLES ABC WITH ALTITUDE h AND BASE c

The points A, B, and C of either triangle in Fig. 219, where the lines intersect, are called the *vertices* of the triangle. The line segments AC, CB, and BA are called the *sides* of the triangle. The notation AC used here designates the line segment from the point A to the point C. The entire triangle is designated as the triangle ABC. The angle "formed" by the sides AB and AC is called the

angle A. Sometimes this angle is termed the *angle CAB*. A triangle with a right angle is called a *right triangle*. The triangle of Fig. 220 is a right triangle with right angle C and sides designated by a, b, and c. The angles A, B, and C are opposite the sides a, b, and c respectively. This is a convention often used in treating geometric figures. We shall also let a denote the length of the side BC. The side c opposite the right angle C is termed the *hypotenuse* of the right triangle. *Right triangle*

We know that $4^2 = 4 \times 4$, or in general terms a^2 (a "squared") means a times a. The *Pythagorean theorem* states that *the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides*. In terms of the right triangle of Fig. 220 this implies that *Pythagorean theorem*

$$a^2 + b^2 = c^2. \tag{11}$$

The Pythagorean theorem is of such importance that proof of it will be given. Since this proof will be based on the use of areas, we shall need a few special geometrical concepts.

The height of a triangle is called the *altitude* of the triangle. The side AC of each triangle in Fig. 219 is the *base*, a term to be used also for the length of the side AC. The vertical line in the left triangle of Fig. 219 cuts this triangle into two right triangles. In Fig. 221 two triangles identical with the triangle of Fig. 220 are adjoined to build up a rectangle. The area of this rectangle is ab. The area of the triangle in Fig. 220 is therefore said to be $\frac{1}{2}(ab)$. *Areas*

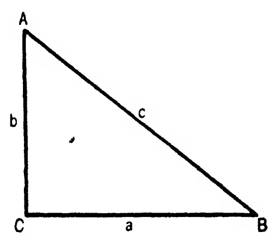


Fig. 220. Right triangle

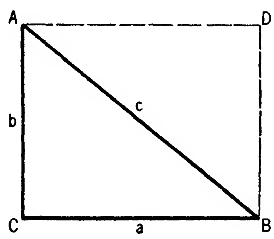


Fig. 221. Rectangle cut into right triangles

The areas of the right triangles in the left triangle of Fig. 219 are $\frac{1}{2}(ah)$ and $\frac{1}{2}(bh)$ respectively. The area of the entire triangle is thus

$$\frac{1}{2}\, h(a + b).$$

A similar treatment holds for the other triangle of Fig. 219. In words, the *area of a triangle is one-half of the altitude times the base*.

If a triangle has a base which is 3 in. in length and an altitude of 4 in., the area of the triangle is $\frac{1}{2} \times 3 \times 4$, or 6 sq. in.

Proof of Pythagorean theorem

In Fig. 222 squares have been placed on the sides of a right triangle. Using the same type of notation for rectangles as for triangles, we designate the square on the hypotenuse to be *ABDE*. This square has the area c^2, whereas the squares *BCHK* and *AFGC* have the areas a^2 and b^2 respectively. To prove that $c^2 = a^2 + b^2$ we therefore wish to show that the area of the square *ABDE* is the sum of the areas of the squares *BCHK* and *AFGC*.

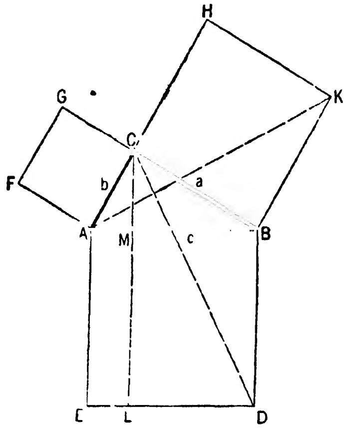


FIG. 222. RIGHT TRIANGLE BOUNDED BY SQUARES

The broken line *CL* is drawn parallel to the line *AE*. Broken lines join the points *A* and *K*, and the points *C* and *D* respectively. The side *DB* of the triangle *DCB* is identical in length with the side *AB* of the triangle *AKB* because they are sides of the same square. Similarly, the side *BC* of the triangle *DCB* is identical with the side *BK* of the triangle *AKB*. The angles *DBC* and *ABK* are the same; that is, one can be superimposed on the other because each is equal to a right angle plus the angle *ABC*.

Geometrical relations

It follows that the triangle *DCB* can be superimposed on the triangle *AKB*, for they are identical. The side *BK* of length a can be taken to be the base of the triangle *AKB*. The altitude of the triangle *AKB* is the distance *CB*, which is also a. The area of the triangle *AKB* is thus $\frac{1}{2}a^2$. The triangle *DCB* has the base *BD* of length c and altitude *LD*. Since this triangle has the same altitude and base as the rectangle *LMBD*, the area of the rectangle *LMBD* is twice the area of the triangle *DCB*, and hence is a^2.

We have thus proved that the area of the rectangle *LMBD* is the same as the area of the square *BCHK*, namely a^2. In the same way we can show that the area of the rectangle *AMLE* is equal to the area of the square *AFGC*, that is, b^2. Since the area of the square *ABDE* is the sum of the areas of the rectangles *AMLE* and *LMBD*, we have shown that $c^2 = a^2 + b^2$, as stated in equation 11, is true.

Q.E.D.

The sides of a right triangle distinct from the hypotenuse are called the *legs*. We recall that $\sqrt{29}$ means the *square root* of 29. The square root of 29, or $\sqrt{29}$, is a number whose square is 29. If the lengths of the legs of a right triangle are 2 and 5 respectively, the Pythagorean theorem states that the hypotenuse of the triangle has the length $\sqrt{2^2 + 5^2}$, that is $\sqrt{4 + 25}$, or $\sqrt{29}$. We shall show later how to approximate $\sqrt{29}$ by a decimal number.

Irrationality of $\sqrt{2}$

It was early discovered that if the legs of a right triangle are both equal to one, the length $\sqrt{2}$ of the hypotenuse cannot be written as a rational number, that is, the quotient of one integer by another. We can prove this as follows.

We start by assuming that $p/q = \sqrt{2}$, where p and q are positive integers with no common factor other than 1. Squaring both sides of the above equation, and then multiplying both sides by q^2, we obtain the equation

$$p^2 = 2\,q^2.$$

If q contains a factor 2, the square of q contains this factor twice. Thus the square of a positive integer can contain only an even number of factors equal to 2. For example, if $q = 2 \times 2 \times 2 \times 3$, the square of q has six factors equal to 2. It follows that $2\,q^2$ has an odd number of factors equal to 2, while p^2 has an even number, if any at all. We have thus arrived at a contradiction, for which reason we conclude that *$\sqrt{2}$ cannot be written as the quotient of two positive integers, and in fact, of any two integers.*

Points on lines

Since there are numbers, like $\sqrt{2}$, which are not rational, the number $\sqrt{2}$ and the rational numbers belong to a larger system which we shall describe but not define because an exact treatment requires some of the most advanced concepts of mathematics. René Descartes (1596–1650), philosopher and mathematician, wished to be able to distinguish between the points on a line just as we differentiate people by their names, or by associating numbers as we do with automobiles in a given state. Of course, you can count all the automobiles, and designate each one by a distinct positive integer. But not all the points on a line can be counted, and it is, therefore, impossible to assign a distinct integer to each point.

Numbering points

Descartes marked off equal distances from a fixed point 0 on a line as shown in Fig. 223, and to each one of these marked points he assigned an integer, namely, the *directed distance* from the fixed

point 0. He used the distance from 0 to the point marked 1 as the unit of measure. The direction from 0 was taken care of by the sign + or − of the integer, the plus sign referring to a point to the right of 0, minus referring to the left. Descartes then further subdivided the line segments between the marked points, assigning to the point midway between 0 and 1 the value $\frac{1}{2}$, to the point midway between 0 and −1 the value $-\frac{1}{2}$, to the point one-third of the distance from 1 to 2 the value $\frac{4}{3}$, and so on. Thus to each rational number there is an assigned point on the line.

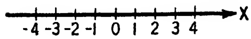


FIG. 223. COORDINATE AXIS

Real numbers

When the rational numbers are connected in this manner to points on the line, there are still points left over in the mathematical sense. There is certainly a point between 1 and 2 to which $\sqrt{2}$ may be assigned, and the number $\sqrt{3}$ also goes with a point in this range. We recall that the length of the circumference of a circle is the length of the diameter times π. This famous number *pi* (3.14159 to five decimal places) must be assigned to a point between 3 and 4.

The numbers thus associated with the points on a line are called the *coordinates* of the points on the line, and the coordinates of the points on a line form what are called the *real numbers*. Real numbers which are not rational are termed *irrational*. The coordinates of points to the right of the *origin* 0 are the *positive numbers*, and coordinates to the left are the *negative numbers*.

Irrationality of π

Whether or not π is rational is a problem which baffled the best mathematicians for two thousand years. Solved only about sixty years ago by Ferdinand Lindemann, who worked with exact determinations of π obtained by means of the calculus, Lindemann showed that π cannot be expressed as a quotient of two integers, and is therefore irrational.

The real number system

In advanced mathematics it is proved that the addition, subtraction, multiplication, and division (division by zero excluded) of real numbers always yields real numbers, so that *the real numbers form a system*. This system satisfied scientists and engineers until the Middle Ages, when imaginary numbers were discovered. Imaginary numbers, associated with the $\sqrt{-1}$, will be discussed later.

New number systems

The process of discovering new systems of numbers is still going on. Such systems were originally and are still likely to be developed only by mathematicians interested mainly in the philosophy of

mathematical concepts. Later such fundamental studies become of material benefit to humanity as practical applications are found by scientists and engineers.

FUNDAMENTAL OPERATIONS OF ALGEBRA

Powers

The product $2 \times 2 \times 2 \times 2 \times 2 \times 2$ is written as 2^6 for the sake of brevity. More generally, a product of n successive a's is said to be the nth power of a, and is written as a^n. In this notation 1,000,000 is written as 10^6.

The product of the nth power of a by the mth power of a is a product of $(n + m)$ factors, all equal to a. This gives the rule:

$$a^n \times a^m = a^{n+m}. \tag{12}$$

For example,

$$5^4 \times 5^3 = (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5) = 5^7.$$

Terms

You have perhaps already become accustomed to the use of the term *factor* for any number multiplied by another. There is also a name for a number added to another, namely, the word *term* itself. In the expression

$$2\,x^2 + 3\,ay + 5$$

the quantities $2\,x^2$, $3\,ay$, and 5 are terms; 2, x^2, 3, a, and y are factors, not of the whole expression, but of these terms.

Letters representing numbers

It is to be emphasized that when letters are used in expressions such as the above, they represent numbers from a system with which we are understood to be working, and not merely abstract symbols. Unless otherwise specified, in this chapter such letters will always represent numbers from the *real number system*. For example, the product ab is the product of a real number a by a real number b, and not the product of one letter by another. Algebra is not the study of letters, but rather the study of numbers. The use of letters merely enables us to treat all numbers from a given class at the same time.

Products

The factors in the product

$$(2 + 6 + 4)(5 + 7 + 1)$$

are 12 and 13 respectively, whence the product has the value 156. If we multiply each term of one factor by each term of the other and add the resulting terms, we have

$$2 \times 5 + 2 \times 7 + 2 \times 1 + 6 \times 5 + 6 \times 7 + 6 \times 1 + 4 \times 5 + 4 \times 7 \\ + 4 \times 1 = 10 + 14 + 2 + 30 + 42 + 6 + 20 + 28 + 4,$$

which also adds up to 156. This illustrates the principle which follows:

To multiply one expression by another, multiply each term in the first expression by each term in the second expression, and add the resulting products.

Thus,

$$(3 + 2x + a)(2x + 4) = 6x + 12 + 4x^2 + 8x + 2ax + 4a$$
$$= 12 + 4a + 14x + 2ax + 4x^2.$$

This rule also applies with negative terms. For example,

$$(2 - 4)(3 + 7 - 1) = 2 \times 3 + 2 \times 7 - 2 \times 1 - 4 \times 3 - 4 \times 7 + 4 \times 1$$
$$= 6 + 14 - 2 - 12 - 28 + 4 = 24 - 42 = - 18.$$

Arrangement of terms in products

If the terms in the factors of a product involve powers of the same number, say x, the multiplication can be conveniently performed by arranging the terms in descending powers of the number, as in the following illustration of the procedure for multiplying $1 + 4x^2 + 7x$ by $x + 4$.

$$\begin{array}{rrrr}
4x^2 & + 7x & + 1 & \\
x & + 4 & & \\
\hline
4x^3 + & 7x^2 + & x & \\
 & 16x^2 + & 28x + & 4 \\
\hline
4x^3 + & 23x^2 + & 29x + & 4
\end{array}$$

It will be observed that in the computation above, terms involving the same powers of x are arranged in the same vertical column for convenient addition. Multiplication of $x - 2 - 3x^2 + x^4$ by $x^2 + 2x^3 + 1$ can be performed as follows.

$$\begin{array}{llllllll}
x^4 & & - 3x^2 & + x & - 2 & & & \\
2x^3 & + x^2 & + 1 & & & & & \\
\hline
2x^7 & & - 6x^5 & + 2x^4 & - 4x^3 & & & \\
 & x^6 & & - 3x^4 & + x^3 & - 2x^2 & & \\
 & & & x^4 & & - 3x^2 & + x & - 2 \\
\hline
2x^7 & + x^6 & - 6x^5 & & - 3x^3 & - 5x^2 & + x & - 2
\end{array}$$

The product of the expressions in this case is

$$2x^7 + x^6 - 6x^5 - 3x^3 - 5x^2 + x - 2.$$

Procedure

In carrying out the multiplication we have first taken the product of $2x^3$ by each term in the first expression, obtaining thus the first line of terms under the long bar. The second line of terms

under this bar is the result of multiplying x^2 by each term of the first expression, whereas the third line of terms, just above the bottom bar, is the product of 1 by the first expression. The bottom row is derived by adding the like terms in each vertical column between the two bars.

Long division

On the left below we have carried out the ordinary long division of 32,342 by 183.

$$
\begin{array}{r|l}
 & 176 \\ \hline
183 & 32342 \\
 & 183 \\ \hline
 & 1404 \\
 & 1281 \\ \hline
 & 1232 \\
 & 1098 \\ \hline
 & 134
\end{array}
\qquad
\begin{array}{r|l}
 & 1 \times 10^2 + 7 \times 10 + 6 + 134/183 \\ \hline
183 & 32342 \\
 & 18300 \\ \hline
 & 14042 \\
 & 12810 \\ \hline
 & 1232 \\
 & 1098 \\ \hline
 & 134
\end{array}
$$

You will find that this procedure of long division is just a system of trial divisions with intermediate subtractions. On the right we have written the quotient in terms of powers of 10. First we find the largest multiple of 10^2 which when multiplied by 183 gives a number not greater than 32,342. This multiple of 10^2 is 1×10^2 because 100×183 is less than 32,342 while 200×183 is greater than 32,342. Subtracting 100×183 from 32,342 we find that we have 14,042 left over. We next find the largest multiple of 10 such that the product of this multiple by 183 is not greater than the remainder 14,042. The desired multiple of 10 is 7×10 because 70×183 is less than 14,042, while 80×183 is more than 14,042. Subtracting 70×183 (or 12,810) from 14,042 we have 1232 left over. Thus far we have 170 in the quotient. We must add another 6 to the quotient since 183 will go into 1232 six times, with a remainder 134. We add the *remainder* 134 divided by the *divisor* 183 to the *quotient*. To check this computation multiply the divisor 183 by the quotient $(176 + \frac{134}{183})$.

Remainder

Division of algebraic expressions

The standard procedure of dividing one algebraic expression by another is like the long division method above. The system of arranging terms according to descending powers of one number, as used in the multiplication of algebraic expressions above, is also helpful in dividing one algebraic expression by another.

We shall divide $1 + x^3 - 3x^2$ by $2x + x^2 + 3$. The computation performed below will be explained step by step.

Algebraic division

$$
\begin{array}{rl}
\text{(divisor)} & \quad\quad\quad x^3 - 2x^2 + x + 1 \quad \text{(quotient)} \\
x^2 + 2x + 3 & \overline{)\, x^5 \qquad\qquad\qquad - 3x^2 \qquad\quad + 1} \quad \text{(dividend)} \\
 & \underline{x^5 + 2x^4 + 3x^3} \\
 & \quad - 2x^4 - 3x^3 \\
 & \quad \underline{- 2x^4 - 4x^3 - 6x^2} \\
 & \qquad\qquad x^3 + 3x^2 \\
 & \qquad\qquad \underline{x^3 + 2x^2 + 3x} \\
 & \qquad\qquad\qquad x^2 - 3x \\
 & \qquad\qquad\qquad \underline{x^2 + 2x + 3} \\
 & \qquad\qquad\qquad\quad - 5x - 2 \quad \text{(remainder)}
\end{array}
$$

The quotient of $1+x^5-3x^2$ by $2x+x^2+3$ is x^3-2x^2+x+1 with the remainder $(-5x-2)$. In the computation above, the dividend $x^5 - 3x^2 + 1$ is written immediately under the bar, and the divisor $x^2 + 2x + 3$ is written to the left of the dividend.

First term of quotient

The first term x^2 of the divisor is divided into the first term x^5 of the dividend yielding the first term of the quotient x^3 written above the bar. The quotient term x^3 is now multiplied by the divisor and the result written under the dividend, a bar being placed beneath. The quantity just above the bar is subtracted from the dividend giving us the *temporary remainder* as indicated below and $-3x^2 + 1$, which is not repeated.

$$
\begin{array}{rl}
\text{(divisor)} & \; x^3 \quad \text{(first term of quotient)} \\
x^2 + 2x + 3 & \overline{)\, x^5 \qquad\qquad\quad - 3x^2 \qquad + 1} \quad \text{(dividend)} \\
 & \underline{x^5 + 2x^4 + 3x^3} \\
 & \quad - 2x^4 - 3x^3 \quad \text{(temporary remainder)}
\end{array}
$$

It is to be emphasized that appropriate *spaces* be left between the terms of the dividend to take care of *missing terms*.

Procedure

The first term x^2 of the divisor is next divided into $-2x^4$, the first term of the temporary remainder, and the result becomes the second term of the quotient. This quotient term $-2x^2$ is multiplied by the divisor and the result placed underneath the temporary remainder $-2x^4 - 3x^3$. Subtraction (top of page) yields $x^3 + 3x^2 + 1$, of which 1 is omitted from the new temporary remainder, and the procedure of division by the first term x^2 of the divisor is repeated. This process is carried on until the remainder $-5x - 2$ is obtained. Since x^2 does not "divide into" $-5x$, the division ends.

Checking a division

To check the above division, simply multiply the *quotient* $x^3 - 2x^2 + x + 1$ by the *divisor* $x^2 + 2x + 3$ and add the *remainder* $-5x - 2$. The result should be the *dividend* $x^5 - 3x^2 + 1$.

Factoring

In dealing with quotients of numbers we showed that fractions may be simplified by removing factors possessed in common by numerator and denominator. It is therefore desirable to have rules for finding the factors of a given algebraic expression. Among these the following are particularly helpful since they occur frequently in engineering calculations:

$$ab + ac = a(b + c) \tag{13}$$

$$a^2 + 2\,ab + b^2 = (a + b)^2 \tag{14}$$

$$a^2 - 2\,ab + b^2 = (a - b)^2 \tag{15}$$

$$a^2 - b^2 = (a - b)(a + b) \tag{16}$$

Proof

Rule 13 has already been used above. We can verify rule 14 by performing the multiplication

$$\begin{array}{r} a + b \\ \underline{a + b \quad\;\;} \\ a^2 + \;\;ab \quad\;\; \\ \underline{ab + b^2} \\ a^2 + 2\,ab + b^2 \end{array}$$

The others may be established in the same way. In Fig. 224 the area of the whole square is $(a + b)^2$. Adding the areas of the regions into which this square is cut up, you will obtain another verification of rule 14,

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$

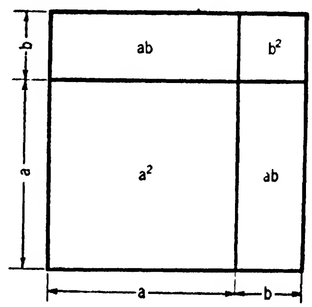


FIG. 224. FIGURE TO ILLUSTRATE
$(a + b)^2 = a^2 + 2\,ab + b^2$

Examples of factoring

Some illustrations of the above rules follow.

$$\begin{array}{rcll} x^2 + 5\,x &=& x(x + 5) & \text{(Rule 13)} \\ xy + y^2 - 2y &=& y(x + y - 2) & \text{(Rule 13)} \\ x^2 + 6\,x + 9 &=& (x + 3)^2 & \text{(Rule 14)} \\ 4\,a^2 - 20\,ab + 25\,b^2 &=& (2\,a - 5\,b)^2 & \text{(Rule 15)} \\ 4\,x^2 - 9 &=& (2\,x - 3)(2\,x + 3) & \text{(Rule 16)} \end{array}$$

The following examples involve more than two factors.

$$\begin{aligned} x^3 + 3\,x^2 - 4\,x - 12 &= x(x^2 - 4) + 3(x^2 - 4) = (x + 3)(x^2 - 4) \\ &= (x + 3)(x - 2)(x + 2) \end{aligned}$$

$$\frac{x^2 - 4}{2\,x + 4} = \frac{(x - 2)\cancel{(x + 2)}}{2\cancel{(x + 2)}} = \frac{x - 2}{2}$$

SQUARE ROOTS

Accuracy

The problem of finding the hypotenuse of a right triangle is only one of many places where the square root of a number is

used. With the slide rule (to be described later), square roots can be obtained to three places at most. We shall describe the standard process for the extraction of the square root of a number to as many places as desired. As in the case of division, this process is one of successive subtractions of properly chosen quantities.

General square

Rule 14 above gives the square of a sum of two terms a and b. This rule can be generalized to apply to the square of the sum of any number of terms, as follows.

The *square of a sum is the sum of the squares of the terms plus the sum of twice all possible products which can be formed by multiplying distinct terms.*

For example,

$$\begin{aligned}(2 + 3 + 5)^2 &= 2^2 + 3^2 + 5^2 + 2(2 \times 3) + 2(2 \times 5) + 2(3 \times 5)\\ &= 4 + 9 + 25 + 12 + 20 + 30 = 100.\end{aligned}$$

Similarly, $(a - 3)^2$ is $a^2 + 9 - 6a$. (See rule 15.)

Extraction of square root

We shall illustrate the process of extracting the square root of a number, and then show why it is correct. You will recall that a positive integer which is the square of another integer is termed a *perfect square*. For instance, 1, 4, 9, and 16 are perfect squares.

To obtain the square root of a number such as 52,346.008 you should perform the steps below.

1. Starting at the decimal point mark off successive pairs of digits in each direction by means of primes. You get

$$5'23'46.00'8.$$

First digit

2. Find the largest perfect square which is less than or equal to the number to the left of the first prime. This perfect square is 4 in our example.

3. Write this perfect square, 4, beneath the number, and again as a divisor to the left below. So far you have

$$\begin{array}{r|l} & \overline{5'23'46.00'8} \\ & 4 \\ \hline 4 & \end{array}$$

4. Place the square root 2 of the perfect square above the given number. You have

$$\begin{array}{r|l} & \ 2 \\ & \overline{5'23'46.00'8} \\ & 4 \\ \hline 4 & \end{array}$$

5. Perform the indicated subtraction and bring down the next two digits. This gives

$$
\begin{array}{r|l}
 & 2 \\
 & \overline{5'23'46.00'8} \\
 & 4 \\
\cline{2-2}
4 & 1\ 23
\end{array}
$$

Second digit

6. Choose a digit x to follow the digit 4 in the divisor so that $(40 + x)$ times x is the largest integer which is still less than or equal to the number beneath the last bar. For the case at hand the number beneath the bar is 123. When x is 2 the product $(40 + x)x$ or 42 by 2 is 84, which is less than 123, whereas the product of 43 by 3 is 129, which is too great. The desired value of x is thus 2.

7. Place the digit 2 after the digit 4 as a new divisor at the left and after the first 2 at the top, and write 2×42 or 84 beneath 123. You then have

$$
\begin{array}{r|l}
 & 22 \\
 & \overline{5'23'46.00'8} \\
 & 4 \\
\cline{2-2}
42 & 1\ 23 \\
 & \underline{\ \ 84}
\end{array}
$$

8. Perform the indicated subtraction. The result is 39. Bring down the next two digits, double the partial root 22 and write the divisor 44 to the left. Hence, you now have

$$
\begin{array}{r|l}
 & 22 \\
 & \overline{5'23'46.00'8} \\
 & 4 \\
\cline{2-2}
42 & 1\ 23 \\
 & \ \ 84 \\
\cline{2-2}
44 & 39\ 46
\end{array}
$$

Third digit

9. Choose a digit x so that $(440 + x)$ times x is no larger than the number 3946 beneath the bottom bar, but $[440 + (x + 1)]$ times $(x + 1)$ is larger than 3946. In this case x is 8, for 448 times 8 is 3584, while 449 times 9 is 4041.

10. Place 3584 beneath 3946, and adjoin the digit 8 to the right of both 44 and 22. Having arrived at the decimal point in the given number, place a decimal to the right of 228. You have completed these steps:

```
       228.
      √5'23'46.00'8
       4
   42 | 1 23
          84
  448 | 39 46
        35 84
```

Meaning of result

Actually, the square of 228 is the largest perfect square that does not exceed 52,346. The next two digits should now be brought down, and the process repeated. The computation carried out to two decimal places is

```
        228.79
       √5'23'46.00'8
        4
    42 | 1 23
           84
   448 | 39 46
         35 84
  4567 | 3 62 00
         3 19 69
 45749 | 42 31 80
         41 17 41
          1 14 39
```

We have stopped at the remainder 11,439. We could have brought down the next two digits, which are zeros, and continued, but such precision would rarely be needed.

Explanation of square root process

In step 2 above, the largest multiple of 100 whose square is not greater than the given number 52,346 is found. This partial root, which we will call a, is 200. In step 5 we subtract a^2 from the original number which gives us the equivalent of this simple calculation:

$$52{,}346.008 - 200^2 = 12{,}346.008.$$

In step 6 we find the largest multiple of 10, which we shall call b, so that $(a + b)^2$ with $a = 200$ does not exceed 52,346.008. This value of b is 20. You will recall that $(a + b)^2 = a^2 + 2\,ab + b^2$.

Successive subtraction

The number 420 (of which the zero is not written) in step 7 is the term $(2\,a + b)$, whereas 8400 is $(2\,a + b)b$, that is, the sum $2\,ab + b^2$. When we arrive at step 9 we have removed $a^2 + 2\,ab + b^2$ by subtraction from the given number, subtracting first a^2 and then $2\,ab + b^2$, with $a = 200$ and $b = 20$. The process of the extraction of the square root is thus a method of finding $a, b, c, \cdots$

which are successive multiples of powers of 10, and the multiples of the numbers 1, $\frac{1}{10}$, · · · so that

$$52{,}346.008 = (a + b + c + \cdots)^2.$$

A second example of this procedure for obtaining the square root follows.

```
      0.067
    ________
   √0.00'45'37
    6 | 45
        36
  127 | 9 37
        8 89
        ----
          48
```

ROUNDING OFF NUMBERS

Use of decimal numbers

The computations and answers to engineering problems are usually in terms of rational numbers. Instead of using $\sqrt{2}$, it is customary to employ 1.414, which is an approximation to $\sqrt{2}$. In place of π the number 3.1416, or even $\frac{22}{7}$, is often employed. The representation 3.1416 is a "rounded off" form of the actual decimal representation of π. The latter representation requires an unlimited number of decimal places, and therefore cannot even be written as a decimal number. Although it is necessary to approximate irrational numbers by rational numbers, it is customary also to approximate rational numbers by simpler rational numbers with fewer decimal places. Thus in place of 5.73634219 an engineer would probably use 5.736. He might even be able to get along with 5.7 if the number represented cubic yards of earth or tons of coal. This process of dropping decimals in a number is called *rounding off* the number.

Method of rounding off numbers

The American Standards Association recommends the following method of rounding off numbers:

(1) *If a digit larger than* 5 *is dropped, the preceding digit is increased by* 1. Hence 3788 and 3.146 are rounded off to 3790 and 3.15 respectively when the last digits are dropped. We cannot drop the zero in 3790 because we need it to locate the position of the decimal point. We shall, however, speak of going from a digit not zero to zero as dropping the given digit.

(2) *If a digit smaller than* 5 *is dropped, the preceding digits are left unchanged.* Thus 3782 and 15.3 can be rounded off to 3780 and 15 respectively.

The digit 5

(3) *If the last digit in a number is* 5, the number can be rounded off to two equally close numbers. Thus 96.45 is midway between the numbers 96.4 and 96.5. If we always dropped the 5, we would invariably decrease numbers ending in 5 and adversely effect the accuracy of summations. A much more desirable method is that of making the preceding digit an even number.

Thus in dropping the digit 5 the numbers 34500, 37.05, 347500, and 0.0135 can be rounded off to 34000, 37.0, 348000, and 0.014 respectively. In dropping the 5 from the number 0.00995 we notice that the digit 9 preceding 5 becomes 10; but this increases the preceding 9 to 10 giving 0.01 as the rounded or approximate value.

Significant digits

You should notice that we did not round off the number 37.05 to 37 because the 0 in 37.0 is *significant* in that it tells us the digit to the right of the decimal place cannot be 1, for example, which fact cannot be inferred from the number 37 alone. In fact, 37 would be the number obtained by rounding off 36.6, 36.7, 36.8, 36.9, 37.1, 37.2, 37.3, or 37.4. In working with numbers you should be sure that each digit is significant and needed.

If x is an approximation to a number between $0.95\ x$ and $1.05\ x$, the number x is said to be accurate to 5 per cent. Again, when 20 is an approximation to a number between 15 and 25, the number 20 is said to be accurate to 25 per cent.

Number of digits to be kept

The distance an automobile travels in 3 hours is measured and found to be 200 miles. The quotient of the distance 200 by the time 3 gives the speed of the automobile in miles per hour. It would be absurd to perform the division and say that the automobile averaged 66.666667 m.p.h. for the 3 hours. There are errors in measuring both the distance and time, so that the first two or three digits are probably all that should be kept, the result in the case of three digits being 66.7 m.p.h.

Simple rules for the number of digits to be dropped in computations cannot be given in such a way as to apply to all situations. From experience you will be able to know how many "significant digits" to retain in a particular engineering problem. That rules can be developed to cover individual computations will be illustrated with a couple of examples.

Suppose that we are interested in the product AB of numbers A and B where there are inaccuracies in the values of A and B. These numbers may, for example, be the lengths of the sides of a

rectangle, and AB is the area. We assume that A and B are known to be accurate to x per cent and y per cent respectively. The number A is then an approximation of a number between *Error in product*

$$A - \frac{x}{100}A \quad \text{and} \quad A + \frac{x}{100}A.$$

Similarly B is an approximation of a number between

$$B - \frac{y}{100}B \quad \text{and} \quad B + \frac{y}{100}B.$$

The product AB becomes then an approximation of a number between

$$(17) \quad \left(A - \frac{x}{100}A\right)\left(B - \frac{y}{100}B\right) \quad \text{and} \quad \left(A + \frac{x}{100}A\right)\left(B + \frac{y}{100}B\right).$$

The first expression in (17) is obtained by replacing A and B by the minimum values $A - \frac{x}{100}A$ and $B - \frac{y}{100}B$ respectively. The other expression is derived from the maximum values.

We subtract AB from the expressions in (17) and obtain the following pair of limits after simplification: *Range of error*

$$(18) \quad -\left(\frac{x}{100} + \frac{y}{100} - \frac{xy}{10{,}000}\right)AB \quad \text{and} \quad \left(\frac{x}{100} + \frac{y}{100} + \frac{xy}{10{,}000}\right)AB.$$

The error in AB due to the errors in A and B lies between the expressions of (18).

Since $xy/10{,}000$ is usually small compared to the terms $x/100$ and $y/100$, we are ordinarily justified in dropping the term $xy/10{,}000$ from the expressions in (18). The expressions in (18) then become

$$(19) \quad -(x + y)\% \text{ of } AB \quad \text{and} \quad (x + y)\% \text{ of } AB.$$

In this case we have: *If the factors A and B are accurate to x per cent and y per cent respectively, the product AB is accurate to $(x + y)$ per cent.*

For example, let us suppose that a rectangular lot is paced off and found to be about 100 ft. by 200 ft. We write $A = 100$, and $B = 200$. Let us suppose further that A and B are paced by different persons known to be accurate in their pacing to 4 per cent and 5 per cent respectively. *Error in area calculation*

By the above rule the product AB is accurate to $(4 + 5)$ per cent, that is, 9 per cent. Since the area AB in square feet is $AB = 100 \times 200 = 20{,}000$, the number AB may be in error by

$$\tfrac{9}{100} \times 20{,}000 = 1{,}800.$$

The answer 20,000 sq. ft. may thus be in error by as much as 1800 sq. ft. The true area then lies between 18,200 and 21,800 sq. ft.

Check

We shall carry through the computation of this example directly. The true width and length of the rectangle may be as much as 104 and 210 ft. respectively. In this case the area is 104×210, or 21,840 sq. ft. Subtracting 20,000 from the last figure, we find that the error in AB is 1840. If on the other hand the true width and length are 96 and 190 respectively, the correct area of the rectangle is 18,240 sq. ft. The product AB is now in error by 1760. You can see that 1800 is a good approximation to 1840 and 1760, the approximation being due to our use of the limits (19) rather than the exact expressions (18).

Error in quotient

By methods similar to those used above, we can show that *if A and B are accurate to x per cent and y per cent respectively, the quotient A/B is accurate to $(x + y)$ per cent.* To get this result we have neglected terms such as $xy/10{,}000$, which are ordinarily small compared to the terms retained. Hence, for practical purposes we may say in the case of either a quotient or a product that the possible percentage error in the answer is the sum of the possible percentages of error in the terms.

Error in speed

As an example, let us suppose that the distance an airplane traveled in 2 hours at constant speed was measured and found to be 300 miles. We shall find the possible error in the speed obtained as 150 m.p.h. by dividing 300 by 2 (distance by time), upon the basis that the distance is accurate to 2 miles, and the time to 1 min.

The distance is now accurate to $\frac{2}{300}$, or about 0.67%, whereas the time is accurate to $\frac{1}{120}$, or approximately 0.83%. The speed 150 m.p.h. is thus accurate to 0.67% + 0.83%, or 1.5%. Taking 1.5% of 150 we have 2.25. The error in the speed may thus be as much as 2.25 m.p.h., so that the true speed lies between 147.75 and 152.25 m.p.h.

Several significant digits

It is sometimes necessary in engineering calculations to work with several significant digits. This is especially true when we must employ a small difference of large values. Such an example arises when one considers an ordinary compressed spring in the shape of a coil. For this spring Hooke's law

$$F = ks \tag{20}$$

applies, where for a change F in the force compressing the spring, the length of the spring changes by a distance s. The constant k

here is simply called the *spring constant.* If in a given state of compression the compressive force of the spring is measured and found to be 210.8 lb., and a decrease of this force to 210.5 lb. causes an increase of $\frac{1}{100}$ in. in the length of the spring, the constant k is $(210.8 - 210.5) \div 0.01 = 30$ (in pounds per inch), so that an inch change in the length of the spring corresponds to a 30-lb. change in the compressive force. *It would be clearly absurd here to round off the values of the compressive force to three figures* before computing the value of k.

Problems

1. What kinds of numbers are the following:

$$\sqrt{3}, \quad 10^2, \quad 13.076, \quad \tfrac{7}{5}?$$

2. Perform the indicated operations:

(a) $\frac{2}{3} + \frac{3}{7}$; (b) $\frac{5}{8} - \frac{3}{16}$; (c) $\frac{4}{5} \times \frac{25}{16}$; (d) $\dfrac{\frac{28}{15}}{\frac{7}{15}}$.

Ans. (a) $\frac{23}{21}$; (b) $\frac{7}{16}$; (c) $\frac{5}{4}$; (d) 4.

3. Give the values of:

(a) 0×2; (b) $\frac{3}{0}$; (c) $\frac{0}{2}$; (d) $0 + 8$; (e) $0 - 5$; (f) $\dfrac{1}{4 - 4}$.

Ans. (a) 0; (b) no meaning; (c) 0; (d) 8; (e) -5; (f) no meaning.

Fractions

4. Write as quotients of two integers:

(a) 0.046; (b) -4; (c) 7; (d) $1\frac{1}{2}$; (e) $0.666 \cdots$; (f) 14%.

Ans. (a) $\frac{23}{500}$; (b) $\dfrac{-4}{1}$; (c) $\dfrac{7}{1}$; (d) $\frac{3}{2}$; (e) $\frac{2}{3}$; (f) $\frac{14}{100}$.

5. Simplify these expressions:

(a) $\dfrac{\frac{1}{2} - \frac{3}{4}}{\frac{4}{8} + \frac{7}{6}}$; (b) $1 + \dfrac{1}{1 + \dfrac{1}{1 + 2}}$; (c) $\dfrac{1 - \frac{3}{4} \times \frac{12}{5}}{\frac{1}{5}}$; (d) $\frac{192}{144}$.

Ans. (a) $-\dfrac{3}{20}$; (b) $\frac{7}{4}$; (c) -4; (d) $\frac{4}{3}$.

6. Perform the indicated operations:

(a) $\dfrac{-4}{-12}$; (b) $(-2)(-3)(-\frac{1}{2})$; (c) $\dfrac{(-3)(4)(-4)}{(-2)(-8)}$.

Ans. (a) $\frac{1}{3}$; (b) -3; (c) 3.

7. If a rectangular sheet of steel has sides 3 in. and 4 in. long, what is the length of the diagonal of the sheet? *Ans.* 5 in.

8. A man has tools weighing $\frac{1}{2}$ lb., $\frac{7}{8}$ lb., 1.14 lb., and $\frac{15}{4}$ lb. respectively. What is the total weight of the tools? *Ans.* 6.26 lb.

Percentage

9. If 2% of 20,000 manufactured pieces are faulty, and 90% of the faulty pieces are thrown away, how many pieces are discarded? *Ans.* 360.

10. If 15 out of every 500 pistons crack under test, what percentage of the pistons crack? *Ans.* 3%.

Roots and powers

11. Extract the square roots of the following numbers, using 6 digits in the answer.

(a) 0.000307; (b) 108,000; (c) 1634.15.

Ans. (a) 0.017521; (b) 328.634; (c) 40.4246.

12. Simplify the expressions:

$2^3 \times 2^4 \times 2^8$; a^2a^5; $(x^2y^3)($ *Ans.* 2^{15}; a^7; x^6y^8.

13. Multiply $3x^2 + x^3 - 1$ by $7 + x^2 + 2x$.
Ans. $x^5 + 5x^4 + 13x^3 + 20x^2 - 2x - 7$.

14. Divide $1 + x^4 - 2x^2$ by $1 - x$. *Ans.* $1 + x - x^2 - x^3$.

Factoring

15. Factor the expressions:

(a) $9x^2 - y^2$; (b) $a^2 - b^2 + a - b$;
(c) $x^2 - 6y + 9y^2$; (d) $4x^2 + 28x + 49$.

Ans. (a) $(3x - y)(3x + y)$; (b) $(a - b)(a + b + 1)$; (c) $(x - 3y)^2$; (d) $(2x + 7)^2$.

16. Reduce the fractions:

(a) $\dfrac{a^2 - 25}{a^2 + 10a + 25}$; (b) $\dfrac{x^2 - x}{x^3 - x^2}$; (c) $\dfrac{24a^3b^2(x + 1)^3}{15a^2b^4(x + 1)^2}$.

Ans. (a) $\dfrac{a - 5}{a + 5}$; (b) $\dfrac{1}{x}$; (c) $\dfrac{8a(x + 1)}{5b^2}$.

Ratios

17. If 7 is to 40 as x is to 100, what is x? *Ans.* $17\frac{1}{2}$.

18. What is the ratio 5 : 200 when expressed in per cent? Also for 4 : 130; 7 : 80; and $2\frac{1}{2} : 10\frac{1}{2}$. *Ans.* $2\frac{1}{2}$%; 3.08%; 8.75%; 23.8%.

19. If one plane can travel at 280 m.p.h. and another at 400 m.p.h., what is the ratio of their speeds expressed as a per cent?
Ans. The slower plane can travel 70% as fast as the top speed of the other.

20. The intensity I of illumination on a flat surface due to a small light source varies inversely as the square of the distance d from the light source to the surface. If the intensity is 20 foot-candles at a distance of 10 ft., what is the intensity when d is 30 ft.? *Ans.* $\frac{20}{9}$ foot-candles.

21. The force F of attraction between two electrical charges e_1, e_2 varies directly as the product of these charges, and inversely as the square of the distance d they are apart. If $F = 10$ when $e_1 = 2$, $e_2 = 3$, and $d = 5$, what is the constant k of proportionality, and what is the force when $e_1 = 1$, $e_2 = 10$, and $d = 1$?
Ans. $k = \frac{125}{3}$, $F = \frac{1250}{3}$.

22. If the compressive force of an ordinary spring is 100 lb. when the spring is 2 in. long, and 200 lb. when the spring is 1 in. long, what is this force when the spring is $1\frac{3}{4}$ in. long? *Ans.* 125 lb.

Rounded numbers

23. Divide 7 by 11 and round off the answer to 4 decimal places.
Ans. 0.6364.

24. Round off the following numbers by dropping the last three digits:
0.007893; 78,345; 405.01; 404.99; 1.0001; 4342; 18,500; 9500.
Ans. 0.008; 78,000; 410; 400; 1.0; 4000; 18,000; 10,000.

25. Write the following as multiples of powers of 10: 100,000; 3 million; 18,000.
Ans. 10^5; 3×10^6; 18×10^3.

26. If $A = 15$, $B = 25$, and A is accurate to 1% while B is accurate to 0.5%, to which number should the product AB be rounded off? *Ans.* 380.

27. If a steamer is found to travel 32 nautical miles in 3 hours, values being off by at most 1.6 mi. and 1.2 min., what is the maximum possible error in the computed speed 32/3? *Ans.* About 5.67%.

)LUTION OF EQUATIONS

Variables and constants

In the expression $25x$ the number 25 is fixed and is therefore called a *constant.* The symbol x may have different values, and is called a *variable.* In the expression $-6xy^2$ the number -6 is a constant, while x and y are variables. If in $25x$ we replace 25 by a we obtain ax. We still say that a is a constant and x is a variable.

Usually the initial letters of the alphabet, such as a, b, and c, are reserved for constants, and the last letters of the alphabet, such as x, y, and z, for variables. In the expression ab^2x^3z the symbols a and b would be called constants, and x and z variables.

Linear equations

Because of its association with straight lines, to be explained in detail later, an equation of the form

$$ax + by + c = 0 \tag{21}$$

is called a *linear equation.* Any equation reducible to the form of equation 21 by the algebraic operations of addition, subtraction, multiplication, and division (division by zero excluded) is also linear. The following are examples of linear equations:

$$2x + 5 + 4y = 0, \quad 6x + 7 = 4x - 2 + x, \quad x + y = 2x - 7y.$$

To solve a linear equation, such as

$$4x - 7 + 5x = 9 - 4x + 2x + 6 \tag{22}$$

you will find it convenient to follow the steps listed below.

Method of solving linear equations

1. *Collect the terms in x* on each side of the equation, and do the same for the constants. You obtain

$$9x - 7 = 15 - 2x.$$

2. Using addition and subtraction *change the equation into the form* $ax = b$ with a term in x on one side, and a constant on the other. For the example under consideration you can do this by first adding $2x$ to both sides, and then by adding 7 to both sides. You have $11x = 22$.

3. Finally, *divide both sides of the equation by the same number* to leave only x on one side. This yields

$$x = \tfrac{22}{11} = 2.$$

The solution of equation 22 is thus $x = 2$. In carrying through the steps above, the same operations were performed on the left of the equation as on the right. This must always be done to preserve the equality.

Checking equations

To be sure that no error has been made in the solution of an equation, the result should be checked by substituting the solution in the given equation. In the case above, this substitution yields:

$$8 - 7 + 10 = 9 - 8 + 4 + 6; \text{ or } 11 = 11.$$

Since this is an obvious equality, the solution for x is correct.

As another example, let us solve for x in the equation

$$\frac{a}{x} + b = \frac{a + 3\,b}{x} + 4.$$

Multiplying both sides by x we obtain the linear equation

$$a + bx = a + 3\,b + 4\,x.$$

Subtracting $4\,x$ and a from both sides we have

$$(b - 4)x = 3\,b.$$

Dividing both sides by the coefficient of x produces

$$x = \frac{3\,b}{b - 4},$$

which is the desired solution.

Substitution of this solution for x in the given equation yields

$$\frac{a}{\dfrac{3\,b}{b - 4}} + b = \frac{a + 3\,b}{\dfrac{3\,b}{b - 4}} + 4.$$

Combining terms on the left and right according to the laws for fractions, we have

$$\frac{ab - 4\,a + 3\,b^2}{3\,b} = \frac{ab - 4\,a + 3\,b^2}{3\,b},$$

which results in an identity and completes the check.

Engineering example

Linear equations in one variable occur often in practice. One such example follows. How many tons of steel scrap, 99% iron, must be added to 10 tons of steel, 95% iron, to obtain a steel 98% iron, and how many tons will there be in the final steel billet? To answer these questions let x be the number of tons of steel which is 99% iron. There are then $10 + x$ tons of the final product. The amount of iron in x tons of 99% steel plus the amount of iron in 10 tons of 95% steel equals the amount of iron in $10 + x$ tons of 98% steel. We have the equation

$$\frac{99}{100}\,x + \frac{95}{100}\,(10) = \frac{98}{100}\,(10 + x).$$

Multiplying both sides by 100 simplifies this equation into

$$99\,x + 950 = 980 + 98\,x.$$

Subtracting $98\,x + 950$ from both sides brings the equation into the form

$$x = 30;$$

from which

$$10 + x = 40.$$

Hence, there are 40 tons of the final product. The checking of the solution will be left to you.

GRAPHICAL AIDS FOR SOLVING EQUATIONS

Coordinate systems

In Fig. 225 there is pictured a Cartesian coordinate system for the plane. The horizontal line is called the *x axis*, and the vertical line the *y axis*. Numbers are associated with points on these axes as in Fig. 223. The number connected in this manner with a point on the x axis is called the *x coordinate* of the point, and is understood to be a value of the variable x. In the same way y coordinates are associated with the points on the y axis. In Fig. 225 an arbitrary point P has been chosen in the plane. Perpendiculars have been dropped from P to the x and y axes, giving points P_x and P_y respectively. The x and y coordinates of the point P are by definition the x and y coordinates of the points P_x and P_y respectively. It is customary to indicate a point by its coordinates, for example, P by (x, y). Points $(-2, 3)$, $(-4, -5)$, and $(4, -3)$ are *plotted*, that is, located, on the plane of Fig. 225.

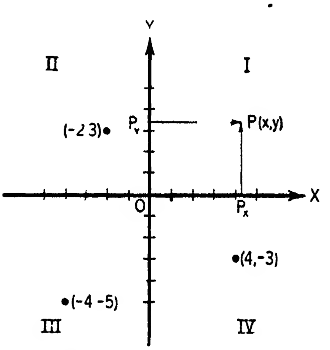


FIG. 225. CARTESIAN COORDINATE SYSTEM IN THE PLANE

Quadrants

The coordinate axes divide the plane into four parts, called *quadrants*. The quadrant in which the points with positive coordinates lie is termed the *first quadrant*. In Fig. 225 this is labeled I. The second, third, and fourth quadrants are labeled II, III, and IV respectively as shown.

The Cartesian coordinate system of the plane gives a unique "name" to each point, namely the pair of coordinates (x, y) of the point. The point $(0, 0)$, denoted by O in Fig. 225, is called the *origin* of the coordinate system. The arrows on the axes in

Fig. 225 designate the positive direction on each of these lines, the *positive direction* being the direction of increasing coordinates.

Graphs

If, in the equation

(23) $$y = x + 2,$$

we assign the value 1 to x, we obtain 3 for y. Other pairs of values of x and y which satisfy this equation can be obtained as shown in the following table:

x	y
1	3
0	2
-1	1
-2	0

Equation 23 connects a given value of x with one and only one value of y. Hence the equation represents only certain pairs of values (x, y), and certain points in the plane of Fig. 227. Here the

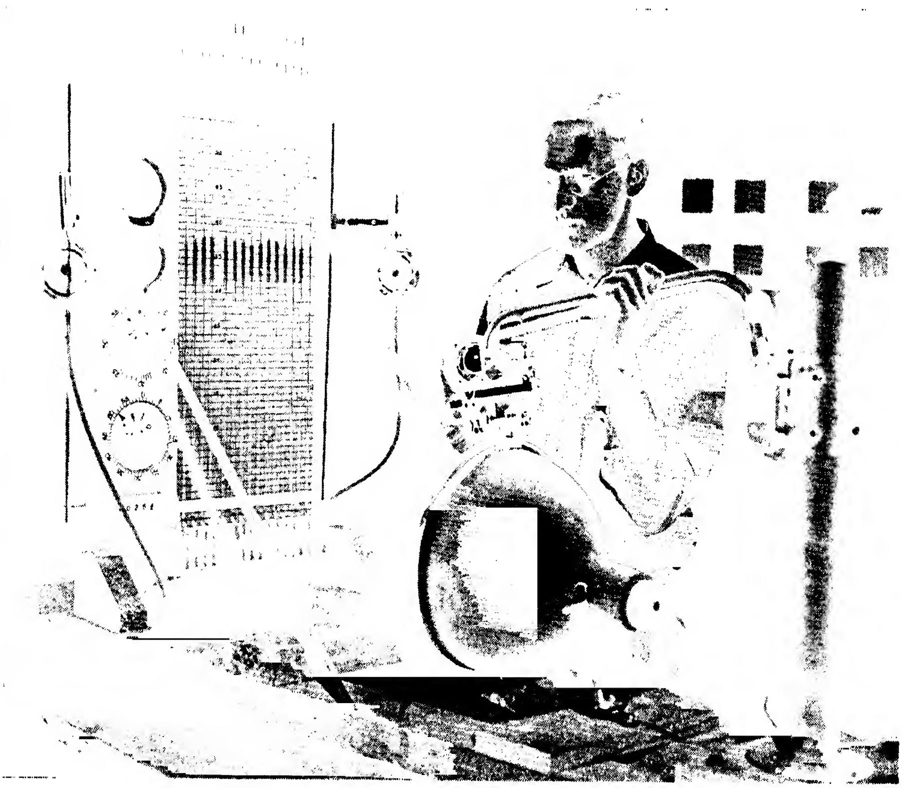


FIG. 226. COORDINATE SYSTEM IN USE EXPERIMENTALLY

points, whose coordinates are the pairs of values from the above table, have been plotted, and a straight line has been drawn through them. We say that equation 23 is the equation of the corresponding straight line of Fig. 227. We shall refer to equation 23 as the *line l* of Fig. 227.

Straight lines

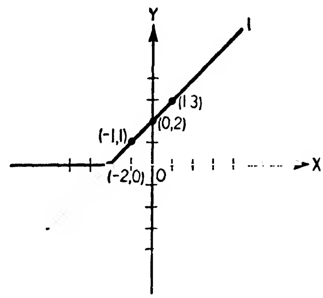


FIG. 227. POINTS ON A LINE l

It will be shown later that a linear equation in x and y yields a straight line when "plotted" on a plane with a Cartesian coordinate system, and conversely, the points of a straight line on a Cartesian coordinate system satisfy a linear equation. The points where a straight line cuts the coordinate axes are called the *intercepts* of the line. The intercepts of the line of equation 23 are the points (0, 2) and (− 2, 0) respectively. Since a straight line is determined or located by any two points on the line, we can locate a straight line by means of its intercepts.

SOLVING SIMULTANEOUS EQUATIONS

Intersecting lines

If we set y equal to zero in the equation

$$2x + 3y = 4, \tag{24}$$

we find that x is 2. This value 2 is said to be the x-intercept of the line represented by equation 24. The y-intercept of this line is $\frac{4}{3}$.

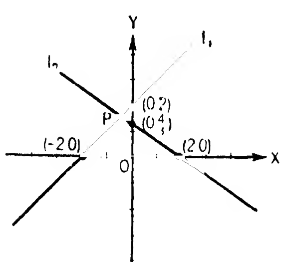


FIG. 228. INTERSECTION OF TWO LINES AS THE SOLUTION OF LINEAR EQUATIONS

By using these intercepts and drawing a line through these points, we obtain the line l_2 in Fig. 228, which line represents equation 24. Similarly, the line l_1 represents equation 23. The lines l_1 and l_2 intersect in a point designated by P. Since P is on the line l_1 the coordinates x, y of P satisfy equation 23. Similarly, x and y satisfy equation 24. Equations 23 and 24 determine P when taken simultaneously. They are therefore called *simultaneous equations.*

Solution by substitution

If you substitute $x + 2$ for y (which comes from equation 23) into equation 24, you obtain an equation in x only which is

$$2x + 3(x + 2) = 4.$$

Solution of this equation by the usual methods gives $x = -\frac{2}{5}$. This must be the x-coordinate of the point P in Fig. 228 since it is obtained by simultaneous solution of equations 23 and 24. Substitution of this value of x in equation 23 yields $y = \frac{8}{5}$, the other coordinate of P. The point of intersection of the lines 23 and 24 is the point $(-\frac{2}{5}, \frac{8}{5})$.

The x-coordinate of a point P is often termed the *abscissa* of P, whereas the y-coordinate is called the *ordinate*.

Solution by elimination

A second method of solving simultaneous linear equations will be illustrated with an example. The given equations are

$$3x + 3y = 2,$$
$$4x - 5y = 7.$$

Multiplying the first equation by 4 and the second equation by 3 produces the new equations

$$12x + 12y = 8,$$
$$12x - 15y = 21.$$

Subtracting both sides of the second equation from the corresponding sides of the first equation yields the equation

$$27y = -13$$

in the variable y. Solution of this equation for y gives $y = -\frac{13}{27}$. Substituting this value of y in the first equation produces an equation in x only

$$3x - \frac{39}{27} = 2.$$

The solution for x gives us $x = \frac{31}{27}$.

To be sure that the computation is correct you should, of course, substitute the solutions for x and y in the original equations.

Application

Two weights A and B attached to a bar balance each other when the bar is supported at a point P for which the ratio of the distance from A to P to the distance from P to B is 3:4. (See Fig. 229.) Together A and B weigh 300 lb. The problem is to determine the weights of A and B individually.

FIG. 229. WEIGHTS A AND B ON LEVER WITH FULCRUM AT P

Setting up equations

We let x be the weight of A and y the weight of B. By the law of the lever from mechanics the weight of A is to the weight of B as the distance from B to P is to the distance from A to P. There follows our first equation,

$$\frac{x}{y} = \frac{4}{3} \quad \text{or} \quad 3x = 4y.$$

Since A and B together weigh 300 lb. we have the second of these simultaneous equations,

$$x + y = 300.$$

Solution of the first equation for x and substitution in the second gives an equation in y alone

$$\frac{4y}{3} + y = 300.$$

Solution of this equation for y gives $y = \frac{900}{7}$. By the first equation we then get $x = \frac{1200}{7}$.

Equations in three variables

The methods of solution above can often be employed for *equations in several variables*. We shall treat the system of three variables x, y, z defined by the following equations

$$\begin{aligned} 3y + 2x + 7 &= 0, \\ z + 3x + 4 &= y - 1, \\ 5 - 4x + y &= z. \end{aligned}$$

It is convenient to arrange the terms so that terms in the same variable occur in the same column. This rearrangement gives

$$\begin{aligned} 2x + 3y \phantom{{}+z} &= -7, \\ 3x - y + z &= -5, \\ -4x + y - z &= -5. \end{aligned}$$

Equalizing coefficients

Multiplying the first equation by 6, the second by 4, and the third by 3 brings these into the system

$$\begin{aligned} 12x + 18y \phantom{{}+4z} &= -42, \\ 12x - 4y + 4z &= -20, \\ -12x + 3y - 3z &= -15, \end{aligned}$$

where the coefficient of x is the same in each equation except for sign. Subtracting the second from the first gives us

$$22y - 4z = -22.$$

Addition of the second to the third yields

$$-y + z = -35.$$

Solution

The solution of the system has now been simplified to

$$\begin{aligned} 11y - 2z &= -11, \\ -y + z &= -35, \end{aligned}$$

which can be treated like the pairs of simultaneous equations above. After the values of y and z are determined from these equations their values may be put back into any one of the three original equations to obtain the value of x.

SOLVING QUADRATIC EQUATIONS

Single variable squared

For a simple equation of the type $y^2 = 17$ we do not need any special methods because the answer is seen at once to be $y = \pm\sqrt{17}$. Here the $\sqrt{17}$ means the square root obtained by the root-extraction process described earlier in this chapter. Approximately, $\sqrt{17} = 4.1$. Since the equation $y^2 = 17$ has two roots, we say that 17 has two square roots, one positive and the other negative. When we write $\sqrt{17}$ without a plus or minus sign in front, we mean the plus square root whereas $-\sqrt{17}$ designates the minus square root, or approximately -4.1. As a consequence $\sqrt{9}$ is 3, while $-\sqrt{9}$ is -3.

Inequality sign

The symbol $\neq$ means "not equal to." Thus $a \neq 2$ means "a not equal to 2."

The quadratic equation

An equation which is of the form

$$ax^2 + bx + c = 0, \quad (a \neq 0) \tag{25}$$

or one that can be reduced to this form by rearrangement of terms, is called a *quadratic equation*. Evidently the value of a should not be zero because a quadratic equation must have terms in the square of the variable. A quadratic equation has no terms of "higher power" than the square. The expression "higher power" refers to the fact that x^m is a higher power than x^n if m is greater than n. For example, x^4 is a higher power of x than x^2 or x^3.

Solution by completing the square

We shall derive a simple formula, called the *quadratic formula*, by means of which you will be able quickly to solve quadratic equations. To derive this formula we describe the general *method of completing the square*.

1. Transpose the term c in equation 25 to the right by subtracting c from both sides of the equation, giving

$$ax^2 + bx = -c.$$

2. Divide both sides by the coefficient a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing square

3. Add $[\frac{1}{2}(b/a)]^2$, or $b^2/4a^2$, to both sides to obtain

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

4. Rewrite the left side to show that it is a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

5. Collect the terms on the right, which gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

6. Extract the square root of both sides. To perform this step go back to rule 16 under factoring. With this rule you can solve the equation $x^2 = 4$, for example. To do this you first subtract 4 from both sides obtaining $x^2 - 4 = 0$. Then factor the left-hand member to get $(x - 2)(x + 2) = 0$. Since *a product of numbers can vanish (become zero) only if at least one of the factors vanishes*, the solution of the quadratic equation $x^2 - 4 = 0$ reduces to the solution of two linear equations, namely, *Product becomes zero*

$$x - 2 = 0, \quad x + 2 = 0.$$

From these equations $x = 2$ and $x = -2$. For simplicity of notation these two solutions are written as $x = \pm 2$, meaning *x is plus or minus 2*. This method shows that if a is a positive number, the solution of $x^2 = a$ is $x = \pm\sqrt{a}$, where by $\sqrt{a}$ we mean the ordinary positive square root of a. From step 5 you have by this procedure *Two roots*

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

7. Finally, subtract $b/2a$ from both sides, getting

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}, \tag{26}$$

which is the solution. There are two roots (they may be coincident) of the quadratic equation 25, namely

$$-\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}}; \quad \text{and} \quad -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

Since

$$\left(\sqrt{\frac{a}{b}}\right)^2 = \frac{a}{b} \quad \text{and} \quad \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{a}{b},$$

Square root of quotient

we have the rule

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \tag{27}$$

which applies to the numbers a and b. Using this rule the solution of equation 25 becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{28}$$

Quadratic formula

This is the quadratic formula.

By means of the quadratic formula you can at once write down the solution of

$$2x^2 + 8x + 3 = 0.$$

Here $a = 2$, $b = 8$, and $c = 3$. The solution is

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 2 \times 3}}{2 \times 2},$$

or

$$x = \frac{-8 \pm \sqrt{64 - 24}}{4} = \frac{-8 \pm \sqrt{40}}{4} = -2 + \frac{1}{4}\sqrt{40} \quad \text{or} \quad -2 - \frac{1}{4}\sqrt{40}.$$

We see therefore that there are always two values of x, equal or unequal, that will satisfy a quadratic equation in x. Often in practical engineering problems one of these solutions is the correct one while the other is impractical or unreasonable. For example, if x is the speed of a car in miles per hour and the two values of x are found to be 32 and 45,200, it is evident that $x = 32$ is the correct solution.

Example of a quadratic equation

A simple problem in shop production involving a quadratic equation will be solved. A sheet of metal is 24 by 36 in. It is desired to remove a uniform border from the metal to obtain a smaller sheet with area 400 sq. in. What must be the width of the border?

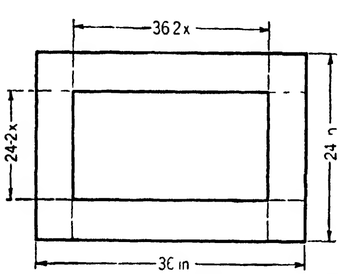


FIG. 230. SHEET OF METAL WITH A BORDER OF WIDTH x INCHES

Let x designate the width of the border as shown in Fig. 230. The reduced sheet has dimensions $24 - 2x$ and $36 - 2x$. It follows that

$$(24 - 2x)(36 - 2x) = 400.$$

Multiplying out the indicated product on the left, dividing the equation by 4, and collecting all terms on one side, we have

$$x^2 - 30x + 116 = 0.$$

Quadratic equation

Solution by the quadratic formula gives

$$x = 15 \pm \tfrac{1}{2}\sqrt{436}.$$

To an accuracy expressed by three significant digits

$$\sqrt{436} = 20.9.$$

The solutions for x are thus approximately

$$x = 25.4;\ 4.6.$$

The first answer is impossible, and is therefore thrown away. The second solution is the desired one; whence the border must be 16 in. in width.

If the term $b^2 - 4\,ac$ is negative, the quadratic formula has no significance when we restrict ourselves to real numbers. In 1572 Raphael Bombelli of Bologna, Italy, published a memoir in which he endeavored to show that square roots of negative numbers can be given a meaning which is useful and logical. Later such numbers were called *imaginary*. *Imaginary numbers*

Meaning is given to imaginary numbers such as $\sqrt{-5}$ by adopting the convention that whenever $(\sqrt{-5})^2$ occurs we may replace this term by -5. For example, $(\sqrt{-5})^4$ is $(\sqrt{-5})^2(\sqrt{-5})^2$, or $(-5)(-5)$, which is $+25$.

If we are willing to employ imaginary numbers, the quadratic formula is still valid when the term $b^2 - 4\,ac$ is negative. The solution of

$$x^2 + x + 1 = 0$$

for example, gives

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Numbers of the form $a + b$, where a is real and b is the square root of a negative number, or zero, are *complex numbers*. Complex numbers, like rational and real numbers, form a system because the common operations of addition, subtraction, multiplication, and division (division by zero excluded) with these complex numbers always again yield complex numbers. *Complex numbers*

Simple physical interpretations for complex numbers can be given. These numbers have been employed with considerable success in many branches of engineering, especially in treating equations describing motions of material things, such as the rotation of a flywheel in a diesel engine, the flow of water through a pipe, the flight of a shell from a gun, and the transmission of electricity along a wire. *Some uses of complex numbers*

Let us consider equations which are sums of terms set equal to zero, where each term is a real number times a power of a variable, say x^n. Linear and quadratic equations in x are examples of such equations. Carl Friedrich Gauss (1777–1855) proved that each such equation has a complex root. This statement means that all algebraic equations in a single variable raised to any powers can be solved by the use of complex numbers.

Graphs of quadratic equations in two variables

The equation

$$u^2 + 2u + v^2 = 4$$

contains squares of the variables u and v, but no higher powers, and is an example of a *quadratic equation in two variables.* By giving values to u and solving the equation for v, you can derive the following table of pairs of values of u and v. As an example of the

u	v
0	± 2
1	± 1
2	Not real
-1	$\pm\sqrt{5}$
-2	± 2
-3	± 1
-4	Not real

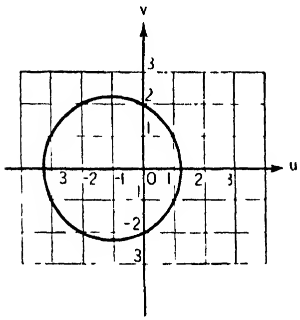


FIG. 231. A CIRCLE AS THE GRAPH OF A QUADRATIC EQUATION

computation involved, when $u = 2$ the equation becomes

$$4 + 4 + v^2 = 4 \quad \text{or} \quad v = \pm\sqrt{-4}$$

so that v is imaginary. In Fig. 231 the points which go with the values in the table are plotted on a u, v Cartesian coordinate system. Through these points a curve is drawn. This is the curve of the given quadratic equation. By analytic geometry we can later show that this is a circle.

Problems

1. Plot the points $(1, -2)$, $(-3, -3)$, $(5, -9)$, and $(-1, 4)$ on a Cartesian coordinate system. In what quadrants do these points lie?

2. Graph the line represented by the equation $2x = 4 - 3y$ by plotting some points and then drawing the line through these points. What are the intercepts of the line? *Ans.* The intercepts are $(2, 0)$ and $(0, \frac{4}{3})$.

3. Graph the curve $x^2 = 2y$.

Simultaneous equations

4. Graph the lines $x = 5 - 3y$, $2x + 5y = 2$.
Find the point of intersection, and check by solving the equations algebraically. *Ans.* $(-19, 8)$.

5. What is the solution of the equations $x + y = 2$, $2x = 1 - 2y$? *Ans.* There is none. Why?

6. Solve the equations below when y and z are variables.

$$2z = 2y - 4,$$
$$z + 2 = y.$$

Ans. These are equations of the same line. Any pair of values of (y, z) which satisfy one equation also satisfy the other.

7. Solve the equations below for a, b, and c.

$$2a + 3b = 14 + c,$$
$$4 + c - 3a = -b,$$
$$a + b + c = 4,$$

Ans. $a = 2$, $b = 3$, $c = -1$.

Quadratic equations

8. Solve the equations below for x.

(a) $\frac{1}{x} + 3 = \frac{x-1}{2x}$, $(x \neq 0)$

(b) $\frac{x^2 - 1}{(x+1)^2} = 2$, $(x \neq -1)$

(c) $3x^2 = 6 - 15x$,

(d) $\frac{3}{x} + x = -1$. $(x \neq 0)$

Ans. (a) $-\frac{3}{5}$; (b) -3; (c) $\frac{-5 \pm \sqrt{33}}{2}$; (d) $\frac{-1 \pm \sqrt{-11}}{2}$.

Linear equations in applications

9. The perimeter of a certain rectangle is 100 ft. and the altitude is three fourths of the base. Find the dimensions of the rectangle. *Ans.* $\frac{150}{7}$, $\frac{200}{7}$.

10. A weight of 200 lb. is placed on a lever 10 ft. from the point of support. How far from the point of support on the other side must a weight of 100 lb. be placed to produce equilibrium, that is, to balance the first load? *Ans.* 20 ft.

11. How many gallons of a solution containing 80% alcohol should be added to 10 gal. of a 30% solution, to give a mixture which is 40% alcohol?

Hint. Let x equal the gallons of 80% solution; then $10 + x$ equals the gallons of 40% solution. Alcohol in final mixture equals $0.80x + (0.30)10$ or $0.8x + 3$. But this is also expressed as $0.40(10 + x)$. Hence, you want to solve the equation $0.8x + 3 = 0.4x + 4$. *Ans.* $2\frac{1}{2}$ gal.

12. How many ounces of pure silver must be added to 100 oz. of silver, 50% pure, to give a mixture 60% pure? *Ans.* 25 oz.

13. A man has one investment yielding 4%, another giving 6%, and a third producing 7%. The total amount of money invested is given to be \$50,000, and the annual interest is \$2800. What is the amount of money invested at each rate if the amount invested at 6% equals the sum of the amounts invested at the other rates? *Ans.* \$25,000 at 6%; \$15,000 at 4%; \$10,000 at 7%.

14. A tank can be emptied by means of either of two orifices in the bottom or by both. If by opening the first orifice alone the tank can be emptied in 3 hours, and by opening the second alone the tank can be emptied in 4 hours, in how many hours can the tank be emptied when both orifices are used?

Hint. In t hours the first orifice empties t thirds of the tank, and the second orifice empties t fourths. *Ans.* 1 hour, $42\frac{6}{7}$ min.

Linear equation, speed problems

15. Two automobiles traveling in opposite directions at 50 and 70 m.p.h., respectively, started from the same place at the same time. After how many hours will they be 300 miles apart?

Hint. In a time t hours, one automobile travels $50t$ miles, and the other $70t$ miles. *Ans.* 2 hours, 30 min.

16. A boat traveling at constant speed down a river takes one hour to cover 10 miles, and 3 hours to travel the same distance going against the stream. What is the speed of the boat in still water, and the speed of the stream?

Hint. Let x be the speed of the boat in still water, and y the speed of the stream in miles per hour. The speed of the boat downstream relative to the shore is $x + y$ m.p.h., and against the stream $x - y$ m.p.h.

Ans. Boat speed: $6\frac{2}{3}$ m.p.h.; stream speed: $3\frac{1}{3}$ m.p.h.

17. A freight train leaves Chicago for New York at a speed of 35 m.p.h. Three hours later an express train leaves Chicago for New York at a speed of 75 m.p.h. When will the express train catch up with the freight train?

Hint. Let t be the time in hours from the departure of the freight until the express train overtakes the freight. The distance the freight covers in the time t is $35\,t$ (rate *times* time). This is equal to the distance $75(t - 3)$ covered by the express in the time t. *Ans.* 5 hours, $37\frac{1}{2}$ min.

Quadratic equations in applications

18. A rectangular piece of property is 40 ft. long and 30 ft. wide. The property is increased to a larger rectangle, of double the original area, by adding a border of uniform width. How wide is the border? *Ans.* $\dfrac{-35 + \sqrt{2425}}{2}$ or 7.12 ft.

19. The diagonal of a certain square is 2 ft. longer than a side of the square. Find the length of a side. *Ans.* $2 + 2\sqrt{2}$ or 4.83 ft.

EXPONENTS AND RADICALS

You have seen how powers of numbers arise in mathematical considerations. To use these powers with efficiency you will need some rules for manipulation, which will be outlined. When we write a power as a^n the number n is the *exponent* while a is the *base*. In the power 12^6 the exponent is 6 and the base 12.

Fractional exponents

Thus far we have treated only powers with positive integral exponents. It is convenient, however, to assign a meaning to a power with exponent n, where n is any real number. In fact, for many mathematical studies it is necessary to use a^n where n is an imaginary number.

We shall restrict ourselves to real exponents. We begin with reciprocals of integers, such as $\frac{1}{2}$ and $\frac{1}{4}$. Since the law (12), $a^n \times a^m = a^{n+m}$, concerning products of powers of the same base is valid for positive exponents which are positive integers, it is logical to restrict the meaning of all exponents so that this rule of multiplication (12) is always satisfied. For example, the power $2^{\frac{1}{2}}$ must be such that

Equivalence with roots

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2,$$

from which it is clear that

$$2^{\frac{1}{2}} = \sqrt{2}.$$

You will recall that the nth root of a number a is a number, written $\sqrt[n]{a}$, such that

Existence of roots

(29) $$(\sqrt[n]{a})^n = a.$$

The *cube root* of 2 written $\sqrt[3]{2}$, that is the third root, is a number such that

$$(\sqrt[3]{2})^3 = 2.$$

In more advanced mathematics it is proved that there are three cube roots of 2, two of which are imaginary. When we use the term *cube root* in what follows we mean the *real* cube root; in fact, we shall restrict ourselves to real roots only, unless there is no real root of the given number.

Roots of positive and negative numbers

There is always a positive nth root of a positive number no matter what integer is assigned to n. Examples of positive roots of positive numbers follow:

$$\sqrt[4]{16} = 2; \quad \sqrt[3]{64} = 4.$$

There is a negative nth root of a negative number when n is odd. Examples are

$$\sqrt[3]{-64} = -4, \quad \sqrt[5]{-32} = -2.$$

However, there are no real even roots of a negative number. Thus there is no real square root of -4, no real fourth root of -5, and so forth.

Since by application of rule 12 a product of n factors each equal to $a^{\frac{1}{n}}$ is a, we have

(30) $$\left(a^{\frac{1}{n}}\right)^n = a \quad \text{and} \quad a^{\frac{1}{n}} = \sqrt[n]{a}.$$

More generally, since by rule 12 a product of n factors each equal to a number $a^{\frac{m}{n}}$ is a^m and $(\sqrt[n]{a^m})^n$ is a^m, we write

(31) $$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

For example,

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$$

Quotients of powers

In the following illustration ,one power of 2 is divided by another:

$$\frac{2^5}{2^2} = \frac{\cancel{2} \times \cancel{2} \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2}} = 2 \times 2 \times 2 = 2^3.$$

This example is an illustration of the law

(32) $$\frac{a^n}{a^m} = a^{n-m} \quad (a \neq 0)$$

when n is greater than m. Formula 32 is proved by noting that when m a's are divided out of the numerator and denominator of the left member of equation 32, the denominator becomes 1 and the n-m factors equal to a remain in the numerator.

Zero exponent

We shall define the rest of our exponents in such a way that equation 32 is satisfied. In particular, consider the quotient $3^2/3^2$. Subtracting exponents this fraction becomes 3^0, whereas direct division gives 1. Therefore, $3^0 = 1$, and since any number except 0 might have been used in place of the number 3, we conclude that

$$(33) \qquad a^0 = 1 \quad (a \neq 0).$$

The restriction that a be different from zero is made because division by zero is not permitted, and rule 33 is based on division with some power of a. It is customary to say that 0^0 "has no meaning."

Negative exponents

By making use of rule 12 we may write

$$a^n a^{-n} = a^{n-n} = a^0 = 1.$$

But we also find that

$$a^n\left(\frac{1}{a^n}\right) = \frac{a^n}{a^n} = 1.$$

We therefore identify

$$(34) \qquad a^{-n} = \frac{1}{a^n}.$$

Examples follow:

$$(\tfrac{1}{2})^{-3} = 8; \qquad 2^{-2} = \frac{1}{2^2} = \frac{1}{4}; \qquad (-3)^{-3} = \frac{1}{(-3)^3} = \frac{1}{-27} = -\frac{1}{27};$$

$$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

We have defined a^n where n is any rational number. A definition of a^n where n is irrational, such as $n = \pi$, will not be given, although a number such as 2^π is real and can be determined to any number of decimal places.

Rules for exponents

A summary of useful rules for handling exponents follows.

(35) Rule for *multiplication:* $a^n a^m = a^{n+m}$.

(36) Rule for *division:* $\dfrac{a^n}{a^m} = a^{n-m}$.

(37) Rule for finding a *power of a power:* $(a^n)^m = a^{nm}$.

(38) Rule for finding a *power of a product:* $(ab)^n = a^n b^n$.

(39) Rule for finding a *power of a quotient:* $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

The rules above will be applied to some examples, which follow:

$$x^4x^{10} = x^{14}; \qquad 2^{25} \times 2^{100} = 2^{125}; \qquad 3^2 \times 3^3 \times 3^4 = 3^9;$$

$$(a^4)^3 = a^{12}; \qquad (2\,b)^5 = 2^5\,b^5 = 32\,b^5; \qquad \frac{x^{12}}{x^2} = x^{10};$$

$$\frac{2^5 \times 3^{12}}{2^8 \times 3^4} = \frac{3^8}{2^3}; \qquad \left(\frac{2^3}{3^4}\right)^3 = \frac{2^9}{3^{12}}; \qquad (a^2bc^3)^2 = a^4b^2c^6.$$

A common error

You will observe that a product or quotient of two powers can be combined into one only if the *bases are the same*, in which case rule 35 or 36 applies, or if the *exponents are the same*, when rules 38 and 39 can be used. It is impossible to combine $2^3 \times 3^2$ by multiplying the bases or adding the exponents, a mistake often made by the novice.

Radicals

The symbol $\sqrt[n]{a}$ is called a *radical*. It has the same value as $a^{\frac{1}{n}}$. A radical is a root written as indicated. The number n is called the *index* of the radical. By convention, the index is not written in the case of square roots.

Rules of radicals

Some properties of radicals may be obtained from the Rules for Exponents above.

(40) $\quad (\sqrt[n]{a})^n = \left(a^{\frac{1}{n}}\right)^n = a.$ $\quad$ (From 37)

(41) $\quad \sqrt[n]{a^m} = a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$ $\quad$ (From 37)

(42) $\quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$ $\quad$ (From 38)

[Except when n is even and a, b are both negative]

(43) $\quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$ $\quad$ (From 39)

(44) $\quad \sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[m]{\sqrt[n]{a}}.$ $\quad$ (From 37)

The first property (40) is merely the definition of the nth root of a. The first equality of (41) is the general law connecting roots and fractional exponents.

Limitation on the root or power of a product

By rule 38 for exponents $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. Since $a^{\frac{1}{n}} = \sqrt[n]{a}$, we have $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$. These rules of radicals and exponents break down when a and b are both negative, and n is even. For example, by definition

$$\sqrt{-2}\,\sqrt{-2} = -2.$$

If we apply rule 42 for radicals we have

$$\sqrt{-2}\,\sqrt{-2} = \sqrt{(-2)(-2)} = \sqrt{4} = +2.$$

We have arrived at a contradiction, whence the above restriction on a, b is needed in the statement of the third property (42). The same restriction of course applies to rule 38 for exponents when a and b are negative, and the exponent is one of the common fractions ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$) having an even number as a denominator.

To use rules 42 and 43 it is necessary to have the *same index* on each radical. Thus, to multiply $\sqrt{2}$ by $\sqrt[3]{2}$, we first change $\sqrt{2}$ to $\sqrt[6]{8}$ and $\sqrt[3]{2}$ to $\sqrt[6]{4}$, so that

$$\sqrt{2}\,\sqrt[3]{2} = \sqrt[6]{8}\,\sqrt[6]{4} = \sqrt[6]{32}.$$

Examples of radicals

Examples of the above properties of radicals follow.

$$(\sqrt[3]{4})^3 = 4; \qquad \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4; \qquad \sqrt[3]{2}\,\sqrt[3]{4} = \sqrt[3]{8} = 2;$$

$$\sqrt[4]{3}\,\sqrt[3]{4} = \sqrt[12]{27}\,\sqrt[12]{256} = \sqrt[12]{6912}; \qquad \frac{\sqrt[3]{-24}}{\sqrt[3]{3}} = \sqrt[3]{-8} = -2;$$

$$\sqrt[3]{\sqrt{8}} = \sqrt{\sqrt[3]{8}} = \sqrt{2}; \qquad \frac{\sqrt{8}}{\sqrt{2}} = \sqrt{4} = 2.$$

Simplification of radicals

The third property (42) is particularly helpful in treating radicals. For example,

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9}\,\sqrt{2} = 3\,\sqrt{2}.$$

Let us suppose that you wish to use a numerical value for the fraction $\frac{1}{\sqrt{2}}$. An obvious method is to extract the root of 2, and then divide the result, say 1.414, into 1. It is much simpler to follow another procedure where the fraction is first *rationalized*, that is, the denominator is freed of radicals. This can be done in the example at hand by multiplying above and below by $\sqrt{2}$ as shown.

Rationalization of fractions

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414\cdots}{2} = 0.707\cdots.$$

In the following illustrations fractions have been rationalized.

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}; \qquad \frac{1}{\sqrt[3]{9}} = \frac{1}{\sqrt[3]{9}}\,\frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{\sqrt[3]{3}}{\sqrt[3]{27}} = \frac{\sqrt[3]{3}}{3};$$

$$\frac{1}{1-\sqrt{2}} = \frac{1}{(1-\sqrt{2})}\,\frac{(1+\sqrt{2})}{(1+\sqrt{2})} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} = -(1+\sqrt{2}).$$

Care must be exercised in treating sums of radicals, for such sums cannot be combined into one unless the terms can be reduced

to expressions in the *same radical.* We cannot combine $\sqrt{2}+\sqrt{3}$ into, for example, $\sqrt{5}$ or $\sqrt{6}$. The terms

$$\sqrt{12}+2\sqrt{3}-\sqrt{1200}$$

can be united into one term by removing factors from the radicals with the aid of property 42 for radicals, giving $2\sqrt{3}+2\sqrt{3}-20\sqrt{3}$, that is, $-16\sqrt{3}$. *Addition of radicals*

To simplify

$$\sqrt{\frac{9\,a}{b}}+3\sqrt{a^3b}+\sqrt{\frac{16\,a^7}{b^3}}+\sqrt{\frac{4\,b^5}{a}}$$

we remove from each radical the factors which are perfect squares to obtain

$$\frac{3\sqrt{ab}}{b}+3\,a\sqrt{ab}+\frac{4\,a^3}{b^2}\sqrt{ab}+\frac{2\,b^2\sqrt{ab}}{a}.$$

Combining terms yields

$$\frac{(3\,ab+3\,a^2b^2+4\,a^4+2\,b^4)}{ab^2}\sqrt{ab}.$$

LOGARITHMS

In view of rules 35 and 36 for multiplication and division of powers, it is a simple matter to multiply or divide numbers when they are written as powers of the same base. In fact, then *multiplication reduces to addition and division reduces to subtraction.* To Henry Briggs (1556–1631) there occurred the idea of writing the positive real numbers as powers of 10, and of working with the corresponding exponents instead of the numbers. Briggs constructed tables of these exponents, so that to multiply two numbers, such as 179 and 43.85, one need only look up in these tables the exponents *2.25285* and *1.64197* (since $179 = 10^{2.25285}$ and $43.85 = 10^{1.64197}$), then add these exponents to get *3.89482*, and finally look up in the tables the number which corresponds to $10^{3.89482}$. This number is found to be 7849.2, which is therefore the desired product of 179 and 43.85. *Logarithms as exponents*

If a number m is written as $m = a^n$, we say that *n is the logarithm of m to the base a.* For example, *2* is the logarithm of 100 to the base 10 since $10^2 = 100$, and *5* is the logarithm of 32 to the base 2 since $2^5 = 32$. The expression "logarithm" is often abbreviated to "log."

Natural logarithms

John Napier (1550–1617) is probably the inventor of logarithms. It is after him that logarithms to a base called e, arising automatically in the branch of mathematics termed the *calculus*, are called *Napierian logarithms*. They are also termed *natural logarithms*. But in what follows "logarithm" will mean *logarithm to the base* 10 as in the first example above.

Decimals and logarithms

If $a = 10^n$, $b = 10^m$, rule 35 implies that $ab = 10^{n+m}$. Since $n = \log a$, $m = \log b$ and $n + m = \log ab$, we have

$$\log ab = \log a + \log b. \tag{45}$$

Thus the *logarithm of a product is the sum of the logarithms of the factors*. Since the logarithm of 10 is *1* and the logarithm of 2 is *0.30103* (from tables), the logarithm of 20, that is, 2×10, is *1.30103*. The decimal parts of the logarithms of 2 and 20 are the same. Likewise you can verify that the logarithm of 200 is *2.30103*, the logarithm of 2000 is *3.30103*, and so on.

The number 0.2 is $2 \times \frac{1}{10}$. The logarithm of $\frac{1}{10}$ is -1 since $10^{-1} = \frac{1}{10}$. It follows that the logarithm of 0.2 is $-1 + 0.30103$. This logarithm is usually written as $9.30103 - 10$. Similarly, the logarithm of 0.02 is $-2 + 0.30103$, written as $8.30103 - 10$, and the logarithm of 0.002 is $-3 + 0.30103$, written as $7.30103 - 10$.

Mantissa and characteristic

From the argument above you can see that the decimal part of the logarithm of any number having 2 as the only nonzero digit is always *0.30103*. This decimal part, termed the *mantissa* of the logarithm, corresponds to the digit 2, whereas the rest of the logarithm, called the *characteristic*, relates to the position of the decimal point.

The logarithms of 721.1, 72.11, 7.211, 0.7211, 0.07211, 0.007211, and 0.0007211 are *2.85800*, *1.85800*, *0.85800*, $9.85800 - 10$, $8.85800 - 10$, $7.85800 - 10$, and $6.85800 - 10$ respectively. The mantissa of each of these numbers is *85800*. The characteristic is *2*, *1*, *0*, $9 - 10$, $8 - 10$, $7 - 10$, and $6 - 10$ respectively. It will prove convenient to use $9 - 10$, $8 - 10$, and so on, as characteristics in place of negative numbers.

Since the *mantissa* is independent of the position of the decimal point in the given number, tables of logarithms actually give only mantissas, the determination of the *characteristic* being left to the computer.

Finding the mantissa in the tables

The log tables given at the back of this book are five-place tables, that is, five places of the mantissa are given. This corresponds to four significant digits in the number whose mantissa is given in

the tables. The first three digits of this number are to be found in the column headed N, and the last digit in the row containing the same letter N. The mantissa of the number is to be found in the row containing the first three digits of the number and the column headed at the top by the remaining digit. Thus, to get the mantissa of 2,437,000 you use only the four digits 2437. You locate 243 in the column headed N, and 7 in the row of N. In the same row as the one in which 243 occurs and in the column headed 7 you will find *686*, which are the last three digits of the mantissa. The first two will be found above and at the left, namely *38*, so that the mantissa of 2,437,000 is *38686*. The first two digits of the mantissa are the same for several successive mantissas in the tables, and therefore these digits are not constantly repeated, in fact, they are given only once. Thus all of the mantissas after *38021* in the tables up to *39005* have *38* as the first two digits.

Turn to page 528 of the tables

A star in front of the last three digits of a mantissa means that the mantissa's two beginning digits are the next first two digits occurring in the tables. The use of these stars saves printing space. In particular, the mantissa *39005* comes from **005* and the *39* of *39094*. The mantissa preceding *38021* is thus *38003*, not *37003* since the digits **003* are starred.

Starred values in tables

From the examples above you will see that the *characteristic of a number bigger than or equal to* 1 *is one less than the number of digits to the left of the decimal place*. The characteristic of 2,437,000 is *6*. Therefore, log 2,437,000 = *6.38686*.

Computation of characteristic

To get the logarithm of 89.01 you must first look up 890 in the column headed N, and 1 in the row marked N. The mantissa of the number is *94944*. The characteristic of the number is *1*. The desired logarithm is thus *1.94944*.

From the discussion of characteristic above you will conclude that the *characteristic of a positive number less than* 1 *is* − n, *where* n *is one more than the number of zeros between the decimal point and the first nonzero digit to the right of the decimal point*. The characteristic of 0.68433 is thus − *1*, or *9* − *10*, and the characteristic of 0.00038 is − *4*, or *6* − *10*. The logarithm of 0.000008901 is *4.94944* − *10*.

By means of the tables headed *proportional parts* you can use five digits of the given number instead of four. These tables are based on the fact that for small differences in the number the differences in the mantissas of the numbers are proportional to the differences in the numbers. Thus the number 34586 is six-tenths

Interpolating log tables

of the way between 34580 and 34590. The mantissa of 34586 is therefore six-tenths of the way between the mantissa *53882* of 34580 and the mantissa *53895* of 34590. The mantissa of 34586 is thus *53890*. The following relation is approximately true.

$$\frac{53890 - 53882}{53895 - 53882} = \frac{6}{10}.$$

The mantissa *53890* can be obtained directly from a table headed *proportional parts*. The difference between the mantissas of *34580* and *34590* is 13. If you locate the fifth digit 6 of *34586* to the left of the proportional-parts column headed 13 you will find 7.8, which is 0.6×13, in the same row. The entry 7.8 when rounded off to 8 gives the correction which must be added to the mantissa *53882* of 3458 to obtain the mantissa *53890* of 34586.

Antilogarithms

A number whose logarithm is n is termed the *antilogarithm* of n. Thus the antilogarithm of *0.30103* is 2. The process of finding the antilogarithm of a number by means of the tables is, of course, the opposite to that described above for determining the logarithm of a number. For example, let us find the antilogarithm of *8.46752 − 10*. The mantissa *46752* is between the entries *46746* and *46761* in the tables on page 529. Since the difference between *46746* and *46761* is 15, we must use the table of proportional parts headed 15. The difference between the mantissas *46746* and *46752* is 6. We look up this difference among the logarithmic corrections in this table. The corresponding antilogarithmic correction is 4. Since the mantissa *46746* belongs to the sequence of digits 2934, the mantissa *46752* belongs to 29344. From the characteristic *8 − 10* we find that the antilogarithm of *8.46752 − 10* is 0.029344. That is, the logarithm of 0.029344 is *8.46752 − 10*.

Example of multiplication with logs

To multiply 534.87 by 0.046832 we look up the logarithms of these numbers in the tables. They are *2.72825* and *8.67054 − 10* respectively. Addition yields *11.39879 − 10*, which we abbreviate to *1.39879*. The antilogarithm of *1.39879* is 25.049, which is the desired product.

Logarithm of quotient

If $a = 10^n$, $b = 10^m$, rule 36 implies that $a/b = 10^{n-m}$. Since $n = \log a$, $m = \log b$, $n - m = \log a/b$, we have

$$\log \frac{a}{b} = \log a - \log b. \tag{46}$$

It follows that the *logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.*

To find the quotient of the number 0.046832 by the number 534,870,000 we write the logarithm of 0.046832, given above, as *18.67054 − 20.* We subtract as indicated, the logarithm of 534,870,000 being *8.72825.*

$$\begin{array}{r} 18.67054 - 20 \\ \underline{8.72825} \\ 9.94229 - 20 \end{array}$$

The antilogarithm of *9.94229 − 20* is 0.000,000,000,087,556, or $87{,}556 \times 10^{-15}$.

With $a = 10^n$, so that n is the logarithm of a, rule 37 implies that $a^m = 10^{nm}$, so that $\log a^m = nm$. But, since $n = \log a$, we have the rule which follows: *Logarithm of a power*

$$(47) \qquad \log a^m = m \log a.$$

Thus the *logarithm of a power of a number is the exponent times the logarithm of the number.*

To find the value of $(534.87)^{\frac{3}{4}}$ we first multiply the logarithm of 534.87, which is *2.72825,* by $\frac{3}{4}$ to get *2.04619.* The antilogarithm of *2.04619* is 111.22, so that

$$(534.87)^{\frac{3}{4}} = 111.22.$$

With logarithms you can extract roots of numbers which you could not otherwise obtain in reasonable time. Thus to get the real 25th root of 24, you write $\sqrt[25]{24} = 24^{\frac{1}{25}}$, to make use of rule 47. Multiplying the logarithm of 24 which is *1.38021* by $\frac{1}{25}$, you obtain *0.055208,* whose antilogarithm is 1.1355. Therefore, $\sqrt[25]{24} = 1.1355$. *25th root*

The slide rule, covered in a later chapter, enables you to add and subtract logarithms mechanically. The scales on the slide rule which you ordinarily use for multiplication of numbers are called *logarithmic scales,* or simply *log scales.* The distance from the left end of the log scale to the number 2 on this scale is not 2 units as on an ordinary rule, but rather the logarithm *0.30103* of 2 (for an appropriate unit of measure), the distance from the left end to the number 3 is the logarithm *0.47712* of 3, and so forth. *Slide rule*

When we multiply 2 by 3 on the slide rule we add the distance 0.30103 on one scale to the distance 0.47712 on another by laying off these distances end to end. The result is a distance 0.77815, marked 6 on the scale where the answer is read. The logarithm of 6 is *0.77815.* Thus the addition of logarithms is accomplished on the slide rule by the addition of distances. *Multiplication with the slide rule*

We have used five-place logarithms for purposes of explanation. Actually, the slide rule is accurate to three places only. If you need more accuracy you will have to employ some other method such as the use of logarithmic tables.

Problems

1. Write the following as powers of 2:

$$32;\ 16;\ \tfrac{1}{4};\ 1;\ 256;\ \sqrt[3]{4};\ \sqrt[5]{2}.$$

Ans. $2^5;\ 2^4;\ 2^{-2};\ 2^0;\ 2^8;\ 2^{\frac{2}{3}};\ 2^{\frac{1}{5}}$.

Powers and roots

2. Perform the indicated operations:

(a) a^4a^6; (b) $x^m x^{2-m}$; (c) $(2\,x)^4$; (d) $(y^4)^4$; (e) $(2^4)^n$; (f) aa^2a^3; (g) $(-\,3\,z^2)^3$; (h) $x^2(xy)^3$; (i) $\dfrac{x^6}{x^2}$; (j) $\dfrac{a^6}{a^{10}}$; (k) $\left(\dfrac{-\,2\,a}{bc}\right)^3$; (l) $\left(\dfrac{3\,a^{\frac{1}{3}}}{y^{\frac{2}{3}}}\right)^3$; (m) $\dfrac{2^0\,a^3}{a^{\frac{3}{2}}}$; (n) $8^{\frac{2}{3}}$; (o) $(.001)^{\frac{2}{3}}$; (p) $(-\,\tfrac{1}{8})^{\frac{4}{3}}$; (q) $\sqrt[4]{\sqrt{64}}$; (r) $\sqrt[3]{8\,a^4}$.

Ans. (a) a^{10}; (b) x^2; (c) $16\,x^4$; (d) y^{16}; (e) 2^{4n}; (f) a^6; (g) $-\,27\,z^6$; (h) x^5y^3; (i) x^4; (j) a^{-4}; (k) $\dfrac{-\,8\,a^3}{b^3c^3}$; (l) $\dfrac{27\,a}{y^2}$; (m) $a^{\frac{3}{2}}$; (n) 4; (o) 0.01; (p) $\frac{1}{16}$; (q) $\sqrt[4]{8}$; (r) $2\,a\sqrt[3]{a}$.

3. Find the value of each of the radicals:

$$\sqrt{36};\ \sqrt[4]{81};\ \sqrt[5]{32};\ \sqrt{.0009};\ \sqrt[5]{-\,32};\ \sqrt{\tfrac{1}{9}};\ \sqrt[3]{-\,8}.$$

Ans. $6;\ 3;\ 2;\ 0.03;\ -\,2;\ \tfrac{1}{3};\ -\,2$.

4. Find the value of each of the following expressions by changing each negative power to a positive one, and each fractional power to a radical

$$9^{\frac{1}{2}};\ (-\,2)^{-3};\ 6^{-1};\ (.0001)^{\frac{1}{4}};\ 27^{-\frac{1}{3}}.$$

Ans. $3;\ -\,\tfrac{1}{8};\ \tfrac{1}{6};\ .1;\ \tfrac{1}{3}$.

5. Rationalize the denominators of the fractions and simplify:

$$\frac{1}{\sqrt{5}};\ \frac{1}{\sqrt{8}};\ \sqrt{\tfrac{2}{3}};\ \sqrt[3]{\tfrac{1}{4}};\ \frac{1}{1+\sqrt{3}};\ \sqrt{\frac{3}{2\,ab^3}}.$$

Ans. $\dfrac{\sqrt{5}}{5};\ \dfrac{\sqrt{2}}{4};\ \dfrac{\sqrt{6}}{3};\ \dfrac{\sqrt[3]{2}}{2};\ \tfrac{1}{2}(\sqrt{3}-1);\ \dfrac{\sqrt{6\,ab}}{2\,ab^2}$.

6. Simplify the expressions:

(a) $\sqrt[3]{8\,a^3b^9}$; (b) $\sqrt{45}+\sqrt{20}$; (c) $\sqrt{\tfrac{1}{2}}-\sqrt{8}$; (d) $2\sqrt{x}-\sqrt{4\,x}+\sqrt{9\,x}$; (e) $\sqrt{12}+\sqrt{18}$; (f) $3\,x\sqrt{16\,x}-\sqrt{81\,x^3}$; (g) $\sqrt{2\,x^3}\,\sqrt{2\,x}$; (h) $\sqrt{\sqrt[3]{16\,x^2}}$; (i) $\dfrac{\sqrt{2}}{\sqrt[3]{4}}$; (j) $\sqrt{2}\,\sqrt[3]{3}$.

Ans. (a) $2\,ab^2$; (b) $5\sqrt{5}$; (c) $-\,\tfrac{3}{2}\sqrt{2}$; (d) $3\sqrt{x}$; (e) $2\sqrt{3}+3\sqrt{2}$; (f) $3\,x\sqrt{x}$; (g) $2\,x^2$; (h) $\sqrt[3]{4\,x}$; (i) $\dfrac{\sqrt[6]{32}}{2}$; (j) $\sqrt[6]{72}$.

Logarithms

7. Find the logarithms of 81, $\frac{1}{9}$, 243, $\frac{1}{27}$, 1 to the base 3.

Ans. 4; − 2; 5, − 3; 0.

8. What are the characteristics of the logarithms of the following numbers?

3,743.4; 1,000,000; 0.001; 0.236

Ans. 3; 6; 7 − 10; 9 − 10.

9. If the mantissas of 25.84, 0.5801, 1342 are *41229, 76350, 12775* respectively, what are the logarithms of the following?

$$25.84;\ 0.5801;\ 1342;\ 25.84 \times 0.5801;\ 25 \times 84 \times 0.5801 \times 1342;$$
$$\frac{25.84}{1342};\ \frac{1342}{.5801};\ \frac{25.84 \times 1342}{.5801};\ (1342)^4;\ \sqrt[5]{1342}.$$

Ans. *1.41229; 9.76350 − 10; 3.12775; 11.17579 − 10; 6.21347; 8.28454 − 10; 3.36425; 4.77654; 12.51100; 0.62555.*

10. By means of the tables find the logarithms of the following numbers: *Log tables*

8,234; .00068; 568,430; 7.6835; 0.89999.

Ans. *3.91561; 6.83251 − 10; 5.75467; 0.88555; 9.95424 − 10.*

11. By means of the tables find the antilogarithms of the numbers:

8.66666 − 10; 2.31156; 0.11111; 9.12315 − 10.

Ans. 0.046415; 219.56; 1.2916; 0.13288.

12. By means of logarithms compute the values of the following expressions to five significant digits.

(a) $\sqrt[30]{3.6875}$; (b) $\frac{37.687 \times 0.04351}{376.88}$; (c) $(3.1416)^7$; (d) $10^{\frac{7}{8}}$; (e) $\sqrt[3]{(1.1212)^2}$; (f) $\left(\frac{56,792}{400,150}\right)^4$.

Ans. (a) 1.0445; (b) 3020.3; (c) 1.0793; (d) 0.0043509; (e) 7.4990; (f) 0.00040573.

13. The resistance of air to a certain ball varies directly as the three-halves power of the speed for the range of speeds of this problem. When the speed of the ball is 2500 ft. per minute, the air resistance is exactly 125 lb. By the use of logarithms find the air resistance of the ball when the speed of the ball is 3,341.5 ft. per minute. *Ans.* 193.16 lb.

TRIGONOMETRIC FUNCTIONS OF ANGLES

Angles

The word *trigonometry* comes from the Greek and means "the measure of triangles." Angles have already been used in the treatment of the Pythagorean theorem. For trigonometric studies it will be necessary to give meaning to the "size" of angles, and to the concepts of "positiveness" and "negativeness" of angles.

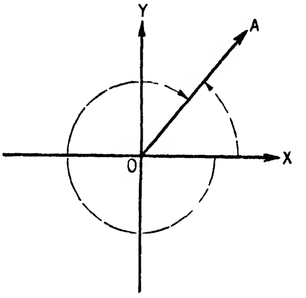


FIG. 232. TWO ANGLES TERMINATING AT OA

Starting with a ray in the position OX of Fig. 232, and keeping the initial point O of the ray fixed, we generate an angle by rotating this ray in the plane to a position OA. In generating the angle, the direction of rotation is for convenience kept constant, that is, the ray is rotated clockwise, or counterclockwise. By *clockwise* is meant the direction in which the hands of a clock revolve.

Positive and negative angles

If the angle is generated by rotating the ray counterclockwise, it is said to be a *positive angle*, and if the ray is rotated clockwise, it generates a *negative angle*. In Fig. 232 two angles are indicated, both having OX as the initial side and OA as the terminal side. The angle obtained by the smaller rotation is positive, and the other is negative.

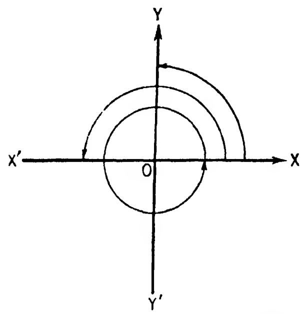


FIG. 233. ANGLES OF 90°, 180°, AND 360°

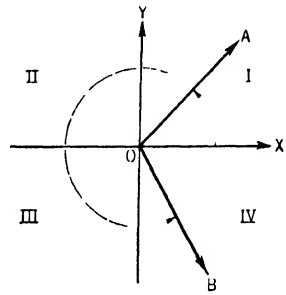


FIG. 234. ADDITION OF ANGLES XOA AND AOB

The initial and terminal sides of the angle may be identical. In Fig. 233 such an angle is indicated, formed by one complete revolution. Such an angle is said to be an angle of 360 degrees, written 360°. One half of this complete rotation produces a *straight angle* (of 180°); and one quarter of a complete rotation generates a *right angle*, that is, an angle of 90°. These are also illustrated in Fig. 233.

Addition of angles

Two angles are added by placing the initial side of the second angle on the terminal side of the first. The initial side of the sum is the initial side of the first angle, and the terminal side of the sum is the terminal side of the second angle. (See Fig. 234.) If both of the given angles are generated by the process of rotating rays counterclockwise, the sum of these angles is understood to be generated by rotating a ray continuously in the same manner. In Fig. 235 we have added the positive angle XOA to the negative angle AOB to obtain the negative angle XOB. The ray was rotated first positively (counterclockwise) and then negatively (clockwise). Addition of an angle of 400° to an angle of − 700° is an angle of − 300°. Evidently an angle may exceed one revolution which is 360° as shown in Fig. 236.

Minutes and seconds

The right angle is divided into 90 equal parts, each part being equal to one degree. Each degree is the sum of 60 equal parts,

termed *minutes*, and each minute is subdivided into 60 equal parts, called *seconds*. One minute is written as $1'$, and one second as $1''$. One minute is $60''$, and one degree is $60'$.

In the United States Army the unit of angular measure is the *mil*. This is $\frac{1}{6400}$ of one revolution, or approximately $\frac{1}{1000}$ of a radian. *Mils*

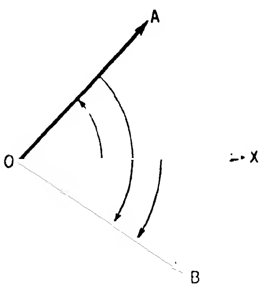


Fig. 235. Addition of a negative angle to a positive one

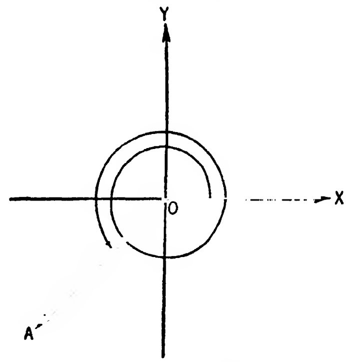


Fig. 236. Angle of more than one revolution

In Figs. 232–236 angles have been drawn with OX as the initial side. This was done for convenience, although the initial side can be taken anywhere. When the initial side of the angle is taken to be OX, the angle is said to be in *standard position*. *Standard position of angles*

If the terminal side of an angle is in a given quadrant, the angle is said to be in that quadrant. As shown in Fig. 234 the angle 45° is an angle in the first quadrant, whereas − 60° is in the fourth. The angle of 0°, which has the same initial and terminal sides, is in the first quadrant. *Quadrants*

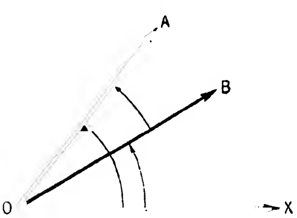


Fig. 237. Subtraction of angles

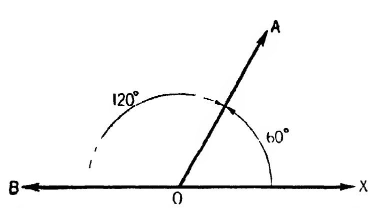


Fig. 238. Subtraction of − 120° from 60°

From the addition of angles the concept of subtraction follows immediately. In Fig. 237 the angle XOB is the angle XOA minus the angle BOA. It will be seen that we have the following rule: *Subtraction of angles*

To subtract an angle BOA from an angle XOA, place the terminal side of BOA on the terminal side of XOA. The *initial side of the new angle* representing the difference of these angles is the initial side OX of the angle XOA, and the terminal side of the new angle is the initial side OB of the angle BOA. In Fig. 238 the angle $-120°$ is subtracted from 60° to yield the angle XOB of angular measure 180°.

Examples of the addition and subtraction of angles follow.

Examples of addition and subtraction of angles

EXAMPLE 1. *Addition.*

$$\begin{array}{r} 20°\ 50'\ 37'' \\ \underline{400°\ 40'\ 34''} \\ 420°\ 90'\ 71'' \end{array}$$

Since 71″ is 1′ 11″ and 90′ is 1° 30′, the answer should be written as 421° 31′ 11″.

EXAMPLE 2. *Subtraction.*

Suppose that you wish to subtract 87° 30′ 30″ from 135° 14′ 22″. Since 30″ cannot be subtracted from 22″ to give a positive result, one minute is borrowed from the 14′ and added to the 22″ to give 82″. The larger angle is now written as 135° 13′ 82″. For the same reason that 22″ was changed to 82″, one degree is borrowed from 135° and added to 13′ so that the larger angle becomes 134° 73′ 82″. The subtraction proceeds as indicated.

$$\begin{array}{r} 134°\ 73'\ 82'' \\ \underline{87°\ 30'\ 30''} \\ 47°\ 43'\ 52'' \end{array}$$

Negative result

EXAMPLE 3. *Subtraction.*

The result of subtracting 347° 2′ 50″ from 13° 8′ 10″ is negative. The angle 13° 8′ 10″ can therefore be subtracted from 347° 2′ 50″ and a negative sign placed before the answer. We write 347° 2′ 50″ as 346° 62′ 50″, and subtract as shown.

$$\begin{array}{r} 346°\ 62'\ 50'' \\ \underline{-\quad 13°\ \ 8'\ 10''} \\ 333°\ 54'\ 40'' \end{array}$$

Hence, the answer is $-$ 333° 54′ 40″.

ANGLE, ARC, AND RADIAN

Arc of circle

When you move from A to B on a circle as in Fig. 239, we say that you *generate an arc* of the circle. The circumference of a circle

covers an arc of one revolution. If you move halfway around a circle, you generate an arc of a half revolution, and if you go around a circle twice, you generate an arc of two revolutions. These examples serve to show that an arc may not only be part of the circumference of a circle, but even more than the entire circumference.

Arc subtended by angle

In Fig. 239 an angle AOB is drawn with vertex at the center of a circle of radius r. When the angle AOB is generated, the side OA is rotated counterclockwise to the position OB as indicated. In this generation the point A sweeps over an arc of the circle, called the arc *subtended* by the angle AOB. This arc is called the *arc* AB. The arc AB is said to have the same angular measure as the angle AOB. An angle of 50° with vertex at the center of a circle subtends an arc of 50° on the circumference of the circle. The complete circumference of a circle is an arc of 360°.

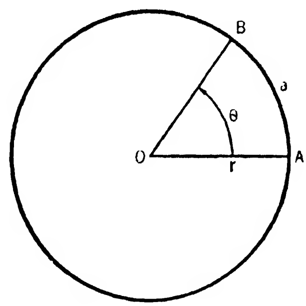


Fig. 239. Arc AB subtended by the angle AOB

Length of arc

We recall that the length of the circumference of a circle is π times the length of the diameter. The idea of the length of an arc has thus already risen in this chapter. We let a and b denote arcs on a circle. If the number of degrees in the arc a is k times the number of degrees in the arc b, the arc a is k times as long as b. The circumference of a circle is thus four times as long as an arc of the circle subtended by an angle of 90° with vertex at the center of this circle. In a circle of radius 1 in. the circumference is 2π in.; for a radius of 10 in. the length of this arc is 20π in.

Radian measure

If the angle AOB in Fig. 239 subtends an arc AB whose length equals the radius of the circle, the angle AOB is said to be an angle of *one radian*, and the arc AB is said to be an arc of one radian. For most mathematical considerations the radian is a more convenient unit of measure than the degree. For this reason radian measure is said to be the natural system of measuring angles.

Radian and degree

An angle of 360° is an angle of 2π radians since the circumference of a circle of radius r is $2\pi r$. It follows that an angle of 1° is an angle of $\pi/180$ radians. The quotient $\pi/180$ is approximately 0.01745, so one degree is 0.01745 radian.

In changing from degrees, minutes, and seconds to radians with-

out the use of tables for this purpose, it is convenient to change minutes and seconds to decimal parts of a degree. One minute is $(\frac{1}{60})°$, or 0.0166 · · ·°, and one second is $(\frac{1}{3600})°$, or 0.000277 · · ·°. Let us change 514° 15′ 30″ to radians. We have

$$15' = 0.25°; \qquad 30'' = 0.0083°$$

so that

$$514°\,15'\,30'' = 514.258°.$$

Multiplying 514.258° by 0.01745 we obtain 8.9738 radians as an approximate equivalent to 514° 15′ 30″.

Applications of radian measure

Using the notation θ for the measure of the angle AOB in radians, r for the length of the radius OA, and a for the length of the arc AB as shown in Fig. 239, we have the relation

$$\theta = \frac{a}{r}.$$

The Greek letter θ is read "theta." The angle θ in radians is the length a of the arc divided by the length r of the radius, or in usual terminology the angle is the arc divided by the radius.

EXAMPLE 1.

Length of arc

Consecutive spokes make an angle of 10° at the center of a wheel of diameter 4 ft. The problem is to find the length of the arc subtended by the angle.

The angle 10° is first changed to $\pi/18$ radians. Multiplying $\pi/18$ by the radius 2 (ft.), the length of the arc is found to be $\pi/9$ ft. To obtain the answer to four figures you can divide 9 into 3.1416, or you can change 10° to 0.1745 radian and multiply this figure by 2. By either method the answer is 0.349 ft.

EXAMPLE 2.

Linear speed

A pulley 8 in. in diameter attached to an electric motor is revolving at an *angular speed* of 1800 revolutions, that is $1800 \times 2\pi$ or 3600π radians, per minute. A belt is attached to the pulley. We shall find the speed at which a point on the belt is traveling, that is, the speed of the belt. The speed of a body traveling along a straight line is called a *linear speed*, which we wish to compute here.

In 1 min. the pulley rotates through 3600π radians. A point on the circumference 4 in. from the center travels 3600π times 4, or $14{,}400\pi$ in. per minute. Since the belt is always in contact with the pulley, a point on the belt travels at the same speed as a point on the circumference of the pulley. The speed of the belt

is thus $14{,}400\,\pi$ in. per minute. This amounts to 3770 ft. per minute. Multiplying by 60, the number of minutes in an hour, and dividing by 5280, namely the number of feet in a mile, we find that the belt speed is approximately 43 m.p.h.

EXAMPLE 3.

Angular speed

A locomotive travels 110 m.p.h. One wheel of the locomotive is 6 ft. in diameter. We shall determine the angular speed of the wheel in revolutions per minute, in radians per second, and in degrees per second.

Multiplying 110 (miles per hour) by 5280 and dividing by 60 we find that the speed of the locomotive is 9680 (feet per minute). The circumference of the wheel is π times the diameter or $6\,\pi$ ft. Since a point on the circumference of a wheel travels at the same speed as the locomotive, the angular speed of the wheel in revolutions per minute (abbreviated r.p.m.) is $9680/6\,\pi$, or approximately 514 r.p.m. Each revolution is 360° and there are 60 sec. in a minute. Hence, the angular speed is also $514 \times 360 \div 60$ or 3084 degrees per second. Since there are $2\,\pi$ radians in one revolution and 60 sec. in 1 min., the speed in radians per second is $\dfrac{514 \times 2\,\pi}{60}$, or 53.7.

The above examples illustrate the use of radian measure in the treatment of circles. Since circles play a dominant role in both mathematics and engineering, radian measure is of considerable importance.

SIMILAR TRIANGLES

Measuring angles

In the treatment of similar triangles to follow, the angles are measured counterclockwise so that they are all positive. The angular measure of each is taken to be the smallest positive measure the angle can have.

In Fig. 240 the angles A, B, C of the triangle ABC are equal respectively to the angles A', B', C' of the triangle $A'B'C'$. The triangles are therefore said to be *similar*.

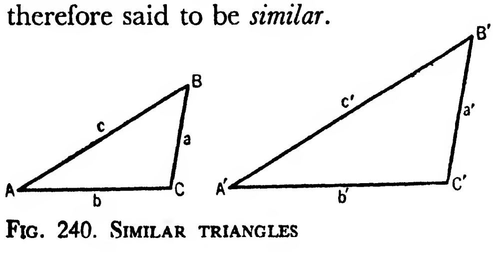


FIG. 240. SIMILAR TRIANGLES

Proportionality of sides

We shall show that corresponding sides of similar triangles are proportional, that is, for Fig. 240,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Sum of the angles in a triangle

A rectangle has four right angles formed by its perpendicular sides. These four right angles add up to 360°. From Fig. 221 it appears that the sum of the angles A, B, and C in the right triangle ABC is one-half of the sum of the angles of the rectangle. It follows that the angles in a right triangle add up to 180°. Any triangle can be cut up into two right triangles as shown in Fig. 219. The sum of the angles in the right triangles ABD and DBC of the left figure is 360°. The angle ADC is a *straight angle* having a measure of 180°. This angle is not one of the angles of the triangle ABC, although it is the sum of two angles of the right triangles ABD and DBC. The angles in the whole triangle ABC therefore add up to an angle of 180°. Hence, *the sum of the angles in any triangle equals a straight angle.*

Parallel lines

Two lines in the same plane which never meet are said to be parallel. We cut two lines l and l' by a line m, as shown in Fig. 241.

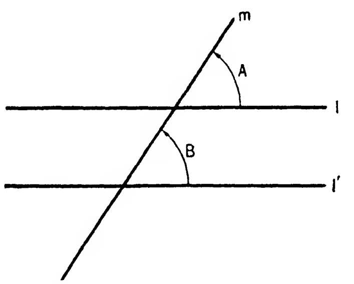


FIG. 241. PARALLEL LINES CUT BY A THIRD LINE

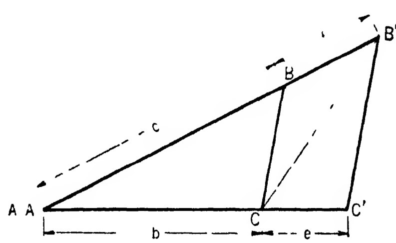


FIG. 242. SUPERIMPOSED SIMILAR TRIANGLES

From propositions of geometry it can be proved that the lines l and l' are parallel if and only if the angles A and B are identical.

The triangles of Fig. 240 are superimposed as in Fig. 242 so that A and A' coincide, whereas the side AB lies on $A'B'$ and AC on $A'C'$. Because the angles C and C' are equal, the sides BC and $B'C'$ are parallel. We draw the dotted lines CB' and BC'.

The triangles ABC and CBC' have bases AC and CC' respectively and the same altitude. Hence it follows that

$$\frac{\text{Area of } ABC}{\text{Area of } CBC'} = \frac{b}{e}.$$

Similarly,

$$\frac{\text{Area of } ABC}{\text{Area of } CBB'} = \frac{c}{f}.$$

Proportionality of sides

The triangles CBC' and CBB' have the common base CB and the same altitude. These triangles therefore have the same area. In the equations above, b/e and c/f are now seen to be equal to the same quantity; hence, we write

$$\frac{b}{e} = \frac{c}{f}. \tag{48}$$

Adding 1 to both sides (e/e on the left and f/f on the right) we have

$$\frac{b}{e} + \frac{e}{e} = \frac{c}{f} + \frac{f}{f}; \quad \text{or} \quad \frac{b+e}{e} = \frac{c+f}{f}; \quad \text{which with (48) yields} \quad \frac{b'}{b} = \frac{c'}{c}$$

because $b' = b + e$ and $c' = c + f$. Hence, the following is proved: *If two triangles are similar, corresponding sides are proportional.* Conversely, it can be proved that if corresponding sides of two triangles are proportional, the triangles are similar. To say that two triangles are similar is the same as saying that they have the same shape.

FUNCTIONS OF ANGLES

Acute angles

An angle with positive measure less than 90° is termed an *acute angle*. Given any acute angle A, a right triangle can be constructed with A as one of its angles. The measure of the remaining acute angle in this triangle is 90° minus the measure of the angle A. All angles of the right triangle are thus determined by A. As a result, the shape of a right triangle is fixed by the measure of one of its acute angles. In this sense an acute angle A corresponds uniquely to all similar right triangles containing A. Any property which all of these triangles possess in common is thus a property of A. In particular, the ratio of one side of a right triangle containing the angle A to another side of this triangle is the same as the ratio of corresponding sides of any other similar right triangle containing the angle A. These ratios are the *trigonometric functions* of the angle A. They are called *functions* because they depend for their values on the measure of the angle A.

Suppose we let θ be an acute angle in a right triangle. The trigonometric functions of θ are defined to be:

Trigonometric functions of acute angles

(49) $$\text{sine of } \theta = \sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}.$$

(50) $$\text{cosine of } \theta = \cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

(51) $$\text{tangent of } \theta = \tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}.$$

(52) $$\text{cotangent of } \theta = \text{ctn}\,\theta = \frac{\text{side adjacent}}{\text{side opposite}}.$$

(53) $$\text{secant of } \theta = \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}}.$$

(54) $$\text{cosecant of } \theta = \text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

Here sin is the abbreviation of sine; the word "cotangent" is often shortened to *cot* instead of *ctn*. In the right triangle of Fig. 220 we have

$$\sin A = \frac{a}{c}.$$

Functions of 45°

A right triangle with equal legs has acute angles each equal to 45°. Such a triangle is shown in Fig. 243. In this case the side

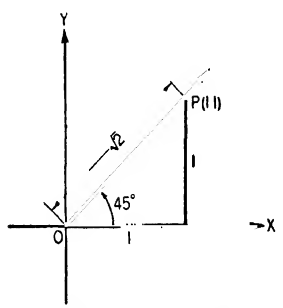


FIG. 243. ANGLE OF 45°

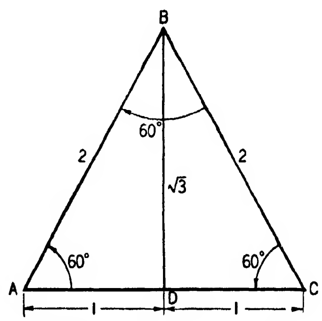


FIG. 244. EQUILATERAL TRIANGLE

opposite the indicated angle of 45° is 1, the side adjacent to this angle is 1, and the hypotenuse is $\sqrt{2}$. We therefore have the following:

$$\sin 45^\circ = \frac{\sqrt{2}}{2}; \qquad \tan 45^\circ = 1; \qquad \sec 45^\circ = \sqrt{2};$$

$$\cos 45^\circ = \frac{\sqrt{2}}{2}; \qquad \text{ctn}\, 45^\circ = 1; \qquad \text{cosec}\, 45^\circ = \sqrt{2}.$$

A triangle with equal angles is called an *equilateral triangle.* (See Fig. 244.) Such a triangle also has equal sides. Since the angles add up to 180°, each angle is one of 60°. We will let such a triangle have a side 2 units in length, as shown in Fig. 244. By

the Pythagorean theorem, applied to the right triangle ABD, the altitude BD of the triangle has length $\sqrt{3}$. The angle ABD is one-half of the angle ABC, and is therefore an angle of 30°. To obtain

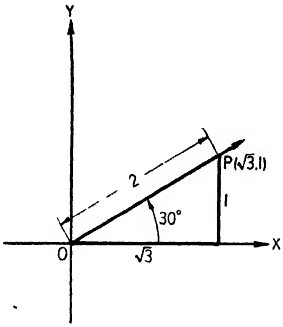


FIG. 245. ANGLE OF 30°

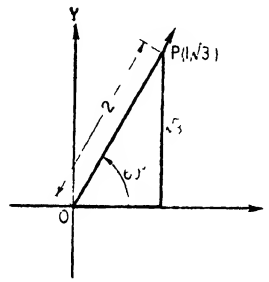


FIG. 246. ANGLE OF 60°

the trigonometric functions of 30° we may therefore use a right triangle with legs 1, $\sqrt{3}$, and hypotenuse 2, as shown in Fig. 245. We can see that:

Functions of 30°

$$\sin 30° = \tfrac{1}{2}; \qquad \tan 30° = \frac{\sqrt{3}}{3}; \qquad \sec 30° = \frac{2\sqrt{3}}{3};$$

$$\cos 30° = \frac{\sqrt{3}}{2}; \qquad \text{ctn } 30° = \sqrt{3}; \qquad \text{cosec } 30° = 2.$$

You can readily verify from Fig. 246 that

Functions of 60°

$$\sin 60° = \frac{\sqrt{3}}{2}; \qquad \tan 60° = \sqrt{3}; \qquad \text{cosec } 60° = \frac{2\sqrt{3}}{3};$$

$$\cos 60° = \tfrac{1}{2}; \qquad \cot 60° = \frac{\sqrt{3}}{3}; \qquad \sec 60° = 2.$$

Trigonometric functions of nonacute angles

The trigonometric functions of a nonacute angle θ cannot be obtained in the manner considered above because there is no right triangle with θ as one of its angles. For this reason the angle θ is placed in standard position on a Cartesian coordinate system as shown in Fig. 247. We select any point P on the terminal side of θ. We let (x, y) be the coordinates of P, and r the distance from the origin O to P. The definitions of the trigonometric functions of θ follow:

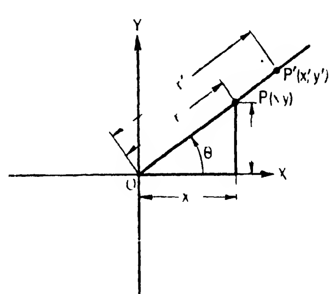


FIG. 247. ANGLE IN STANDARD POSITION ON A CARTESIAN COORDINATE SYSTEM

Principal functions of angles

(55) $$\sin\theta = \frac{\text{ordinate}}{\text{distance } r} = \frac{y}{r}.$$

(56) $$\cos\theta = \frac{\text{abscissa}}{\text{distance } r} = \frac{x}{r}.$$

(57) $$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}.$$

(58) $$\text{ctn}\,\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}.$$

(59) $$\sec\theta = \frac{\text{distance } r}{\text{abscissa}} = \frac{r}{x}.$$

(60) $$\text{cosec}\,\theta = \frac{\text{distance } r}{\text{ordinate}} = \frac{r}{y}.$$

The values of the trigonometric functions depend on θ only, not on the position of the point P on the terminal side of θ.

In Figs. 243–246 angles discussed above have been placed in standard position. From the coordinates of the points P in these figures and definitions 55 to 60, the trigonometric functions of these angles may be computed.

Functions of 0°

We use the point (1, 0) as in Fig. 248 to get the trigonometric functions of 0°. It follows from definitions 55, 56, 57, and 59 that

$\sin 0° = \frac{0}{1} = 0$; $\cos 0° = \frac{1}{1} = 1$; $\tan 0° = \frac{0}{1} = 0$; $\sec 0° = \frac{1}{1} = 1$.

Since division by 0 is not allowed, ctn 0° and csc 0° have no meaning. Sometimes the sign for infinity, ∞, is written for these functions, meaning that as an acute angle of $x°$ gets smaller and smaller,

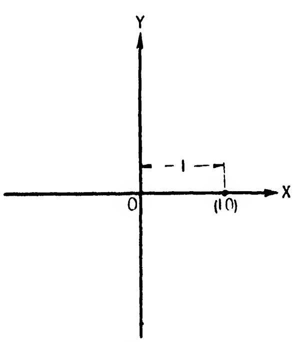


FIG. 248. ANGLE OF 0°

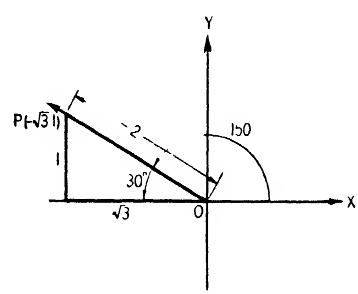


FIG. 249. ANGLE OF 150°

ctn $x°$ and cosec $x°$ get larger and larger; in fact as x goes to 0, the functions ctn $x°$ and cosec $x°$ increase beyond all bounds.

Functions of 150°

The study of the trigonometric functions of an angle over 90° in standard position can always be reduced to that of the acute

angle whose terminal side is identical with that of the given angle, and whose initial side is along the one half of the x axis. Thus if $150°$ is subtracted from $180°$, the acute angle $30°$ of Fig. 249 is obtained. The right triangle of Fig. 245 containing $30°$ is used in Fig. 249 to obtain the trigonometric functions of $150°$. From the point $P(-\sqrt{3}, 1)$ on the terminal side of $150°$ and from definitions 55 to 60 we have the following results:

$$\sin 150° = \tfrac{1}{2}; \qquad \tan 150° = -\frac{\sqrt{3}}{3}; \qquad \sec 150° = -\frac{2\sqrt{3}}{3};$$

$$\cos 150° = -\frac{\sqrt{3}}{2}; \qquad \text{ctn } 150° = -\sqrt{3}; \qquad \text{cosec } 150° = 2.$$

Solution of triangles in surveying

The sides and angles of a triangle are called the *parts* of the triangle. Trigonometric functions are particularly useful in finding unknown parts of a triangle. We can illustrate with an example.

EXAMPLE. *To find a leg of a right triangle.*

A man whose eye level is 6 ft. above the ground stands 150 ft. from a building. The measure of the *angle of elevation* of the building is exactly $30°$, which is the angle between the horizontal line of sight and the line of sight from the man to the top of the building. How high is the building?

We can let y be the height of the building, $y - 6$ being the vertical distance from the level of the man's eye to the top of the building, as shown in Fig. 250. By definition

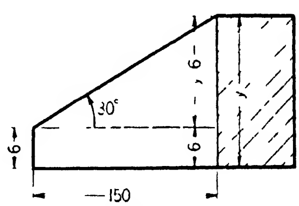


FIG. 250. MAN ABOUT 6 FEET HIGH AND BUILDING y FEET HIGH

$$\tan 30° = \frac{y-6}{150}; \qquad y - 6 = 150 \tan 30°; \qquad \text{or} \quad y = 150 \tan 30° + 6.$$

Substituting the value of $\tan 30°$, which is $\sqrt{3}/3$ or 0.577, we have

$$y = 150 \times 0.577 + 6 = 92.55.$$

The height of the building is therefore 92 ft. 7 in.

Dependence of trigonometric functions

The trigonometric functions of an angle are not independent. In fact, all of them are determined by any one trigonometric function and the quadrant in which an angle lies, when placed in standard position. For example, let us suppose that θ is an angle lying in the third quadrant for which $\sin \theta = -\frac{1}{3}$. We draw a circle of radius 3 with center at the origin as shown in Fig. 251. We also construct the line $y = -1$ parallel to the x axis and one

unit below. The point P where this circle and line intersect in the third quadrant is a point on the terminal side OP of θ. By the Pythagorean theorem, applied to the right triangle formed by dropping a perpendicular from P to the x axis, the point P has coordinates $(-2\sqrt{2}, -1)$. The trigonometric functions of θ are therefore:

$$\cos\theta = -\frac{2\sqrt{2}}{3}; \qquad \text{ctn}\,\theta = 2\sqrt{2}; \qquad \text{cosec}\,\theta = -3.$$

$$\tan\theta = \frac{\sqrt{2}}{4}; \qquad \sec\theta = -\frac{3\sqrt{2}}{4}.$$

Signs

The point P' of Fig. 251 is on the terminal side OP' of an angle in the *fourth quadrant* also having the sine $-\frac{1}{3}$. Since a trigonometric function has the same plus or minus sign in two quadrants, an angle with one known trigonometric function may be located as lying in either of two possible quadrants. For example, since the coordinate x has the same sign as y in the first and third quadrants, the tangent is positive for all angles lying in these quadrants. It follows that any angle for which the tangent is positive lies in the first or third quadrant. Then, if this same angle is known to have a negative sine, it must be a third-quadrant angle because only angles in the third and fourth quadrants have negative sines.

Third and fourth quadrant ∡s

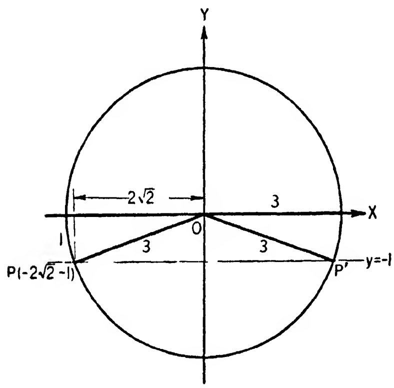


Fig. 251. Angle XOP determined by its sine and quadrant

Trigonometric identities

From the definitions of the trigonometric functions we have:

$$\text{cosec}\,\theta = \frac{1}{\sin\theta}; \quad \sin\theta = \frac{1}{\text{cosec}\,\theta}; \quad \tan\theta = \frac{1}{\text{ctn}\,\theta}; \quad \tan\theta = \frac{\sin\theta}{\cos\theta};$$

$$\sec\theta = \frac{1}{\cos\theta}; \quad \cos\theta = \frac{1}{\sec\theta}; \quad \text{ctn}\,\theta = \frac{1}{\tan\theta}; \quad \text{ctn}\,\theta = \frac{\cos\theta}{\sin\theta}.$$

The above relations are true for any value of θ, and are therefore called *identities*. To abbreviate notations we write $(\sin\theta)^2 = \sin^2\theta$, also $(\cos\theta)^2$ as $\cos^2\theta$, and so on. Since $\sin\theta = y/r$ and $\cos\theta = x/r$ we have

$$\sin^2\theta + \cos^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2+y^2}{r^2}.$$

By the Pythagorean theorem $x^2 + y^2 = r^2$, whence we have proved that

$$\sin^2\theta + \cos^2\theta = 1. \tag{61}$$

In the same way we can show that

$$1 + \tan^2\theta = \sec^2\theta; \qquad 1 + \operatorname{ctn}^2\theta = \operatorname{cosec}^2\theta. \tag{62}$$

The above identities are called the *fundamental trigonometric identities* because they are used so often, but there are countless other identities involving the trigonometric functions of θ.

Simplifying expressions

The fundamental identities 61 and 62 are useful in simplifying trigonometric expressions. For example, we may write

$$(\sec^2\theta - 1)\operatorname{cosec}^2\theta = (\tan^2\theta)\operatorname{cosec}^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\times\frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

In carrying out computations it is much easier to use $\sec^2\theta$ than the original expression.

Computing trigonometric functions

These identities can also be used to find the trigonometric functions of an angle, given one such function of the angle. As an illustration let us suppose that θ is an angle in the fourth quadrant with $\operatorname{cosec}\theta = -\frac{5}{3}$. Since the sine is the reciprocal of the cosecant, we have $\sin\theta = -\frac{3}{5}$. From $1 + \operatorname{ctn}^2\theta = \operatorname{cosec}^2\theta$ and the fact that the cotangent is negative in the fourth quadrant, we have $\operatorname{ctn}\theta = -\sqrt{\frac{25}{9} - 1} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$. Since the tangent is the *reciprocal* of the cotangent we have $\tan\theta = -\frac{3}{4}$. From the relationship $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ we determine that $\cos\theta = \dfrac{\sin\theta}{\tan\theta} = \dfrac{-\frac{3}{5}}{-\frac{3}{4}} = \frac{4}{5}$. From the relation $\sec\theta = 1/\cos\theta$, we have $\sec\theta = \frac{5}{4}$.

Sines and tangents of small angles

When an angle θ is small, the value of θ in radians and the values of $\sin\theta$ and $\tan\theta$ are approximately the same in the sense that as θ goes to 0, the fractions $\sin\theta/\theta$ and $\tan\theta/\theta$ go to 1. Thus the sine and tangent of one-tenth radian equal $\frac{1}{10}$ to an accuracy that is satisfactory for most engineering calculations. A particularly important application is to the angle of deflection or slope of a loaded beam. This angle is usually small and so its sine and tangent are equal to its angular value in radians to reasonable accuracy.

Problems

1. Perform the indicated operations graphically, using a protractor:
   $90° + 405°$; $90° - 135°$; $315° - 90°$; $180° - 45°$; $270° + 135°$.

2. Change $190° 40' 20''$ to its decimal equivalent. *Ans.* $190.6722\cdots°$.

3. Perform the indicated computations:
   $400° 57' 38'' + 210° 33' 50''$; $100° 13' 9'' - 45° 40' 40''$; $10° 15' 50'' - 100° 20' 10''$.
   Ans. $611° 31' 28''$; $54° 32' 29''$; $-90° 4' 20''$.

4. Change 87.345° to degrees, minutes, and seconds. *Ans.* 87° 20′ 43″.

5. In what quadrants do the following angles lie?

50°, 400°, 700°, 200°, − 200°, − 50°, − 500°.

Ans. I, I, IV, III, II, IV, III.

Radians

6. The radius of the earth is approximately 4000 miles. Two cities on the earth's surface are 1000 miles apart, this distance being measured on the surface. How many degrees, minutes, and seconds are they apart, and what is the difference in their sun times if one is directly east of the other?

Hint. Twenty-four hours corresponds to 360°.

Ans. 14° 19′ 26″; 57 min. 18 sec. in solar time.

7. Change the following angles to radians: 37.125°; 300° 10′ 40″.

Ans. 0.648; 5.239.

8. Change these angles to radians:

100°; − 47°; 30°; 45°; 60°; 90°; 120°; − 135°; − 150°.

Ans. $0.55\,\pi$; $-.26\,\pi$; $\frac{\pi}{6}$; $\frac{\pi}{4}$; $\frac{\pi}{3}$; $\frac{\pi}{2}$; $\frac{2\pi}{3}$; $-\frac{3\pi}{4}$; $-\frac{5\pi}{6}$.

9. Change the following angles from radians to degrees, minutes, and seconds:

7π radians; 3 radians; $\frac{\pi}{7}$ radians.

Ans. 1260°; 171° 53′ 14″; 25° 41′ 51″.

10. The bob of a pendulum 1 ft. long swings through an arc of 2 in. Through what angle does the pendulum swing? *Ans.* $\frac{1}{6}$ radian, or 9° 32′ 57″.

Revolving machines

11. The rim speed of a flywheel is 100 ft. per sec. Find the angular speed of the flywheel in radians per second if its diameter is 3 ft. *Ans.* $66\frac{2}{3}$ radians per second.

12. A wheel in a slow-moving mechanism revolves with an angular speed of 100° per minute. Find the rim speed in feet per second if the radius of the wheel is 2 in. *Ans.* 0.00485 ft. per sec.

13. A belt in contact with a pulley travels with a speed of 10 m.p.h. The radius of the pulley is 3 in. Find the angular speed of the pulley in revolutions per minute, and in radians per second. *Ans.* 560 r.p.m.; $58\frac{2}{3}$ radians per second.

Trigonometric functions

14. Find the trigonometric functions of 90°, 120°, 135°, 180°, 225°, and 270°.

Ans.

Angle	*Sin*	*Cos*	*Tan*	*Ctn*	*Sec*	*Cosec*
90°	1	0	No meaning	0	No meaning	1
120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
180°	0	-1	0	No meaning	-1	No meaning
225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
270°	-1	0	No meaning	0	No meaning	-1

15. An angle θ is an acute angle of a right triangle having a hypotenuse of 4 ft. The side adjacent to the angle is 3 ft. Find the trigonometric functions of the angle.

$$\textit{Ans.}\ \sin\theta = \frac{\sqrt{7}}{4};\ \cos\theta = \frac{3}{4};\ \operatorname{cosec}\theta = \frac{4\sqrt{7}}{7};\ \sec\theta = \frac{4}{3};$$
$$\tan\theta = \frac{\sqrt{7}}{3};\ \operatorname{ctn}\theta = \frac{3\sqrt{7}}{7}$$

16. An angle θ is in the third quadrant. Given that $\sin\theta = -\frac{5}{13}$, find the other trigonometric functions of θ.

Ans. $\cos\theta = -\frac{12}{13}$; $\tan\theta = \frac{5}{12}$; $\operatorname{ctn}\theta = \frac{12}{5}$; $\sec\theta = -\frac{13}{12}$; $\operatorname{cosec}\theta = -\frac{13}{5}$.

17. By the fundamental identities simplify the expression

$$\frac{\cos\theta}{1-\sin\theta} - \frac{1-\sin\theta}{\cos\theta}.$$

Hint. Multiply the first fraction above and below by $1+\sin\theta$. *Ans.* $2\tan\theta$.

Solving triangles

18. The line of sight of a man to an airplane makes an angle of 45° with horizontal. The airplane is 2 miles up. How far is the airplane from the man?

Ans. $2\sqrt{2}$ miles, or approximately 2.83 miles.

19. A sheet of metal is in the shape of a right triangle with a 30° angle. The side adjacent to the angle is 30 in. long. What is the area of the metal sheet?

Ans. 260 sq. in., or 1.80 sq. ft.

Properties of trigonometric functions

20. What are the plus or minus signs of trigonometric functions of angles in the fourth quadrant? *Ans.* sin −; cos +; tan −; ctn −; sec +; cosec −.

21. Prove that $1 + \cot^2\theta = \sec^2\theta$.

22. What is the range of values of the sine, cosine, tangent, and secant?

Ans. The sine and cosine are between −1 and 1; the tangent can be any real number; and the secant is greater than 1 or equal to 1, or less than or equal to −1.

23. What trigonometric functions of an acute angle θ increase with θ?

Ans. Sine, tangent, secant.

USE OF TRIGONOMETRIC TABLES

Trigonometric tables

You have already learned how to compute the trigonometric functions of certain angles, such as 30°, 45°, and 60°. It would be both tedious and difficult for you to derive the value of a trigonometric function of an angle each time you had to use it. For this reason tables of values of trigonometric functions have been computed. Such tables will be found at the back of this book. These tables can be constructed from exact formulas derived from the calculus for each trigonometric function. These formulas enable one to find a trigonometric function of any angle to any desired accuracy. As an example of such an expression we have for θ in radians

$$\sin\theta = \theta - \frac{\theta^3}{1\times2\times3} + \frac{\theta^5}{1\times2\times3\times4\times5} - \frac{\theta^7}{1\times2\times3\times4\times5\times6\times7} + \cdots.$$

The first extensive tables of trigonometric functions were developed by the Germans in the fifteenth and sixteenth centuries, prominent among the Germans being John Mueller, called *Regiomontanus* (1436–1476).

Explanation of trigonometric tables

With values of the angles in degrees given at the tops of the pages of the tables go the minute column at the left and the trigonometric function headings at the top. For example, sin 26° 17′ is 0.44281. On page 560 corresponding to 26° we locate 17 in the minute column, and the entry 0.44281 is found in the same row as 17 but under the column headed Sin. Similarly, we find tan 26° 17′ to be 0.49387.

Complementary angles

Two angles which add up to 90° are said to be *complementary*. The angles 30° and 60° are complementary. The sine of 30° is the cosine of 60°. Because of this property the cosine is said to be the *cofunction* of the sine. Similarly, the cofunctions of the cosine, tangent, cotangent, secant, and cosecant are the sine, cotangent, tangent, cosecant, and secant respectively. From the definitions of the trigonometric functions of acute angles in right triangles you can easily see that the *function of an angle is the cofunction of its complement*. The tangent of 20° is thus the cotangent of 70°. The origin of the prefix "co" in cosine, cotangent, and cosecant is therefore clear.

Tables for angles over 45°

Since a trigonometric function of an angle between 45° and 90° is the cofunction of the complement, the complement being between 0° and 45°, separate tables are not given for the acute angles between 45° and 90°. Instead the angles between 45° and 90° will be found listed at the bottoms of the tables. With these angles go the headings at the bottom and the minute columns on the right.

For example, consider the angle 63° 43′ whose complement is 90° − (63° 43′) = 26° 17′. Cos 63° 43′ will be found on page 560 listing 63° at the bottom and 26° at the top. Since we are interested in a cosine we look in the column headed Cos at the bottom and in the horizontal row which has 43 in the minute column on the right. The value of the cosine of 63° 43′ is 0.44281. This is the same value obtained for sin 26° 17′ found above. These cofunctions of complementary angles should be identical.

Interpolation

The tables are graduated in minutes. An angle given to a fraction of a minute, such as 26° 17′ 20″, will not be found in these tables. It is therefore necessary to *interpolate*. It will be observed

that for small differences in the angles near a given angle the differences in the angles are proportional to the differences in the sines of these angles. For angles near 26° 17′ the difference in the sine is 0.00026 for a difference of 1′ in the angle. Thus the sine of 26° 18′ is 0.00026 more than the sine of 26° 17′. See page 560.

To get the sine of 26° 17′ 20″ we observe that this angle is $\frac{20}{60}$ or one-third of the way between 26° 17′ and 26′ 18′. The same is true of the sine. We therefore take one-third of 0.00026, namely 0.00009, and add this to 0.44281, the sine of 26° 17′. Hence we find that

$$\sin 26^\circ\, 17'\, 20'' = 0.44290.$$

Sine and tangent increase with angle

From the definitions of sine and tangent, or from the tables, it can be seen that for acute angles the *sine and tangent increase with the angle*. The interpolation method described above for the sine thus works equally well for the tangent. As an illustration we shall find tan 75° 13′ 15″. From the tables (page 554) we have

$$\tan 75^\circ\, 13' = 3.7893; \quad \tan 75^\circ\, 14' = 3.7938.$$

The difference between the entries is 45 in the last two places. The given angle is $\frac{15}{60}$ or one-fourth of the way between 75° 13 and 75° 14′. We take one-fourth of 45, namely 11, and add this to the last two places of tan 75° 13′ to obtain

$$\tan 75^\circ\, 13'\, 15'' = 3.7893 + 0.0011 = 3.7904.$$

Interpolation for cosine and cotangent

The *cosine and cotangent of an acute angle decrease with an increase in this angle*, so the method of interpolation used above must be modified for these functions. The small differences in angles near a given angle are still proportional to the corresponding differences in the cosines and cotangents of these angles. However, when the correction is obtained it must be subtracted from the entry corresponding to the smaller angle.

For example, let us find cos 14° 30′ 10″. From the tables (page 554) we have

$$\cos 14^\circ\, 30' = 0.96815; \quad \cos 14^\circ\, 31' = 0.96807.$$

The difference between the entries is 8 in the last place. Since the given angle is $\frac{10}{60}$ or one-sixth of the way between 14° 30′ and 14° 31′ we take one-sixth of 8, namely 1, and subtract this from cos 14° 30′. The result is

$$\cos 14^\circ\, 30'\, 10'' = 0.96814.$$

As a second example let us find ctn 80° 11′ 23″. From the tables we obtain

$$\text{ctn } 80^\circ\, 11' = 0.17303; \quad \text{ctn } 80^\circ\, 12' = 0.17273.$$

The difference between the entries is 30 in the last two places. We take $\frac{23}{60}$ of 30, namely 12, and subtract this result from the last two places of ctn 80° 11′. Thus we obtain

$$\text{ctn } 80^\circ\, 11'\, 23'' = 0.17303 - 0.00012 = 0.17291.$$

Functions of angles over 90°

Except for a plus or minus sign, the trigonometric function of an angle greater than 90° is equal to the same trigonometric function of the acute angle formed by the terminal side of the given angle and the x axis. For example,

$$\sin 225^\circ = -\sin 45^\circ; \quad \cos 300^\circ = \cos 60^\circ; \quad \tan 100^\circ = -\tan 80^\circ.$$

Finding angles from functions

The tables can also be used to determine angles from their functions. We shall illustrate the process with acute angles.

EXAMPLE 1.

Angle from its sine

Given $\sin \theta = 0.50423$, we shall find θ, where θ is acute. From the tables we have

$$\begin{aligned} \sin 30^\circ\, 17' &= 0.50428 \\ \sin \theta &= 0.50423 \\ \sin 30^\circ\, 16' &= 0.50403. \end{aligned}$$

The middle sine is $\frac{20}{25}$, that is, four-fifths of the way between the lower sine and the upper sine. The same ratio applies to the angles, so that the angle correction is

$$\tfrac{4}{5} \times 60'' = 48''.$$

Adding 48″ to 30° 16′, we have $\theta = 30^\circ\, 16'\, 48''$.

EXAMPLE 2.

Angle from its cosine

Given $\cos \theta = 0.51700$, we shall find an acute angle θ having this cosine. From the tables we have

$$\begin{aligned} \cos 58^\circ\, 53' &= 0.51678 \\ \cos \theta &= 0.51700 \\ \cos 58^\circ\, 52' &= 0.51703. \end{aligned}$$

The middle cosine is $\frac{3}{25}$ of the way between the lower cosine and the upper cosine. Since the same ratio applies to the angles, we compute the angle correction to be

$$\tfrac{3}{25} \times 60'' = 7''.$$

Adding 7″ to 58° 52′ we have $\theta = 58^\circ\, 52'\, 7''$.

The procedure for interpolating the value of an angle from the table of tangents or cotangents is the same as that used above for the sine and cosine respectively.

SOLVING RIGHT TRIANGLES

We shall illustrate the solution of right triangles with the aid of the tables, using the notation of Fig. 220.

EXAMPLE 1. *To find the acute angles and hypotenuse, given the legs.* *Legs given*

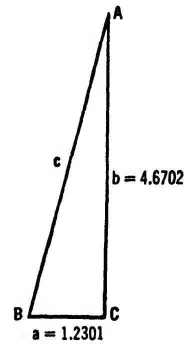


FIG. 252. RIGHT TRIANGLE OF EXAMPLE 1

Given: $\begin{cases} a = 1.2301 \\ b = 4.6702 \end{cases}$

To find: $\begin{cases} c = 4.8297 \\ A = 14^\circ\,45'\,21'' \\ B = 75^\circ\,14'\,39'' \end{cases}$

$$\tan A = \frac{a}{b}, \quad c = \frac{a}{\sin A}, \quad B = 90^\circ - A.$$ *Formulas*

$$\tan A = \frac{a}{b} = 0.26339$$ *Computations*

$$\begin{aligned} 90^\circ &= 89^\circ\,59'\,60'' \\ A &= 14^\circ\,45'\,21'' \\ \hline B = 90^\circ - A &= 75^\circ\,14'\,39'' \end{aligned}$$

From the tables

$$\begin{aligned} \tan 14^\circ\,46' &= 0.26359 \\ \tan A &= 0.26339 \\ \tan 14^\circ\,45' &= 0.26328 \\ \tfrac{11}{31} \times 60'' &= 21'' \\ A &\doteq 14^\circ\,45'\,21'' \end{aligned} \qquad \begin{aligned} \sin 14^\circ\,46' &= 0.25488 \\ \sin A &= 0.25470 \\ \sin 14^\circ\,45' &= 0.25460 \\ c = \frac{1.2301}{.25470} &= 4.8297 \end{aligned}$$

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$ *Check*

$$1.2301^2 = (4.8297 - 4.6702)(4.8297 + 4.6702)$$
$$1.5131 = 1.5123 \text{ (approximately).}$$

The solutions have been recorded in the "To find" section. It is a good idea to enter the results here as they are found. The formula used for the *check* comes from the Pythagorean theorem for $a^2 = c^2 - b^2$ is a direct consequence of $a^2 + b^2 = c^2$.

Hypotenuse and a leg given

EXAMPLE 2. *Given a leg and the hypotenuse of a right triangle, to find the remaining leg and the acute angles.*

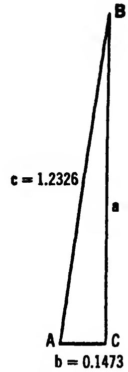


FIG. 253. RIGHT TRIANGLE OF EXAMPLE 2

Given: $\begin{cases} c = 1.2326 \\ b = 0.1473 \end{cases}$

To find: $\begin{cases} a = 1.2238 \\ A = 83^\circ\ 8'\ 12'' \\ B = 6^\circ\ 51'\ 48'' \end{cases}$

Formulas

$$\cos A = \frac{b}{c}, \quad a = b \tan A, \quad B = 90^\circ - A.$$

$$\cos A = \frac{b}{c} = 0.11950$$

Computations

From the tables

$$\cos 83^\circ\ 9' = 0.11927$$
$$\cos A = 0.11950$$
$$\cos 83^\circ\ 8' = 0.11956$$

Correction ratio is $\frac{6}{29}$.

$$\text{Correction} = \tfrac{6}{29} \times 60'' = 12''$$
$$A = 83^\circ\ 8' + 12'' = 83^\circ\ 8'\ 12''$$

$$\begin{aligned} 90^\circ &= 89^\circ\ 59'\ 60'' \\ A &= 83^\circ\ \ 8'\ 12'' \\ \hline B = 90^\circ - A &= 6^\circ\ 51'\ 48'' \end{aligned}$$

$$\begin{aligned} \tan 83^\circ\ 9' &= 8.3245 \\ \tan 83^\circ\ 8' &= 8.3041 \\ \hline &\quad 0.0204 \end{aligned}$$

$$\text{Correction for } 12'' = \tfrac{12}{60} \times 0.0204 = 0.0041$$
$$\tan A = 8.3041 + 0.0041 = 8.3082$$
$$a = b \tan A = 0.1473 \times 8.3082 = 1.2238$$

Check

$$a^2 = c^2 - b^2 = (c-b)(c+b)$$
$$1.2238^2 = (1.2326 - .1473)(1.2326 + .1473)$$
$$1.4977 = 1.4977.$$

Applications

The solution of the right triangle comes into engineering problems in numerous ways of which only a few examples can be given.

Angles of elevation and depression

The acute angle from the horizontal at an observer to a point P above the observer is called the *angle of elevation* of P at the location of the observer. Similarly, the acute angle from the horizontal at an observer to a point P below the observer is termed the *angle of depression* of P at the location of the observer.

EXAMPLE 1.

Surveyor's problem

The base of a warehouse is at the same level as the base of a skyscraper. (See Fig. 254.) The angles of elevation and depression of the top and base of the skyscraper from the top of the warehouse 200 ft. from the skyscraper are 70° 30′ 14″ and 24° 10′ 15″ respectively. We shall find the height of the skyscraper.

Referring to Fig. 254, we have

$$x = 200 \tan 70° \, 30' \, 14'',$$
$$y = 200 \tan 24° \, 10' \, 15''.$$

From the tables, with the aid of interpolation

$$\tan 70° \, 30' \, 14'' = 2.8245,$$
$$\tan 24° \, 10' \, 15'' = 0.44881.$$

Substituting the values of these tangents, we have

$$x = 564.90, \quad y = 89.76.$$

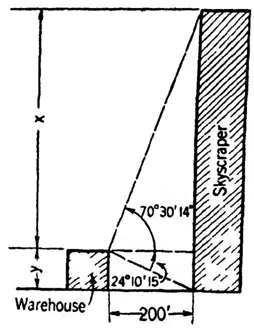


FIG. 254. FINDING THE HEIGHT OF THE SKYSCRAPER

The height $(x + y)$ of the skyscraper is therefore 654 ft. 8 in.

In the example above, the angles were given to 1″, which is much more accuracy than is obtainable with ordinary surveying instruments. An accuracy of 1′ would be adequate for most problems of this type.

Grade and angle of inclination

If a railroad rises x ft. for every 100 ft. along the horizontal, the railroad is said to have a *grade* of x per cent. The angle of inclination of the railroad is the acute angle between the line of the rails and the horizontal. The grade is therefore the tangent of the angle of inclination of the railroad.

EXAMPLE 2.

We assume that the angle of inclination of a railroad is 5° and find the grade of the road. The grade is tan 5°. By the tables, tan 5° = 0.08749. The grade is therefore approximately 9 per cent.

Resultant velocity

We recall that a *parallelogram* is a figure with four sides, opposite sides being parallel. The line segment drawn from a vertex of a parallelogram to an opposite vertex is a *diagonal of the parallelogram.*

The *velocity* of a body in motion is its speed and direction. The velocity is usually represented by an arrow whose length is a convenient measure of the speed of the body, and whose direction is the direction of motion of the body. Thus the velocity of a body

traveling northeast (N.E.) at 10 m.p.h. is represented by an arrow ten units long pointing northeast. If a body is traveling at y miles per hour in one direction and x miles per hour in another, the *resultant velocity* of the body is the diagonal z of the parallelogram as shown in Fig. 255. This statement will not be proved.

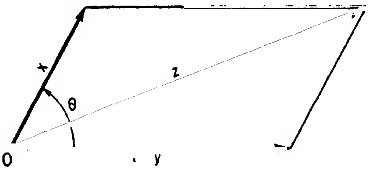


FIG. 255. RESULTANT z OF VELOCITIES x AND y

EXAMPLE 3.

Airplane's velocity

We suppose that an airplane is flying toward the South Pole from the equator at a speed of 400 m.p.h., but is being blown eastward by a hurricane of speed 150 m.p.h. We shall find the actual speed of the plane relative to the surface of the earth, and the direction of travel.

We can solve a right triangle here since the vector arrows 400 and 150 units in length, representing the given velocities, are perpendicular to each other, the first being directed south and the other east, as shown in Fig. 256. The hypotenuse x of the triangle in Fig. 256 has the length $\sqrt{400^2 + 150^2}$, or about 427, which follows from the Pythagorean theorem. The speed of the plane relative to the earth is thus 427 m.p.h. From Fig. 256 we have

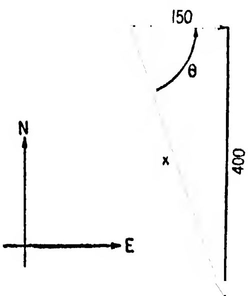


FIG. 256. VELOCITY x OF AIRPLANE HEADING SOUTH AND DRIFTING IN A WIND BLOWING EAST

$$\tan \theta = \tfrac{400}{150} = 2.6667.$$

From the tables we find that θ is approximately 69° 27′. The plane's line of flight thus makes an angle of 69° 27′ with the east direction.

Resultant force

Forces are pictured in the same manner as velocities, and are "added" in the same way. Thus if x and y in Fig. 255 are forces acting on a body, the resultant action of these forces on the body is properly represented by the arrow z.

EXAMPLE 4.

Two men are assumed to be pushing a rock, one with a force of 50 lb. toward the east, and the other with a force of 60 lb. toward the north. We shall find the resultant force.

You are referred to Fig. 257. We have

$$\tan \theta = \tfrac{60}{50} = 1.2000.$$

From the tables we have $\theta = 50° 12'$ approximately. We thus have the direction of the resultant force. Now the resultant

$$x = \frac{50}{\cos \theta}.$$

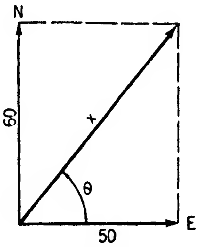


FIG. 257. FORCES ACTING ON ROCK

By the tables $\cos \theta = 0.64011$, whence x is about 78 lb. We may, therefore, for practical purposes replace the two men by one man exerting a force of 78 lb. making an angle of 50° 12′ with the east direction, obtaining thus the same physical effect with one man as with the two given men. The resultant force of 78 lb. can be checked by the Pythagorean theorem since $\sqrt{50^2 + 60^2}$ must also equal the magnitude 78 of the resultant force in lb.

Many-sided figures can often be solved by means of right triangles as the following example shows. *Pentagon*

EXAMPLE 5.

A heavy pentagonal sheet of metal (5 equal sides and angles) 5 ft. on a side is to be flame-cut from a circular piece of metal plate. How large must the circle be?

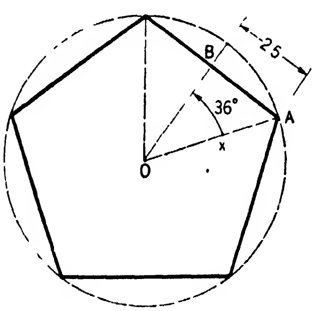


FIG. 258. PENTAGON

We let x be the radius of the circle passing through the vertices of the pentagon as shown in Fig. 258. From the center of the pentagon a perpendicular is dropped to a side of the pentagon meeting this side at a point B. The angle AOB is one-tenth of a revolution, or 36°. From the figure we have

$$x = \frac{2.5}{\sin 36°}.$$

By the tables sin 36° is 0.58779. It follows that the circle must have a radius of 4.25 ft., or 4 ft. 3 in.

SOLVING OBLIQUE TRIANGLES

We shall use the expression *oblique triangle* in referring to a triangle that is not a right triangle. Using the notation of Fig. 259 *Sine and cosine laws*

we have two laws by means of which we can solve all such triangles from known parts. These laws follow.

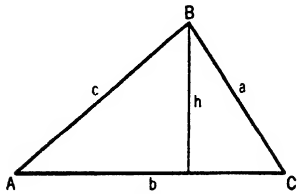


Fig. 259 Oblique triangle

Sine law:

$$(63) \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Cosine law:

$$(64) \qquad a^2 = b^2 + c^2 - 2\,bc\cos A.$$

$$(65) \qquad b^2 = a^2 + c^2 - 2\,ac\cos B.$$

$$(66) \qquad c^2 = a^2 + b^2 - 2\,ab\cos C.$$

The proofs of these laws will be omitted. They follow from properties of right triangles into which oblique triangles can be subdivided by drawing altitudes.

EXAMPLE 1.

Using the cosine law

A face of a metal tank is an oblique triangle with sides 2, 4, and 5 ft. respectively. We wish to find the area of this metal sheet.

Using the notation of Fig. 259 we let

$$a = 2, \quad b = 4, \quad c = 5.$$

From the cosine law

$$\cos A = \frac{4^2 + 5^2 - 2^2}{2 \times 4 \times 5} = \frac{37}{40} = 0.925.$$

By the tables $A \doteq 22° \, 20'$.

With AC as the base, the altitude h of the triangle is such that

$$h = c \sin A.$$

Substituting the value of c, and the value of $\sin A$ from the tables, we have $h = 1.90$.

Now we may write

$$\tfrac{1}{2}\,hb = \tfrac{1}{2} \times 1.90 \times 4 = 3.80,$$

so that the area of the sheet is 3.80 sq. ft.

EXAMPLE 2.

Using the sine law

Points A and B are on opposite sides of a lake as shown in Fig. 260. An observer at the position C is 10 miles from A and 20 miles from B. The angle A is one of $20° \, 10.5'$. We shall find the distance that the locations A and B are apart.

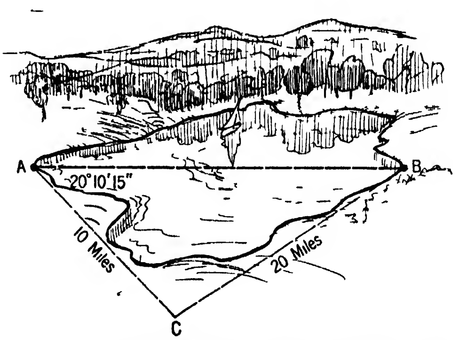


Fig. 260. Points A and B on opposite sides of a lake

By the sine law

$$\sin B = \frac{10 \times \sin 20^\circ\, 10.5'}{20} = 0.17246.$$

By the aid of the tables we find that $B = 9^\circ\, 56'$. Subtracting the sum of the angles A and B from 180°, we have $C = 149^\circ\, 53.5'$. By the sine law

$$AB = \frac{20 \sin 149^\circ\, 53.5'}{\sin 20^\circ\, 10.5'} = \frac{20 \times 0.5016}{0.3449} = 29.0.$$

Thus we have found that the side AB is 29 miles in length.

Problems

1. Find the following trigonometric functions:
$\sin 27^\circ\, 14'\, 42''$; $\cos 60^\circ\, 10'\, 15''$; $\tan 75^\circ\, 1.2'$; $\text{ctn}\, 30^\circ\, 30'\, 30''$.
Ans. 0.45780; 0.49742; 3.7373; 1.6971.

2. Find the acute angles having the following functions:
$\tan\theta = 0.11745$; $\sin\theta = 0.91124$; $\cos\theta = 0.87666$; $\text{ctn}\,\theta = 7.7777$.
Ans. $6^\circ\, 41'\, 58''$; $65^\circ\, 40'\, 40''$; $28^\circ\, 45'\, 30''$; $7^\circ\, 19'\, 35''$.

3. Find the following trigonometric functions:
$\sin 400^\circ\, 10'\, 15''$; $\cos - 331^\circ\, 19'\, 47''$; $\tan - 10^\circ\, 10'\, 10''$; $\text{ctn}\, 800^\circ\, 45'\, 50''$.
Ans. 0.64507; 0.47976; − 0.17656; 0.16261.

4. Find θ for which:
(a) $\sin\theta = -\ 0.31416$, where θ is in the third quadrant.
(b) $\cos\theta = -\ 0.91222$, where θ is in the second quadrant.
Ans. (a) $198^\circ\, 18'\, 37''$; (b) $155^\circ\, 48'\, 50''$.

Solving triangles

5. Solve the right triangle of Fig. 220, where $a = 124$, $b = 178$.
Ans. $c = 216.93$; $A = 34^\circ\, 51'\, 35''$; $B = 55^\circ\, 8'\, 25''$.

6. Solve the right triangle of Fig. 220, where $a = 0.041$, $c = 0.062$.
Ans. $c = 0.0465$; $A = 41^\circ\, 23'\, 52''$; $B = 48^\circ\, 36'\, 8''$.

7. Solve the right triangle of Fig. 220, where $A = 27^\circ\, 10'\, 28''$, $b = 13.58$.
Ans. $c = 29.73$; $a = 26.45$; $B = 62^\circ\, 49'\, 32''$.

8. Solve the right triangle of Fig. 220, where $A = 79^\circ\, 10.8'$, $c = 2340$.
Ans. $a = 2298.4$; $b = 439.29$; $B = 10^\circ\, 49'\, 12''$.

9. Solve the oblique triangle of Fig. 259, where $a = 13.5$, $b = 14.7$, $c = 16.8$.
Ans. $A = 50^\circ\, 12'\, 47''$; $B = 56^\circ\, 47'\, 44''$; $C = 72^\circ\, 59'\, 29''$.

10. Solve the oblique triangle of Fig. 259, where $B = 37^\circ\, 14.1'$, $a = 138$, $c = 176$.
Ans. $b = 106.52$; $A = 51^\circ\, 37'\, 18''$; $C = 91^\circ\, 8'\, 36''$.

Angles of elevation and depression

11. A roadbed has a grade of 2.2%. What is the angle of inclination of the road, and what is the rise at this grade in 30 miles? *Ans.* $1^\circ\, 15'\, 37''$; 3480 ft.

12. From the top of a hill 52 ft. 3 in. above the surrounding plain the angle of depression to a surveyor's bench mark on the plain is $5^\circ\, 53'$. How far is the bench mark from the top of the hill in the horizontal direction? *Ans.* 507.06 ft.

13. An airplane is directly above an observer B. An observer A is 2000 ft. from B on the same level as B. At A the angle of elevation of the plane is $80^\circ\, 14'$. How far is the plane above C? *Ans.* 11,620 ft.

14. From a point 900 ft. above a lake the angle of depression to a point on the near shore is 11° 40′, and to a point directly beyond on the opposite shore is 5° 14′. What is the width of the lake between the points mentioned? *Ans.* 5468 ft.

15. The angle of elevation of a cloud from a point due south is 40°, and from another point 2 miles due east of the first point the angle of elevation is 35°. Find the height of the cloud.

Hint. Let x be the distance the first point is south of the cloud, y the height of the cloud, and obtain two linear equations from right triangles that can be constructed conveniently. *Ans.* 3.03 miles.

Velocity problems

16. If an airplane is traveling north at 200 m.p.h., but is drifting east, due to the wind, at 60 m.p.h., what is the velocity of the airplane?

Ans. 209 m.p.h.; the angle the direction of travel of the plane, relative to the earth, makes with the north is 16° 42′.

17. A boat is traveling across a stream at 6 m.p.h., and is drifting downstream at 5 m.p.h. What is the velocity of the boat?

Ans. 7.8 m.p.h.; the angle the direction of travel of the boat makes with the downstream direction is 50° 12′.

18. A cutting tool on a machine is moving upward at 3 ft. per sec., and sideways at 4 ft. per sec. What is the velocity of the cutting tool?

Ans. 5 ft. per sec.; the direction of motion is 53° 7′ with the vertical.

19. A helicopter is traveling south at 50 m.p.h. A wind from the northwest is blowing at 40 m.p.h. What direction does the wind appear to have to the pilot of the helicopter?

Hint. You must solve an oblique triangle. *Ans.* Toward 52° 5′ east of north.

20. Wind is known to be blowing at 20 m.p.h. from due north. A plane is traveling east. To the pilot the wind appears to be traveling at a speed of 170 m.p.h. What is the speed of the plane? *Ans.* 169 m.p.h.

Forces

21. If a force of 500 lb. acting east is applied to a boat, and a force of 1000 lb. is applied, acting to the north, what is the combined effect of these two forces expressed as a single force? *Ans.* 1118 lb.; 63° 26′ north of east.

Miscellaneous problems

22. If a sheet of plastic is in the form of a parallelogram with sides equal to 5 ft. and 14 ft. respectively, and an angle between two sides is 87° 27′, what is the area of the plastic sheet? *Ans.* 70 sq. ft.

23. A *hexagon* is a figure with six equal sides and six equal angles. The head of a bolt is in the shape of a hexagon. If the area of the head is to be 1 sq. in., what must be the length of a side? *Ans.* $\frac{5}{8}$ in.

24. A pole 30 ft. long is resting against a wall at an angle of 20° 13.6′. The bottom of the pole is drawn 1 ft. farther away from the wall. How far down the wall will the top of the pole fall? *Ans.* 3.28 ft.

25. Points A and B are on opposite sides of a lake as shown in Fig. 260. The point C, however, is 4.6 miles from A and 6.9 miles from B. The angle CAB is 18° 10′ 40″. Find the distance from A to B. *Ans.* 11.1 miles.

POINTS AND STRAIGHT LINES

Distance between points

The use of the coordinate system made possible great strides in geometry. The field of geometry in which a coordinate system is used is called *analytic geometry*. By means of coordinate systems the study of geometry reduces to algebra, so that any statement in algebra can be transformed into a geometrical statement which can be pictured. The remainder of this chapter will be devoted almost entirely to the applications of analytic geometry.

We wish to consider any two points P_1 and P_2 in the plane with coordinates (x_1, y_1) and (x_2, y_2) respectively. A horizontal line through P_1 and a vertical line through P_2 intersect at a point P as shown in Fig. 261. Except for a possible change of sign, the dis-

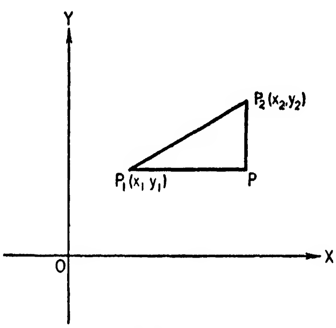


FIG. 261. DISTANCE P_1P_2 BETWEEN POINTS

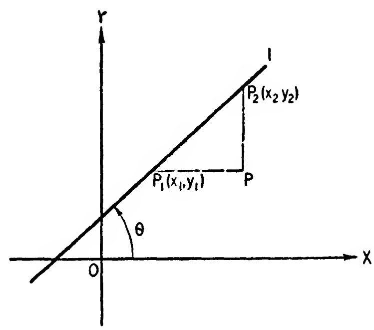


FIG. 262. ANGLE OF INCLINATION θ

tance PP_1 is always $x_2 - x_1$, whereas PP_2 is $y_2 - y_1$. By the Pythagorean theorem the distance P_1P_2 is given by

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{67}$$

Formula 67 gives the *distance between any two points* on the plane.

Negative coordinates

If any coordinate is negative, such as x_1 or x_2, it is introduced in equation 67 with its proper sign. For example, the distance between the point $(3, -4)$ and the point $(5, 7)$ is approximately 11.18, arising from

$$\sqrt{[7 - (-4)]^2 + (5 - 3)^2} = \sqrt{11^2 + 2^2} = \sqrt{125} = 5\sqrt{5}.$$

Angle of inclination

To be able to measure the direction of a straight line in a plane we use the smallest positive angle which this line makes with the positive direction of the x axis. In Fig. 262 we have drawn a line l for which this angle is θ. The angle θ is called the *angle of inclination* of the line. If the line does not intersect the x axis, this angle is 0°.

The angle can be any positive angle up to, but not including 180°. *Two lines have the same direction if and only if they have the same angle of inclination.* Parallel lines, therefore, have the same direction.

Slope of line

In practice it is more convenient to work with a trigonometric function of θ than with θ itself. The function in common use is $\tan \theta$, called the *slope* of the line. For different directions $\tan \theta$ has different values. Since the slope of a line increases without bounds as the line approaches parallelism with the y axis, it is customary to say that the slope of a line parallel to the y-axis is "infinite," and to denote it by the symbol ∞.

Two points determine a straight line, and thus also the slope of the line. We can therefore express the slope of the line in terms of coordinates of two points on the line.

Formula for slope

We let P_1 and P_2 be two points on the line l in Fig. 262. The distance P_1P and PP_2 are $x_2 - x_1$ and $y_2 - y_1$ respectively. From the definition of $\tan \theta$ we find that the slope m of l is given by

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}. \tag{68}$$

You can easily verify that equation 68 holds for any line l and any two points on l.

A line passing through the points $(-2, -3)$, $(4, 5)$ has the slope m, where

$$m = \frac{5 - (-3)}{4 - (-2)} = \frac{5 + 3}{4 + 2} = \frac{8}{6} = \frac{4}{3}.$$

In Fig. 263 we have drawn perpendicular lines l_1 and l_2 with angles of inclination θ_1 and θ_2 respectively. From the figure we see that

$$\theta_2 = \theta_1 + 90°.$$

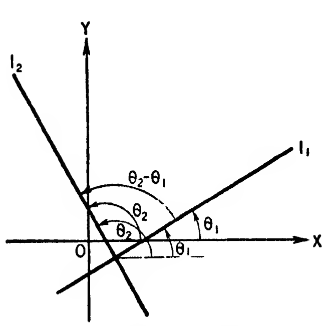


FIG. 263. PERPENDICULAR LINES

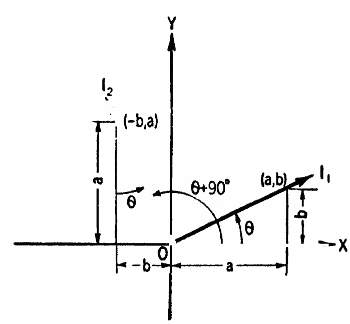


FIG. 264. ANGLE $90° + \theta$

From Fig. 264

$$\tan(\theta + 90°) = \frac{a}{-b} = -\cot\theta = -\frac{1}{\tan\theta},$$

Perpendicular lines

whence

$$\tan\theta_2 = -\frac{1}{\tan\theta_1}.$$

If m_1 and m_2 designate the slopes of l_1 and l_2 respectively, we have

$$m_2 = -\frac{1}{m_1},$$

or simply

(69) $$m_1 m_2 = -1.$$

It can be shown easily that equation 69 applies to any two perpendicular lines l_1 and l_2.

We have proved that *two lines are perpendicular if and only if the product of their slopes is* -1.

Angle between two lines

We let θ be the angle formed by rotating a line l_1 counterclockwise about its point of intersection with a line l_2 until l_1 and l_2 coincide as shown in Fig. 265. The slopes m_1 and m_2 of l_1 and l_2 respectively are related to θ by the formula

(70) $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

The proof of this formula will be omitted.

We consider the example where two lines l_1 and l_2 intersect at the point (1, 1), while l_1 passes through (3, 4) and l_2 through (2, 6). By equation 68 the slopes m_1 and m_2 of l_1 and l_2 are given by

$$m_1 = \frac{4-1}{3-1} = \frac{3}{2}; \quad m_2 = \frac{6-1}{2-1} = \frac{5}{1} = 5.$$

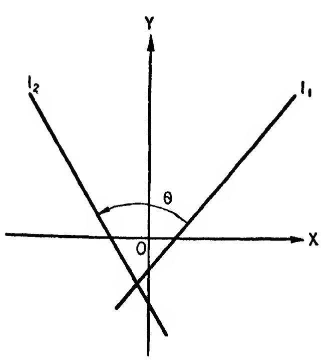


FIG. 265. ANGLE θ BETWEEN TWO LINES

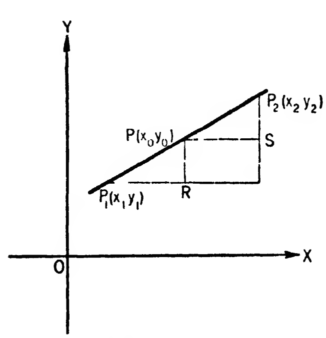


FIG. 266. MIDPOINT P OF LINE SEGMENT P_1P_2

In view of equation 70 we conclude that the tangent of the angle θ between these lines is

$$\tan\theta = \frac{5 - \frac{3}{2}}{1 + 5 \times \frac{3}{2}} = \frac{\frac{7}{2}}{1 + \frac{15}{2}} = \frac{\frac{7}{2}}{\frac{17}{2}} = \frac{7}{17}.$$

Mid-point of line segment

In Fig. 266 we have arbitrarily taken two points P_1 and P_2 in the plane, and located the mid-point P of the line segment joining P_1 and P_2. The coordinates of these points are chosen as indicated. Horizontal and vertical lines are drawn through these points to yield the points R and S as shown. Since P is the mid-point of P_1P_2, the triangles P_1PR and PP_2S are identical. The distance P_1R is the same as the distance PS. It follows that

$$P_1R = \frac{x_2 - x_1}{2}.$$

Adding P_1R to x_1 we have

$$x_0 = x_1 + \frac{x_2 - x_1}{2}.$$

Thus

$$x_0 = \frac{x_1 + x_2}{2}; \quad \text{also,} \quad y_0 = \frac{y_1 + y_2}{2}. \tag{71}$$

The same type of proof applies to any other pair of points P_1, P_2.

The *coordinates of the mid-point* of a line segment are thus the averages of the corresponding coordinates of the end points of the line segment.

The mid-point of the line segment joining $(-4, -8)$ with $(1, 7)$ is thus $(-\frac{3}{2}, -\frac{1}{2})$.

Problems

1. The positions of three cities are plotted on a map and found to have the coordinates $(2, -3)$, $(7, 1)$, and $(-1, 8)$ respectively, the unit of measure being 1 mile. Find the distances between the cities. *Ans.* 6.4, 10.6, and 11.4 miles.

2. Four roads plotted on a map intersect at the points $(-2, -1)$, $(3, 3)$, $(-1, 2)$, $(4, 6)$. Prove by the distance formula 67 that these points are vertices of a parallelogram.

Hint. The opposite sides of a parallelogram are of the same length.

3. Three roads plotted on a map intersect at the points $(0, 4)$, $(1, -1)$, and $(3, 2)$. Prove by the distance formula 67 and the Pythagorean theorem that these points are vertices of a right triangle.

4. Find the slope and angle of inclination of each of the lines through the following pairs of points:

(a) $(-4, -1)$, $(5, -4)$; (b) $(4, 7)$, $(-3, 11)$.

Ans. (a) $-\frac{1}{3}$, 18° 26′ 5″; (b) $-\frac{4}{7}$, 29° 44′ 42″.

5. Three mountain peaks plotted on a map are at the points $(-5, -3)$, $(7, -1)$, and $(4, 5)$. Find the angles of the triangle with these points as vertices. *Ans.* $72^\circ 53' 49''$, $74^\circ 55' 53''$, $32^\circ 10' 18''$.

6. By the use of slopes of sides prove that the figure of problem 2 is a parallelogram.

7. By the use of slopes of sides prove that the figure of problem 3 is a right triangle.

Mid-points of lines

8. The ends of a rod are located on a Cartesian coordinate system, and are found to be at the points $(-4, -5)$, $(2, 7)$. Find the mid-point of the rod. *Ans.* $(-1, 1)$.

9. The point $(3, 4)$ is the mid-point of a rod joining $(-5, 1)$ with another point. Find the coordinates of the other point. *Ans.* $(11, 7)$.

10. A beam indicated on a drawing has slope 2. Find the slopes of the other beams which make angles of 45° with this beam. *Ans.* $\frac{1}{3}$, -3.

11. A searchlight beam directed upwards, when plotted, passes through the points $(1, 5)$ and $(3, 4)$. A second beam in the same vertical plane has exactly 30° as its angle of inclination (with the horizontal). Find the acute angle between these beams. *Ans.* $56^\circ 33' 58''$.

12. A light ray, when plotted, passes through the points $(1, 5)$ and $(8, 4)$. A second ray passes through $(-1, 3)$ and $(6, 8)$. Find the acute angle between these rays. *Ans.* $43^\circ 40' 4''$.

EQUATIONS OF STRAIGHT LINES

By definition *an equation of a straight line is an equation satisfied by the coordinates of each point on the line, and not satisfied by the coordinates of any point off the line.* There are different standard formulas for equations of a straight line depending on the data given. These are called standard forms.

Point-slope form

A line is determined by its slope m and one point (x_1, y_1) on the line. If (x, y) designates any other point on the line, by formula 68

$$m = \frac{y - y_1}{x - x_1},$$

provided $x \neq x_1$. Clearing, we have an equation of the line expressed in *point-slope* form,

$$y - y_1 = m(x - x_1). \tag{72}$$

If a line l passes through the point $(2, 3)$ with slope 5, the points (x, y) on the line satisfy the equation 72 of the line. With the given coordinates and slope substituted, we obtain the expression

$$y - 3 = 5(x - 2),$$

or, after simplification,

$$5x - y - 7 = 0.$$

Line from equation

Any equation of the first degree in x and y which can be put in the point-slope form is an equation of the line with slope m passing through the point (x_1, y_1). Suppose, for example, that $2y - x - 4 = 0$ is the given equation. By algebraic operations this equation can be brought into the form

$$y - 0 = \tfrac{1}{2}(x + 4).$$

This equation is therefore an equation of a line with slope $\frac{1}{2}$ passing through the point $(-4, 0)$.

Two-point form

By rule 68 any point (x, y) on a line passing through the points (x_1, y_1), (x_2, y_2) satisfies the equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

provided x_2 and x are different from x_1. Multiplying by $(x - x_1)$ we have

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{73}$$

The latter is called an *equation of a line in two-point form.*

For example, let us suppose that a straight line passes through the points $(-4, 5)$ and $(6, -1)$. By the two-point formula the line must satisfy the equation

$$y - 5 = \frac{-1 - 5}{6 - (-4)}[x - (-4)].$$

This equation can be simplified into

$$3x + 5y - 13 = 0.$$

Slope-intercept form

We let m be the slope of a line not parallel to the y axis, and b its y-intercept. The line thus goes through the point $(0, b)$. By the point-slope form (72) the line has the equation

$$y - b = m(x - 0).$$

We can write this equation in the *slope-intercept form*

$$y = mx + b. \tag{74}$$

Intercept form

A line which passes through $(a, 0)$ and $(0, b)$ has the x- and y-intercepts a, b respectively. By the two-point form the line has the equation

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{75}$$

This equation is said to be in the *intercept form.* To use equation 75 we must have the condition $a \neq 0$, $b \neq 0$.

Example

A line passes through the points (5, 7) and (3, − 5). We shall find an equation of the line in intercept form. By the two-point form (73)

$$y - 7 = \frac{-5 - 7}{3 - 5}(x - 5).$$

Simplified, this equation becomes

$$6x - y = 23.$$

Further manipulation gives the intercept form where $a = \frac{23}{6}$ and $b = -23$:

$$\frac{x}{23/6} + \frac{y}{-23} = 1.$$

Lines parallel to coordinate axes

The equation $x = -4$ is satisfied by all points with abscissa -4. This relationship is therefore an equation of the line parallel to the y axis, but 4 units to the left of this axis. More generally, the equation $x = a$ is an equation of a line parallel to the y axis, and $y = b$ is an equation of a line parallel to the x axis.

General form

We have shown that *each straight line has an equation of the first degree*. To prove that each equation of the first degree is an equation of a straight line we note that each first degree equation in x and y can be brought by algebraic operations into the form

$$Ax + By + C = 0 \tag{76}$$

where A and B are not both zero. This is the *general form* of an equation of a straight line.

Equivalence of equations and lines

If B is zero in equation 76, we can write this equation as

$$x = -\frac{C}{A}.$$

This is an equation of a straight line parallel to the y axis.

If B is not zero in equation 76, we divide the equation by B to get

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

The latter is an equation of the straight line with slope $-A/B$ and y-intercept $-C/B$. This equation is in slope-intercept form.

We have shown that *every equation of the first degree is an equation of a straight line*, and conversely that every straight line has an equation of the first degree. The study of first-degree equations is thus identical with the study of straight lines.

Problems

1. In each case find an equation of the straight line with the properties given:
   (a) Slope $\frac{1}{2}$, passing through the point $(-3, -5)$.
   (b) Slope $-\frac{3}{4}$, passing through the origin $(0, 0)$.
   (c) Passing through the points $(-7, -1)$, $(8, -9)$.
   (d) Slope $-\frac{1}{4}$ and y-intercept 5.
   (e) Intercept on x axis -2 and y-intercept -3.
   (f) Parallel to the line $x + y + 1 = 0$ and passing through the origin.
   (g) Perpendicular to the line $2x - 3y = 4$ and passing through $(1, 2)$.
   (h) Slope -1 and the same y-intercept as the line $2y = 3 - x$.

   Ans. (a) $x - 2y = 7$; (b) $4y + 3x = 0$; (c) $8x + 15y = -71$; (d) $x + 4y = 20$; (e) $3x + 2y = -6$; (f) $x + y = 0$; (g) $3x + 2y = 7$; (h) $x + y = \frac{3}{2}$.

2. Find the slope and y-intercept of each of the lines:
   (a) $x + 2y + 1 = 0$; (b) $x - y = 1$.

   Ans. (a) $-\frac{1}{2}, -\frac{1}{2}$; (b) $1, -1$.

3. Find the coordinates of the mid-point of the line segment cut out on the line $5x = 8 - 4y$ by the coordinate axes. *Ans.* $(\frac{4}{5}, 1)$.

4. What is the x-intercept of a line with slope $-\frac{3}{7}$ and y-intercept $\frac{1}{4}$? *Ans.* $\frac{7}{12}$.

5. Which pairs of the following lines are parallel and which are perpendicular?
   (a) $2x + 3y + 7 = 0$; (b) $4x = 14 - 6y$; (c) $3x = 2y + 1$; (d) $x + 3y + 2 = 0$; (e) $3x + 2 = y$.

   Ans. (a) is parallel to (b); (a) and (b) are both perpendicular to (c); and (d) is perpendicular to (e).

Applied problems on lines

6. The positions of movable joints A and B in a machine are fixed by their respective distances x and y from a fixed pivot. If x and y are related by the equation $2x + 4y = 5$ due to connections between A and B, when one of the joints is at the pivot, where is the other? Distances are in feet.

   Ans. When A is at the pivot, B is $1\frac{1}{4}$ ft. away, while if B is at the pivot, the joint A is $2\frac{1}{2}$ ft. away.

7. When plotted on a map with 1 mile as the unit of measure a highway has the equation $x - 3y = 1$, and a city is at the point $(3, 5)$. How far is it in a perpendicular direction from the city to the highway?

   Ans. Approximately 4.12 miles.

8. When graphed, two parallel rods in a machine yield the lines $x + 2y = 1$, $x + 2y = 10$, the unit of measure being 1 ft. How far are these rods apart?

   Ans. About 4 ft.

9. The Fahrenheit and Centigrade temperature scales are related at the boiling and freezing points by 100° C. = 212° F. and 0° C. = 32° F. respectively. Find the slope m and intercept b in the equation $T_F = mT_C + b$ relating the Fahrenheit temperature T_F to the Centigrade temperature T_C. Plot this temperature line.

   Ans. Slope 1.8; intercept 32.

Family of lines

10. What property do the lines represented by $x + y = k$ have in common for the different values of k?

    Ans. The x and y intercepts are equal.

THE CIRCLE

The figure formed by the points which satisfy a given condition is said to be the *locus* of these points. Thus the locus of the points which satisfy the equation $5x - 4y + 2 = 0$ is a straight line. *Locus*

A circle is by definition the locus of all points equidistant, that is at the same distance, from a given point called the *center*. If a circle has its center at (2, 4) and a radius 10, any point (x, y) on the circle satisfies the equation *Definition of the circle*

$$\sqrt{(x-2)^2 + (y-4)^2} = 10,$$

because by equation 67 the left-hand expression is the distance (radius) from the point (2, 4) to the point (x, y). Squaring, we have

$$(x-2)^2 + (y-4)^2 = 100.$$

Since the coordinates of a point off the circle cannot satisfy this equation, this must be an equation of the given circle.

More generally, a circle with center (h, k) and radius r has the equation *Equations of circles*

(77) $$(x-h)^2 + (y-k)^2 = r^2.$$

This equation of the circle is said to be in *standard form*. If the circle has its center at the origin the standard form becomes

(78) $$x^2 + y^2 = r^2.$$

The equations of the circle, discussed above, are of the second degree in x and y. It will appear later that not all equations of the second degree represent circles. Exactly which second-degree equations represent circles can readily be determined, however. By expanding the equation of the circle in standard form (77) above, and rearranging terms we have

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is of the form

(79) $$x^2 + y^2 + Dx + Ey + F = 0,$$

where D, E, and F are constants. Note that there is no term in xy.

A term involving the product xy is called a *cross-product* term. We could readily prove the following result. A *second-degree equation in x and y* (having all terms collected on one side) *is an equation of a circle if and only if there is no cross-product term, and the coefficients of the* x^2 *and* y^2 *terms are identical.* *Cross produc*

Problems

1. *Find an equation of the circle with center at $(-2, 4)$ and radius 5.*

Ans. $x^2 + y^2 + 4x - 8y - 5 = 0$.

2. Where does the circle with center at (1, 1) and radius 3 cut the x axis?

Ans. $1 \pm 2\sqrt{2}$.

3. Find the center and radius of the circle with the equation

$$x^2 - 4x + y^2 = 6y + 3.$$

Ans. (2, 3), $r = 4$.

4. Find an equation of the line which passes through the centers of the circles

$$x^2 + y^2 + 2x + 1 = 2y, \quad x^2 + y^2 = 4x + 4y + 1.$$

Ans. $x - 3y + 4 = 0$.

Applications

5. The equation of the rim of a wheel when plotted is $x^2 + y^2 - 2x - 2y = 2$. The wheel is on a machine with an axle at $(-4, -4)$. Find the distance from the axle to the wheel.

Hint. First find the distance from the axle to the center of the wheel.

Ans. $5\sqrt{2} - 2$.

6. When graphed the equations of two wheels are

$$x^2 + y^2 - 2x - 4y + 1 = 0,$$
$$x^2 + y^2 - 10x - 2y + 22 = 0.$$

How far are they apart?

Hint. First find the distance their centers are apart. *Ans.* $\sqrt{17} - 4$.

7. Two points on a machine have the coordinates (1, 2) and $(-1, -2)$. One end of a rod on the machine moves so that the sum of the squares of the distances from this end to the given points is 36. Show that the locus is a circle, and find its center and radius. *Ans.* (0, 0), $r = \sqrt{13}$.

8. One edge of a field when plotted on a map is along the x axis, and a tree in the field is at the point (2, 3). A tractor moves so that the square of its distance from the tree is proportional to its distance from the line of the given edge of the field. Show that the locus is a circle.

Hint. Let the coordinates of the tractor be (x, y). Then its distance from the x axis is y and from the tree is $\sqrt{(x - 2)^2 + (y - 3)^2}$.

CONIC SECTIONS

Right circular cone

On a plane, for convenience taken horizontal, we draw a circle C with center O as in Fig. 267. We choose P vertically above O, and pass through P all possible lines intersecting the circle C. Thus we generate a *right circular cone* as shown in Fig. 267. The line joining P to O is the axis of the cone. In the expression "right circular cone" the word "circular" refers to the property of C being a circle, and the word "right" indicates that the axis PO is perpendicular to the plane of C. The point P is called the *vertex* of the cone. The part of the cone above P is termed a *nappe;* similarly, for the part below. The second sketch of Fig. 267 is a cone that is obviously neither "right" nor "circular."

Conic sections

If we pass any plane through a right circular cone we cut out a *conic section.* This is simply a matter of definition. Planes perpendicular to the axis of the cone cut out circles, so that circles are special cases of conic sections. It is easily seen that planes not

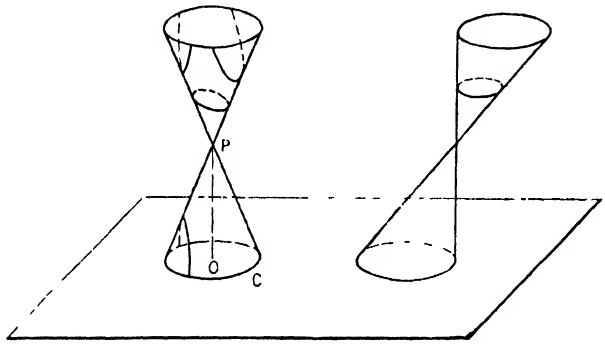


FIG. 267. CONES

perpendicular to the axis cut out conic sections which are not circles. Of course, a plane passing through the vertex perpendicular to the axis cuts through the cone at a single point only.

Functions of second degree

Conic sections were studied extensively by the ancient Greeks. The conics have numerous applications, some of which will be brought out here. They are in many respects the simplest curves after the straight line. They are of particular mathematical interest because it can be shown that the locus of the points whose coordinates satisfy an *equation of the second degree in x and y is always a conic*, provided we include the "degenerate conics" composed of two straight lines (which may be coincident), a single point, or no real locus.

PARABOLA, ELLIPSE, HYPERBOLA

Parabola

The parabola is the locus of all points which are the same distance from a given point as from a given line. An example of a parabola is given in Fig. 268. Here the given point marked F is the point (2, 0) and the given line is the line $x = -2$. The points (2, 4) and (0, 0), as well as (2, $-$ 4), are on this parabola because they are the same distance from the line $x = -2$ as from the point (2, 0).

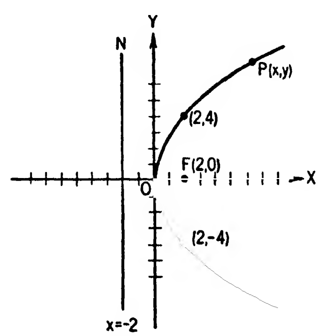


FIG. 268. PARABOLA

Following the notations of Fig. 269 we let $P(x,y)$ be a point on the parabola, d_1 being the distance from P to the line $x = -2$, and d_2 the distance from P to the point (2, 0). From the figure you can see that $d_1 = x + 2$, and from formula 67, $d_2 = \sqrt{(x-2)^2 + (y-0)^2}$. Since by definition we have $d_1 = d_2$, an equation of this parabola is

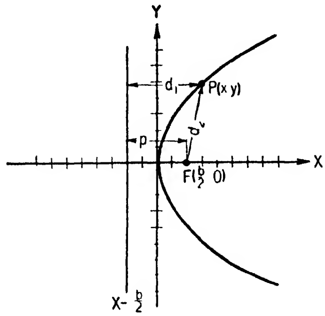


FIG 269 PARABOLA IN STANDARD POSITION

$$x + 2 = \sqrt{(x-2)^2 + y^2}, \quad \text{or} \quad y^2 = 8\,x.$$

Standard equation of the parabola

We return to the general treatment of the parabola. The given point is called the *focus*, and the given line the *directrix*. The point midway between the focus and the directrix is the *vertex*. The distance from the focus to the directrix will be denoted by p.

We shall take the focus F at the point $\left(\frac{p}{2}, 0\right)$ in Fig. 269, and the directrix D to be the line $x = -\frac{p}{2}$. The vertex is now at the origin, as in the example above. The derivation of the equation of the parabola is the same as in the example of Fig. 268. We have

$$d_1 = x + \frac{p}{2}, \quad d_2 = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

Equating the formulas for d_1 and d_2, squaring, and simplifying we have

$$y^2 = 2\,px. \tag{80}$$

The latter is the *standard equation* of the parabola. The parabola of Fig. 269 is in *standard position*. Note again that $\frac{p}{2}$ is the distance from the vertex to the focus.

Axis of the parabola

The line through the vertex and focus is called the axis of the parabola. The parabola is symmetrical with respect to its axis.

As the abscissa of the point P of Fig. 269 increases, the ordinate also increases so that P rises as we move to the right. You can see that P will rise forever as you go to the right. In a similar way the lower half of the parabola diverges steadily from the x axis.

Some uses of the parabola

The cable of a suspension bridge, with weight from the roadway uniformly distributed over the length of the bridge, is in the

form of a parabola. A bomb dropped from a plane falls approximately along a parabolic curve. When a parabola is revolved about its axis, a *paraboloid of revolution* is obtained, the "focus" of this paraboloid being the focus of the original parabola. If the reflector of a searchlight is formed as a paraboloid of revolution, with light source at the focus, the beam of light leaving the searchlight will not spread but will form a beam of parallel rays of light. The lenses of the great reflecting telescopes for observing stars are made in the shape of paraboloids of revolution. The light falling on such a lens is collected at the focus where it is recorded on a photographic plate, or observed.

Application to searchlights

A searchlight with parabolic reflector is to be 2 ft. in diameter and 10 in. deep. We shall find the position at which the light source must be placed in the searchlight.

A cross section of the searchlight taken through the axis of the searchlight yields a parabola which we place in standard position as in Fig. 269. The equation of the parabolic section is thus (80) for some value of p. Using an inch as the unit of measure, the point (10, 12) is known to be on the parabola. Substituting the coordinates of this point in equation 80 and solving for p we obtain $p = \frac{36}{5}$. Then $\frac{p}{2} = \frac{18}{5}$. The light source must be placed at the focus $(\frac{18}{5}, 0)$, which is 3.6 in. from the vertex.

Application to projectiles

Let us suppose that a projectile is fired from a gun with a speed v, the angle of elevation of the gun being θ. We neglect such factors as air resistance and the rotation of the earth. By laws of mechanics and calculus it can be shown that the path of the projectile is part of the parabola

$$(81) \qquad y = x \tan \theta - \frac{32\, x^2}{2\, v^2 \cos^2 \theta}.$$

This parabola is not in standard position as you can see from Fig. 270 where we have graphed the curve.

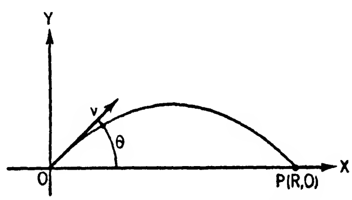


FIG. 270. PATH OF PROJECTILE

Here the gun is at the origin, the horizontal through the gun is the x axis, and the gun is pointed into the first quadrant. The distance R of Fig. 270 is called the *range* of the gun.

Range

We shall find the range of the gun if $\theta = 45°$, and the muzzle speed v is 2400 ft. per sec., as is often the case with a 75-mm. gun. Substituting $v = 2400$ and $\theta = 45°$ in equation 81, we get

$$y = x - \frac{x^2}{180{,}000}.$$

Setting y equal to zero to get the point P of Fig. 270, we have

$$180{,}000\,x - x^2 = 0.$$

Since $x \neq 0$ at P, we can divide through by the factor x, and obtain $x = 180{,}000$. The range is thus 180,000 ft., or 34 miles. If we take air resistance into account the range turns out to be much less.

Ellipse

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{82}$$

where a is not less than b, is the standard equation of a conic, called the *ellipse*, pictured in Fig. 271. This ellipse is in *standard position.*

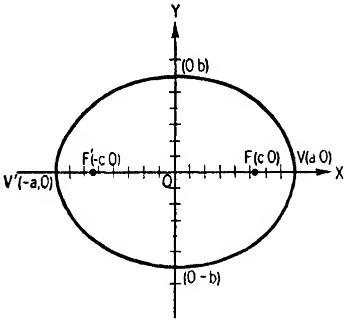


FIG. 271. ELLIPSE IN STANDARD POSITION

The ellipse may be defined by the property that it is the *locus of a point which moves in such a way that the sum of its distances from two fixed points is a constant.*

For the ellipse of Fig. 271 the constant is $2a$, and the fixed points are $F(c, 0)$ and $F'(-c, 0)$, where $c^2 = a^2 - b^2$. The points F and F' are called the *foci* of the ellipse, whereas $V(a, 0)$ and $V'(-a, 0)$ are the *vertices*. The distance $2a$ is the major axis, $2b$ the minor axis; a is the semimajor axis, and b the semiminor axis.

Example

We shall find an equation of the ellipse whose foci are $(\pm 3, 0)$, and vertices $(\pm 5, 0)$. In the notation of Fig. 271 we have $a = 5$, $c = 3$. Since $c^2 = a^2 - b^2$, it follows that $b = 4$. Substituting these values in the standard equation of the ellipse, we obtain the special equation describing this particular ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Places where the ellipse occurs

The planets move around the sun in orbits which are elliptical. To obtain a variable rate of motion from a shaft rotating at constant speed, elliptical gears are sometimes employed. If an ellipse is rotated about an axis, an *ellipsoid of revolution* is generated. Huge elevated tanks often have ellipsoidal bottoms which serve better

than conical or hemispherical bottoms. The two "foci" of the ellipsoid of revolution are the foci of the original ellipse. A whispering gallery is in the shape of an ellipsoid of revolution. A sound leaving one focus is reflected back to the other.

Application to elliptical gears

Let us assume that an elliptical gear has the equation $9x^2 + 16y^2 = 144$, the unit of measure being the inch. We wish to find the lengths of the major and minor axes of the elliptical gear. Dividing by 144 we get the equation in standard form, namely,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

It follows that the gear has a major axis of 8 in. and a minor axis of 6 in., these being the dimensions of the ellipse.

Hyperbola

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{83}$$

is the *standard equation* of the *hyperbola* (*in standard position*). The hyperbola is a curve made up of two sections, called *branches*, as shown in Fig. 272. The points $(a, 0)$ and $(-a, 0)$ are termed the

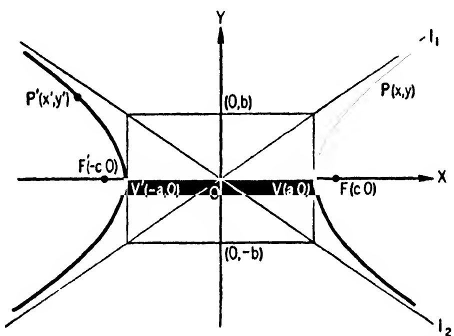


FIG. 272. HYPERBOLA IN STANDARD POSITION

vertices of the hyperbola. The line segment joining the vertices is called the *transverse axis*, and the line segment joining the points $(0, -b)$ and $(0, b)$ is called the *conjugate axis*. It follows that a and b are the semitransverse and semiconjugate axes of the hyperbola.

The hyperbola is by definition the *locus of a point which moves in such a way that the difference of its distances from two fixed points, the foci, is constant*. For Fig. 272 this constant is $2a$, and the foci are $(\pm c, 0)$.

Asymptotes

As the abscissa of a point on the hyperbola in the first quadrant of Fig. 272 increases, the point draws closer to the line l_1. It can be proved that by going out far enough along the hyperbola we

can get as close to the line l_1 as we please. The hyperbola does not, however, cross the line l_1. The same situation applies in the other quadrants. The lines l_1 and l_2 of Fig. 272 are called *asymptotes*, a Greek name suggested by the properties just discussed. The equations of l_1 and l_2 are

$$(84) \qquad y = \frac{b}{a}x, \quad y = -\frac{b}{a}x.$$

Boyle's law

Laws of physics expressed in the form of equations, when graphed, often yield hyperbolas. Particularly well known is Boyle's law mentioned before. We recall that if p is the pressure of a gas at constant temperature and v its volume, the gas satisfies the law

$$(85) \qquad pv = k,$$

where k is a constant depending on the kind and amount of gas used. The graph of equation 85 is a hyperbola, as shown in Fig. 273, with the axes as asymptotes.

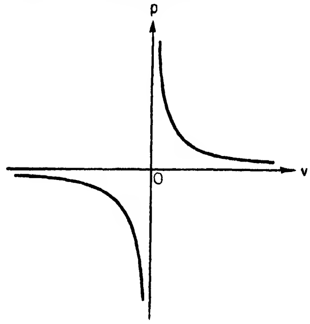


FIG. 273. HYPERBOLA FOR BOYLE'S LAW

Location of a gun

The position of an enemy gun 10 miles away can be located within a radius of about 50 ft. by a method based on the time of travel of sound and the use of hyperbolas. Observers are placed at points F and F' a known distance $2c$ apart as in Fig. 272. We suppose that the sound of fire from the gun reaches F an interval of t seconds earlier than this sound reaches F'. If $P(x, y)$ in Fig. 272 is the position of the gun, the distance $F'P$ is $1120\,t$ ft. greater than the distance FP, *the speed of sound being about 1120 ft. per sec.* The gun is therefore on the hyperbola of Fig. 272 where a is half of $1120\,t$, that is, $560\,t$. The gun is located in the same manner on another hyperbola, which intersects the first. The gun is therefore at one of the points of intersection of these hyperbolas. Since the position of the gun is known roughly in advance, the proper intersection is chosen immediately.

Example of gun location

We suppose that observers at F' and F are 600 ft. apart, and that the sound of gunfire reaches F at an instant one-fifth of a second earlier than F'. We shall find an equation of the locus of position of the gun.

We take one-fifth of 1120, obtaining 224. The gun is therefore

224 ft. closer to the observer at F than the observer at F'. We thus have $a = 112$. Since $c = 300$, and it can be proved that

$$c^2 = a^2 + b^2 \tag{86}$$

for any hyperbola, we have $b^2 = 300^2 - 112^2 = 77{,}456$. By equation 83 the gun is on the hyperbola

$$\frac{x^2}{12{,}544} - \frac{y^2}{77{,}456} = 1.$$

Since the gun is nearer F than F', the gun is on the right branch of the hyperbola.

The ellipse, hyperbola, and parabola are the only nondegenerate conics. The circle is an ellipse with equal major and minor axes. Such special cases of conics as a point, or two straight lines, are called *degenerate conics*. *Nondegenerate conics*

Problems

1. Find the distance from the vertex to the focus of the parabola $y^2 = 8x$. *Ans.* 2.

2. Find an equation of the parabola in standard position with the following properties:
   (a) Focus (3, 0); (b) $p = 5$; (c) Passing through the point (5, 7).
   Ans. (a) $y^2 = 12x$; (b) $y^2 = 10x$; (c) $y^2 = \frac{49}{5}x$.

3. Find equations of ellipses in standard position satisfying the following conditions.
   (a) Foci ($\pm$ 2, 0) and vertices ($\pm$ 3, 0).
   (b) Vertices ($\pm$ 7, 0) and passing through (2, 4).
   (c) Foci ($\pm$ 5, 0) and length of major axis equal to 6.
   Ans. (a) $5x^2 + 9y^2 = 45$; (b) $16x^2 + 45y^2 = 784$; (c) $11x^2 + 36y^2 = 396$.

4. Find equations of hyperbolas in standard position satisfying the following conditions.
   (a) $a = 5$, $b = 12$.
   (b) Foci ($\pm$ 4, 0) and transverse axis equal to 6.
   (c) Asymptotes $y = \pm 2x$, and vertices ($\pm$ 3, 0).
   Ans. (a) $144x^2 - 25y^2 = 3600$; (b) $7x^2 - 9y^2 = 63$; (c) $36x^2 - 9y^2 = 324$.

5. Find the coordinates of the foci, vertices, and equations of the asymptotes of the hyperbola
$$144x^2 - 25y^2 = 169.$$
   Ans. $(\pm \frac{169}{60}, 0)$; $(\pm \frac{13}{12}, 0)$; $y = \pm \frac{12}{5}x$.

6. A reflector is to be in the form of a paraboloid of revolution covered by a plane sheet of glass perpendicular to the "axis" of the paraboloid. The diameter of the glass is to be 1 ft., and the focus of the paraboloid is to be 2 in. from the glass. What must be the depth of the reflector? *Ans.* 4.5 in. *Paraboloid*

Parabolas

7. A parabolic arch has a span of 120 ft. and height of 20 ft. Derive an equation for the parabola and find the heights at points 10 ft., 20 ft., and 30 ft. from the center. *Ans.* $x^2 = -180y$; 10 ft., 20 ft., and 30 ft. from center, height is $\frac{175}{9}$, $\frac{160}{9}$, and 15 ft. respectively.

8. A projectile is fired from a gun with a muzzle speed of 1000 ft. per sec. The gun is inclined at 30° to the horizontal. Find the range over a level plain neglecting air resistance. *Ans.* 26,900 ft.

9. An equation relating the time θ and volume v of a filtrate passing through a filter cloth is a parabola with the equation

$$(v + v_c)^2 = k(\theta + \theta_c),$$

where k, v_c, and θ_c are constants. Let v be in liters and θ in seconds. If $v = 0$ when $\theta = 0$, and $v = 2$ when $\theta = 53.4$, and further $v = 4$ when $\theta = 163.0$, what are the values of the constants k, v_c, and θ_c? After having determined the values of these constants find the amount of the filtrate which will go through the filter cloth in two minutes.

Ans. $k = 0.143$, $v_c = 0.900$, $\theta_c = 5.69$; amount of filtrate is 3.34 liters.

10. For a simple pendulum the time of a half vibration in seconds is related to the length of the pendulum in feet by the equation of a parabola

$$t^2 = \frac{\pi^2}{g} l = \frac{\pi^2 l}{32} \text{ nearly}$$

[g is the acceleration of gravity].
Find the time of half vibration of a pendulum of length 3 ft. *Ans.* 0.96 sec.

11. Towers supporting a suspension bridge are 320 ft. apart and rise 80 ft. above the roadbed. The lowest point of a main cable in the shape of a parabola is 30 ft. above the roadbed. Find an equation for the curve of the cable.

Ans. One possible equation is $x^2 = 512(y - 30)$.

Ellipses

12. The earth's orbit is an ellipse with the sun at one focus, the major axis being 186×10^6 miles, and the "eccentricity" $(\sqrt{a^2 - b^2})/a$ being $\frac{1}{60}$. Find the difference between the greatest and least distance of the earth from the sun.

Ans. 3.10×10^6 miles.

13. The stone arch of a bridge has a span of 150 ft. and a height of 38 ft. The arch is in the shape of half of an ellipse. Find an equation for the ellipse, and locate the foci. *Ans.* $\frac{x^2}{5625} + \frac{y^2}{1444} = 1$; foci are 64.7 ft. from center line.

Hyperbolas

14. The weight W that a simple beam of length l will support varies inversely as the length l, so that we may say $W = k/l$ for a constant k depending on the cross section of the beam, and the material. The curve representing this relation is a hyperbola. If a beam 10 ft. long will support 2 tons, how many tons will a beam 20 ft. long of the same material and cross section support? Plot the hyperbola in this case. *Ans.* 1 ton.

15. Two observers are 1000 ft. apart. The sound of a shot from a gun reaches one observer one-fifth of a second earlier than the other. Find an equation for a hyperbola on which the gun lies.

Ans. In standard position $\frac{x^2}{12,544} - \frac{y^2}{237,456} = 1$.

CURVE FITTING

The curves we have been discussing often arise in the correlation of data obtained from experiments. When the data are plotted as points on a Cartesian coordinate system, and the points lie on a curve, or approximately on a curve, it is usually easier to work with an equation of this curve than with the original points.

Curve fitting

The process of finding a curve to represent a group of points is called *curve fitting*. If there seems to be a choice of different curves to fit a given set of points, one should naturally choose a curve with as simple an equation as possible, preferably, of course, a straight line.

Use of plotted data

Usually a curve approximating the plotted data can be drawn, and an equation of the curve can then be obtained. In Fig. 274 some points have been plotted, and a straight line has been drawn arbitrarily which approximately fits the points. This line is a reasonable choice because it passes near all of the points. Since it passes through the points (0, 0) and (5, 6), by the two-point formula (73) an equation of the line is

$$6x - 5y = 0.$$

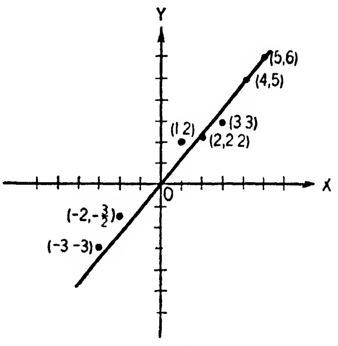


FIG. 274. LINE FITTING POINTS

The disadvantage of an arbitrary choice of a curve is that the result is not *unique*. Different persons may choose different curves to fit the same data. A more scientific procedure in general use will be described.

Least squares

It is not difficult to learn how to fit straight lines to points by the method of least squares published in 1809 by Adrien Marie Legendre (1752–1831).

We suppose that (x_1, y_1), (x_2, y_2), etc., are a set of points to which a straight line with the equation

$$y = a + bx \tag{87}$$

is to be fitted. Remember that a is the intercept on the y axis and b is the slope of the line. For $x = x_1$ the ordinate of the *line* (value of y, not y_1) is $a + bx_1$. The difference between y and y_1, that is between this value of y on the line and the ordinate y_1 of the first point, is called a *residual*, and will be designated by

$$r_1 = y_1 - (a + bx_1). \tag{88}$$

From Fig. 275 you can see that the residual r_1 is the amount whereby the straight line deviates from the point $P_1(x_1, y_1)$ in the y-direction. In the same way we form the residuals r_2, r_3, and so on for all other plotted points. We let S^2 designate the average of the squares of the residuals. If there are n given points we have

$$S^2 = \frac{1}{n}(r_1^2 + r_2^2 + \cdots + r_n^2) = \frac{\Sigma r^2}{n}. \tag{89}$$

Here Σr^2 denotes the sum of the squares of the residuals.

Least sum of squares

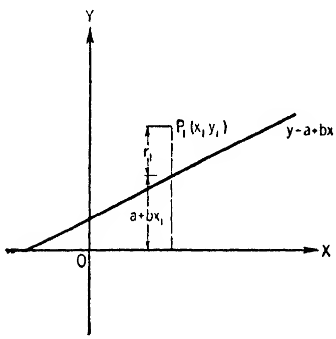


FIG. 275. RESIDUAL r_1 FROM LEAST SQUARES

By the method of least squares the *straight line fitting the given points is that line for which a and b are chosen so that S^2 is as small as possible.* You can see that a line found by the method of least squares can pass through all of the given points if and only if each residual vanishes, that is, $S = 0$. The smaller the residuals are, the smaller is S^2. The quantity S^2 was defined in terms of the squares of the residuals because these are always positive even though the residuals may be negative.

The criterion of least squares, using the relation 89 given above, also applies when a parabola $y = a + bx + cx^2$ or any other more complicated curve, such as $y = a + bx + cx^2 + dx^3$, is fitted to the given points.

Fitting a line of nonzero slope

We shall treat lines of the type $y = a + bx$ for which the slope b is not zero. We let $\Sigma 1$ stand for n, the number of given points, and let Σx, Σy, Σx^2, and Σxy denote the sums

$$x_1 + x_2 + \cdots + x_n, \quad y_1 + y_2 + \cdots + y_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2, \quad x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

respectively. By the methods of the calculus we can show that S^2 takes on its minimum value when a and b satisfy the equations

$$\begin{aligned} a\Sigma 1 + b\Sigma x - \Sigma y &= 0, \\ a\Sigma x + b\Sigma x^2 - \Sigma xy &= 0, \end{aligned} \tag{90}$$

provided that these equations can be solved for a unique pair of values (a, b). The equations 90 are two linear equations in the two unknowns a and b. The values of a and b which satisfy equa-

tions 90 are substituted in equation 87 to give the desired representative line. It is only in exceptional cases that equations 90 have no unique solution (a, b).

Straight line by least squares for plotted data

For the example of Fig. 274,

1	x	y	x^2	xy
1	-3	-3	9	9
1	-2	-1.5	4	3
1	1	2	1	2
1	2	2.2	4	4.4
1	3	3	9	9
1	4	5	16	20
1	5	6	25	30
$\Sigma 1 = 7$	$\Sigma x = 10$	$\Sigma y = 13.7$	$\Sigma x^2 = 68$	$\Sigma xy = 77.4$

The equations 90 for a and b are therefore

$$7\,a + 10\,b - 13.7 = 0,$$
$$10\,a + 68\,b - 77.4 = 0$$

Solving these equations simultaneously, we have $a = 0.4$, $b = 1.1$. By substitution in $y = a + bx$ there follows the desired equation

$$y = 0.4 + 1.1\,x.$$

Horizontal line

If a horizontal line $y = a$ is fitted to the points $(x_1, y_1), \cdots, (x_n, y_n)$ by the method of least squares, we can show that

$$a = \frac{\Sigma y}{n}. \tag{91}$$

Thus a is the average of the sum of the ordinates of the given points.

For the points of Fig. 274 the average a is

$$\frac{\Sigma y}{n} = \frac{13.7}{7} = 2.0$$

correct to one decimal place. The horizontal line which best fits these points by the method of least squares is the line $y = 2.0$. Of course, it is evident that a sloping line such as the one shown in Fig. 274 is a far better representation of these points than any horizontal line. In practice the line $y = a$ should be used only if the points lie approximately along such a line.

Problems

1. When a small valve was opened, the pressure p of a gas in a cylinder in pounds per square inch was observed on a pressure gage for values of the time t at intervals of 10 sec., and the following table was obtained. Fit the lines $p = a$ and $p = a + bt$ to these points by the method of least squares.

p	1000	990	982	969	951	938
t	10	20	30	40	50	60

Ans. $p = \dfrac{2915}{3}$; $p = \dfrac{3047}{3} - \dfrac{44}{35}t$.

2. The increasing speed v of a locomotive in miles per hour was measured at intervals of 10 sec. and the following table relating v and the time t was obtained·

v	0	10	10	15	15	20	25	30	40
t	0	10	20	30	40	50	60	70	80

Plot the points on graph paper and draw a line which seems to represent the points approximately. Then fit the line $v = a + bt$ to the data given, by least squares. Draw the line obtained by the method of least squares on the graph and compare with your approximate line. *Ans.* $v = \frac{4}{3} + \frac{17}{40}t$.

3. Construct a graph on Cartesian coordinate paper for the law that the illumination from a light source varies inversely as the square of the distance from the light.

Hint. Construct a table of values for convenient choices of the distance, and plot.

FAMILIES OF CURVES AND NOMOGRAPHS

Families of curves

A relation between more than two variables cannot be expressed by a single curve. Various methods have been used to picture such a relationship. For example, an equation in three variables, say x, y, and z, is the equation of an unlimited number of curves obtained by assigning different values to z, the curves being plotted on an x, y Cartesian coordinate system. These curves form what is called a *family of curves*. This procedure is used on weather maps. On the temperature maps, which relate temperature to position on the earth's surface (position requiring two coordinates) the curves are termed *isothermal lines*, that is, lines of equal temperature. Usually these lines are so irregular that a simple equation for them cannot be obtained. They represent, however, a relation between three variables. At each point (x, y) on the isothermal line marked 60° the temperature z is supposed to be 60° at the time indicated. On pressure maps the curves of the family are lines of equal pressure, and are called *isobaric lines*.

Nomographs

The nomograph is often much more convenient than other methods of picturing a relation between variables. The ordinary

nomograph (alignment chart) consists of a series of lines (usually, but not necessarily, straight), one for each variable in the given relation. Values of the variable are marked along the line of that variable as on a scale. These markings or scales are made in such a way that by properly laying a straight edge over given values of all variables but one, the corresponding value (or values) of the remaining variable as determined by the given equation may be read off at once.

Usefulness

Nomographs are particularly useful where numerous computations are to be made with the same formula. Nomographs can be constructed for such expressions as

$$z = 0.01\,xy^2; \quad z = \frac{5\,x}{y}; \quad \frac{1}{z} = \frac{1}{x} + \frac{1}{y}; \quad u = x + y - z.$$

The construction of suitable nomographs depends a great deal on the ingenuity of the worker. However, an idea of how to make them can be obtained from the example which follows.

Simple nomograph

In Fig. 276 we have constructed a nomograph for the relation

$$z = xy$$

between the variables x, y, and z. The lines A and B are parallel to C which is midway between A and B. A line perpendicular to

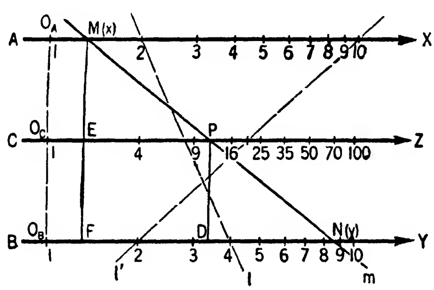


FIG. 276. NOMOGRAPH FOR $z = xy$, OR $y = \frac{z}{x}$

A, B, and C cuts these lines at the "origins" O_A, O_B, and O_C respectively. The scales on A and B are logarithmic scales such as the scales on a slide rule.

Vertical line

A straight line perpendicular to A, B, and C cuts these lines in such a way that the scale values of the points of intersection with A and B are identical, and the point of intersection with C has a scale value equal to the square of the corresponding value on the other scales. In particular, the distance from O_A to the point

marked 3 on A is log 3 in an appropriate unit of measure, the distance from O_A to the point marked 4 on A is log 4, and so on. The line joining the points marked 3 on A and 3 on B cuts C at the point marked 9, that is, the square of 3.

Multiplication and division

It follows that to multiply 4 by 4 we need only lay a vertical line on the points marked 4 on A and B, and read off the result, 16, on C. This method of obtaining products also applies to unequal numbers, as we shall see later. Thus to multiply 2 by 4 we lay the line l on the point of A marked 2 and the point of B marked 4, as shown in Fig. 276. The line cuts the C-scale at the product 8. The line l' of Fig. 276 meets A and B at points marked 10 and 2 respectively, and cuts C at their product 20.

In general terms, to obtain the product xy of numbers x and y, we pass a straight line through the points marked x and y on A and B respectively. The value of xy is the scale value of the point where this line cuts C. Obviously, division may be performed by drawing a line from the *divisor* on A through the *dividend* on C to obtain the *quotient* read off on the scale B.

Proof

To prove the validity of the nomograph of Fig. 276 for computing the product xy, we have drawn Fig. 277. Since the scales A

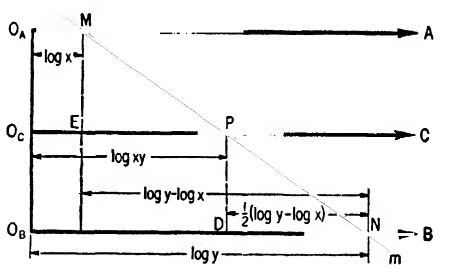


FIG. 277. NOMOGRAPH OF THE EQUATION $z = xy$

and B are *logarithmic*, the distance from the origin to M is $\log x$ and to N is $\log y$. The horizontal distance FN between these points M and N is evidently $\log y - \log x$ and one-half of this distance, DN, is $\frac{1}{2}(\log y - \log x)$. From Fig. 277

$$O_BD = O_BN - DN = \log y - \tfrac{1}{2}(\log y - \log x).$$

By rule 46 we have $\log y - \log x = \log y/x$, and by rule 47, $\frac{1}{2}\log y/x = \log \sqrt{y/x}$. It follows that

$$O_BD = \log y - \log \sqrt{\frac{y}{x}} = \log \frac{y}{\sqrt{\frac{y}{x}}} = \log y \sqrt{\frac{x}{y}} = \log \sqrt{xy}.$$

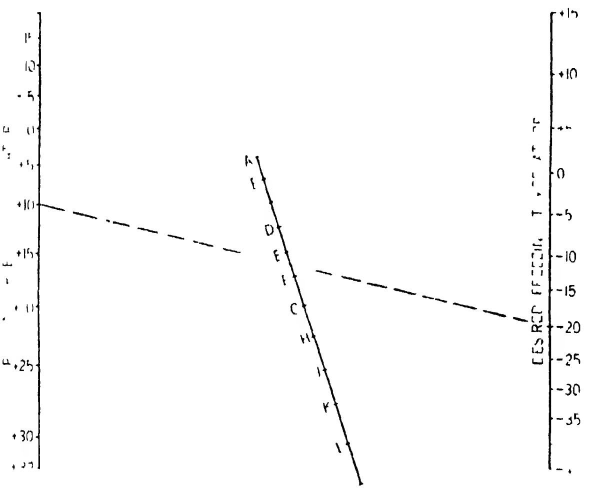
Since the number on the C scale at P is the square of the corresponding number on the B scale at D, we may write

$$O_CP = \log (\sqrt{xy})^2 = \log xy,$$

whence the line m of Fig. 276 cuts the B scale at a number which is the product of x and y, that is, the value z in the equation $z = xy$ for which the nomograph was drawn.

Nomograph for voltage

A nomograph such as Fig. 276 may be used effectively for calculations involving Ohm's law, $E = IR$, which states that the *potential drop* (voltage) E across a *resistance* R is the product of R by the *current* I flowing through the resistor. Letting the upper and



QUARTS TO REPLACE	COOLING SYSTEM CAPACITY					
	8 QTS	12 QTS	16 QTS	20 QTS	24 QTS	28 QTS
A		1		1	1	1
B	1	1	2		2	3
C	1	2	2	3	4	4
D	2	2	3	4	5	6
E	2	3	4	5	6	7
F	2	4	5	6	7	8
G	3	4	6	7	8	10
H	3	5	6	8	10	11
J	4	5	7	9	11	13
K	4	6	8	10	12	14
L	4	7	9	11	13	15

FIG. 278. NOMOGRAPH FOR DETERMINING QUANTITY OF ETHYLENE GLYCOL (PRESTONE) NEEDED TO REDUCE FREEZING POINT OF AUTOMOBILE RADIATOR SOLUTION

Using the chart

To use the nomograph lay the edge of a ruler from the mark on the left-hand scale representing the present protection, to the mark on the right-hand scale corresponding to the freezing protection desired. Read the first letter above or crossed by the ruler on the lettered scale in the middle of the chart. Turn to the table of Cooling System Capacity and in the column corresponding to the capacity of the radiator read the number found on the same line as the first letter appearing on or above the ruler. This number represents the quarts of solution to be drawn off from the cooling system and replaced with undiluted ethylene glycol anti-freeze.

Thus, if the cooling system capacity is 12 quarts, the present protection is 10° F. and protection to − 20° F. is desired, lay a ruler from the + 10 mark on the left-hand scale across to the − 20 mark on the right-hand scale. E is the first letter above the ruler on the lettered scale in the middle. In the "capacity" table on the extreme right, under "12 quarts" and on the same line as E is the number 3, which is the number of quarts to be drawn off and replaced with undiluted ethylene glycol anti-freeze to decrease the freezing point to − 20° F.

lower lines of the nomograph of Fig. 276 represent I and R respectively, we see that the middle line represents their product E. Knowing any two of these variables, the third is found by laying a straight edge across the chart.

Nomograph for an antifreeze solution

The nomograph of Fig. 278 is a more complex type, used for computing the amount of antifreeze needed to reduce the freezing temperature of a radiator solution for any desired protection. Its use is explained underneath the chart.

SCOPE OF MATHEMATICS

In the present chapter we have given the mathematics needed for the understanding and solution of a great many engineering problems. This by no means exhausts the body of mathematics needed for the comprehension and development of scientific fields. In fact, most of the mathematics required for the treatment of engineering problems is still undiscovered, although new mathematical principles are found daily by research mathematicians.

Applied mathematics

Unhappily, to understand one branch of mathematics you must have studied other branches of mathematics as well. In the mathematical approach to scientific problems the applied mathematician can therefore be of great assistance. That much useful work can be done with mathematics alone is emphasized by the branch of science called *astronomy*. Since relatively indirect methods must be used to measure the properties of stars and planets, the astronomer has had to rely mainly on mathematical reasoning to explain the movements and structure of heavenly bodies.

With mathematical reasoning and experimental analysis the dream world of the radio, the telephone, the airplane, and the automobile has become a reality.

The Slide Rule

THE ENGINEER'S COAT OF ARMS

What it is

Most engineering problems require numerical calculations and the slide rule is commonly used for making these calculations. It is an instrument with which you can multiply and divide quickly and accurately; with which you can find a power or root of a number easily; and with which you can find the sine, cosine, or tangent of an angle or the logarithm of a number in less time than you can look it up in a set of tables. The engineer's slide rule is usually straight and consists of three parts as shown in Fig. 279. There is the *frame* or stock; the movable part in the center which is called the *slide;* and the glass or *runner*. The runner has a central vertical line called the *hairline*. This line is used for locating a number on the slide rule. Each row of marks on the slide rule is called a *scale*.

Its evolution

The slide rule is based on logarithms, invented by John Napier and made public in 1614. In 1620, Edmund Gunter invented the logarithmic line or scale, which was simply a scale with lengths proportional to logarithms. The first slide rule, with two or more logarithmic scales contained in a single instrument, appeared about 1630, invented independently by Edmund Wingate and William Oughtred. Since that time many additions and improvements have been made to the slide rule but the essential form remains unchanged. New types of scales have been added for special calculations; better methods of marking the scales on the slide rule make it easier to read; and improved workmanship has increased its accuracy and ease of operation.

Short and long rules

Slide rules for calculation are made in lengths from 5 to 20 in. The shorter ones can be carried more conveniently in a pocket or brief case and are easier to use, while the longer ones are more accurate. Most people use the 10-in. rule which combines the good points of the others. Slide rules of 6- or 8-ft. length that you may have seen in the classroom are for demonstration purposes only.

Read these pages with your slide rule at hand and actually

work through the examples with it. Then, as you finish a paragraph, do the problems at its end. The problems will furnish the practice you need.

THE MANNHEIM SLIDE RULE

Construction

About 1850, Amédée Mannheim, a French artillery officer and later Professor of Mathematics at L'École Polytechnique in Paris, designed the slide rule which bears his name, and which, with some additions, is still in wide use. One improved version is the Polyphase slide rule, a trade name for the standard Keuffel & Esser Company rule. The front of this slide rule is shown in Fig. 279. There are three scales on the front of the stock (the A, D, and K scales), and three on the front of the slide (the B, CI, and C scales). The slide is thinner than the stock and has three scales on the back (the S, L, and T scales), which are ordinarily used by turning the entire slide rule over and pulling the slide to the right.

Advantages

The chief advantages of the Mannheim slide rule are that it is simple, compact, and sturdy. It is smaller than other types of slide rules and yet it has all of the scales necessary for most calculations. These are arranged conveniently and there is not so much chance of confusion in operation as with a slide rule having twice as many scales. The heavy stock is more rigid than those on other types of slide rules and cannot easily get out of adjustment. The runner has only one piece of glass and is not so likely to break when the slide rule is dropped as are the runners on other types. The ruler on the top front edge of the slide rule (divided into sixteenths of an inch) is very useful in making sketches.

Improvements

The only real disadvantage of the Mannheim slide rule appears in comparison with the log-log slide rule which will be discussed later. With the Mannheim, it is not so easy to raise a number to a fractional power or to find the *natural logarithm*

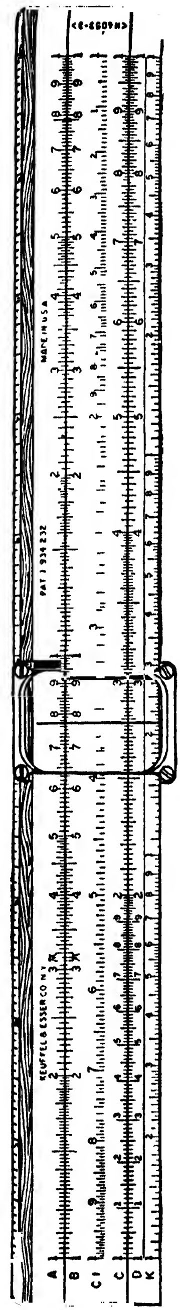


FIG. 279. MANNHEIM SLIDE RULE

of a number. Also, certain specialized calculations take longer, although they can all be done on the Mannheim slide rule.

The scales A scale is a row of marks on the slide rule and each scale is designated by the letter at its end. For instance, notice the A scale on the top of the stock in Fig. 279. There are three general types of scales on the Mannheim slide rule. They are:

1. Logarithmic scales, which are on the front of the slide rule (A, B, CI, C, D, and K).
2. Trigonometric scales, which are on the back of the slide (S and T).
3. The equal-parts scale, which is on the back of the slide (L).

C scale The C scale is representative of all logarithmic scales so it can be used as a model for explanation. The distance from the left end of this scale to the position occupied by a number represents the *mantissa* of the common logarithm of the number. Since this is so, the location of a number on the C scale depends only upon the sequence of numerals in the number and not upon the position of the decimal point in the number. Hence, you can disregard any zeros at the beginning or end of a number. For example, the numbers 0.0278, 27.80, and 27,800 all occupy the same position on the C scale and, therefore, you can think of each as 278 for the purpose of locating it. Then just locate the number as you would on an engineer's scale (decimal system) used for measuring. The runner is set at this number (278) on the C scale in Fig. 279.

Numbers from 1 to 2 The mark at the extreme left end of the C scale is called the *left index*. If the first numeral of the number (disregarding any zeros at the beginning) is 1, the number is located between the left index and the printed numeral 2 which is about three-tenths of the distance to the right end. Within this range you can locate

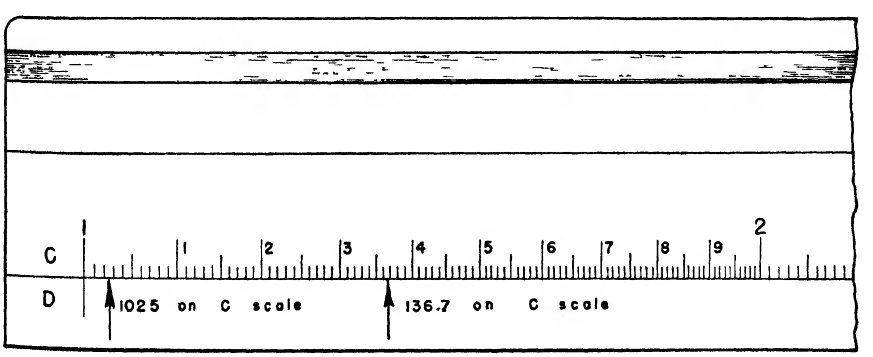


Fig. 280. The C scale between the numerals 1 and 2

the first three numerals of the number exactly and estimate the fourth. None after the fourth can be represented. Correct locations for the numbers 1.025 or 102.5 and 136.7 are shown in Fig. 280.

Numbers from 2 to 4

Between the printed numeral 2 referred to in the paragraph just above and the numeral 4 on the C scale, the marking is a little different. You can always locate the first two numerals of the number exactly, but the third can be located exactly only if it is even. If it is odd, it falls halfway between two marks because each small space represents an increase of two in the third numeral. More than three numerals cannot be represented in this range. The numbers 279 and 0.316 are shown in their proper locations in Fig. 281.

Numbers above 4

To the right of the printed numeral 4 on the C scale, the first two numerals of the number can be located exactly, but the third must usually be estimated. Each small space represents an increase of five in the third numeral. Fig. 282 shows correct locations for the numbers 57.2 and 6570.

D and CI scales

The D scale is an exact duplicate of the C scale; numbers are located on both in the same way. The CI scale differs only in that it is read from right to left instead of left to right in the usual way. It is often called the *reciprocal scale* because the number on it at a given distance from the left end is the reciprocal of the number on the C scale at the same distance from the left end.

Divided scales

The A and B scales are double scales in that each offers a double range of numbers. A given number might be located in either half, and the question of which half to use will be answered in the section where the use of these scales is discussed.

The K scale is a triple scale and a given number might be

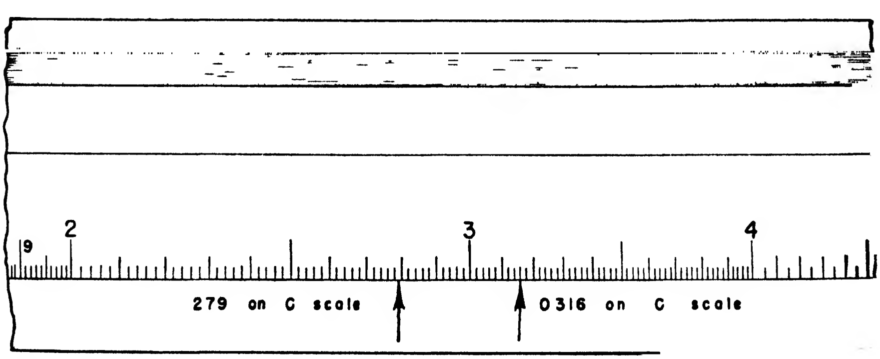


FIG 281. THE C SCALE BETWEEN THE NUMERALS 2 AND 4

located in any one of its three parts. Which part to use in a particular problem will be considered in the section on the use of the K scale.

Trigonometric scales The trigonometric scales are on the back of the slide and are the sine scale (marked S) and the tangent scale (marked T). See Fig. 283. On each, the problem is to locate an angle in degrees and minutes. This is easy if you keep in mind that there are 60 min. in one degree. The sine scale ranges from 0° 34′ to 90° and the tangent scale from 5° 43′ to 45°. All numerals printed on these scales are to be read in degrees except the numerals 40 and 50 at the left end of the sine scale. These two represent, respectively, 0° 40′ and 0° 50′. Each of the first five marks to the right of 70° on the sine scale represents an increase of 2°. Thus the next to the last mark on the sine scale represents 80°. The last mark represents 90°. The last mark on the tangent scale represents 45°. Figure 283 shows part of the tangent scale with correct locations for the angles 16° 25′ and 21° 33′.

Log scale The equal parts scale is on the back of the slide and is usually marked with the letter L. It is called the *log scale* because it is used in finding the common logarithm of a number. Figure 284 shows part of a log scale with correct locations for the numbers 068 and 123.

COMMON OPERATIONS

How to multiply A great many engineering problems include multiplication of one number by another. On the slide rule, multiplication is ordinarily done with the C and D scales. First you move the slide so that one end (index) of the C scale is over the *first number* on the D scale. Next you set the runner so that the hairline of the runner

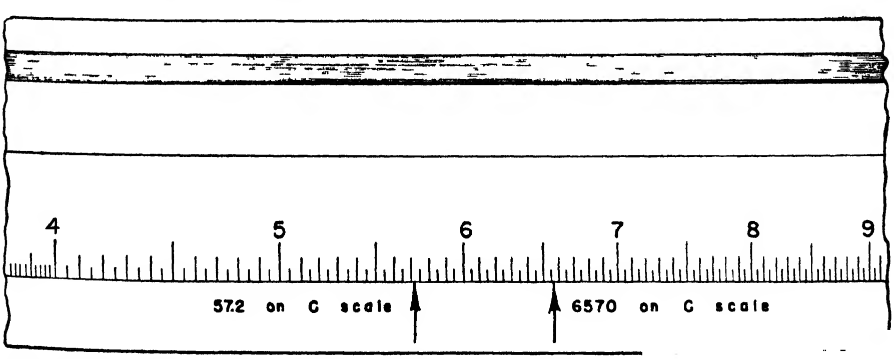


FIG. 282. THE C SCALE BETWEEN THE NUMERALS 4 AND 9

The answer

is over the *second number* on the C scale. Then you read the *answer* on the D scale under the hairline of the runner. Follow a couple of examples on your own slide rule and see how it goes. To multiply 11.6 by 1.25, as in Fig. 285, you set the left index of the C scale over 11.6 on the D scale. Then you set the hairline of the runner to 1.25 on the C scale. This completes the manipulation of the slide rule. The correct setting of the slide and runner is shown in Fig. 285. The answer is read as 14.5 on the D scale under the hairline of the runner.

Decimal points

Of course you do not get the *decimal point* from the slide rule. All it gives you is the sequence of numerals (1, 4, 5) in the answer, but the decimal point can usually be located by exercising judgment as to what would be a reasonable answer. (For instance, you know that 1.45 is too small and 145 is too large for the answer to this problem. Hence it must be 14.5.) If you want a little practice at this point, try the same problem with the numbers in reversed order by multiplying 1.25 by 11.6. Naturally the answer is the same.

Reversing the slide

Now try to multiply 8.5 by 72 in the same way. You cannot do it by the method explained above, because when you have set the left index of the C scale to 8.5 on the D scale, the number 72 on the C scale is far beyond the end of the D scale. When this happens you must solve the problem by moving the slide to the left as in Fig. 286. Set the right index of the C scale to 8.5 on the D scale. Next set the hairline of the runner to 72 on the C scale. Then read the answer as 612 on the D scale under the hairline. (You know that it is 612 because 61.2 is too small and 6120 is too large.)

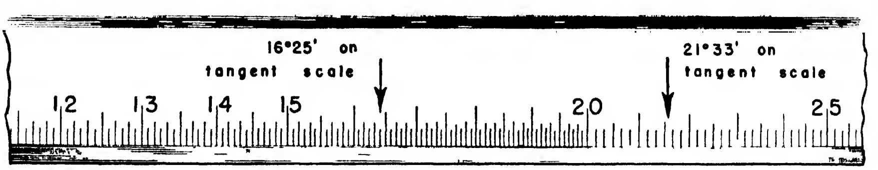


FIG. 283. LOCATING ANGLES ON THE TANGENT SCALE

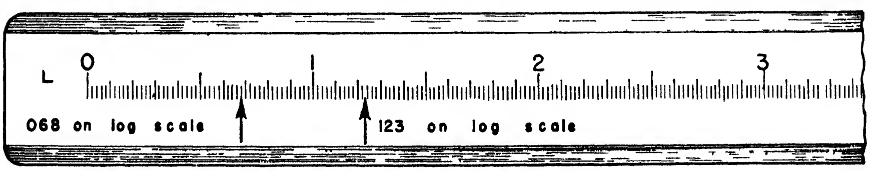


FIG. 284. THE EQUAL PARTS SCALE OR LOG SCALE

At times it is hard to say whether the slide must go to the right or left until you try it. In such a case do not waste time thinking about it. Try it one way and if it does not work, you can always do the problem by moving the slide the other way. You can multiply any number by any other number with the Mannheim slide rule. *Right or left?*

Do not worry if you did not read the answers exactly as given in the preceding examples. In the first one you might have read the answer as 14.49 or 14.52 rather than 14.50, because you might have made a slight error in setting the slide or runner (or I might have). The difference between 14.49 and 14.52, as a slide-rule answer, is not ordinarily important. *Slide-rule answers are accurate but not exact.* Now try solving the following problems in multiplication. Verify each of the answers given. *Accuracy*

Problems

1. 20.7 × 1.63 = 33.8
2. 4.82 × 0.833 = 4.01
3. 0.667 × 58.4 = 38.9
4. 0.707 × 33 = 23.3
5. 1.008 × 0.905 = 0.913
6. 5280 × 22.5 = 118,800
7. 8.32 × 2.77 = 23
8. 0.985 × 6370 = 6280
9. 17.24 × 5.5 = 94.8
10. 0.563 × 0.316 = 0.178
11. 11.04 × 6.97 = 77
12. 0.161 × 594 = 95.5
13. 10.9 × 0.501 = 5.46
14. 35.6 × 16.5 = 588
15. 6.29 × 0.437 = 2.75
16. 0.948 × 0.876 = 0.831
17. 74.8 × 3.64 = 272
18. 0.989 × 31 = 30.7
19. 3.03 × 0.484 = 1.465
20. 963 × 8.94 = 8600

Division is just the reverse of multiplication on the slide rule. First set the hairline of the runner, on the D scale, to the number into which we wish to divide (dividend). Then move the slide so that the number you want to divide by (divisor) is under the hair- *How to divide*

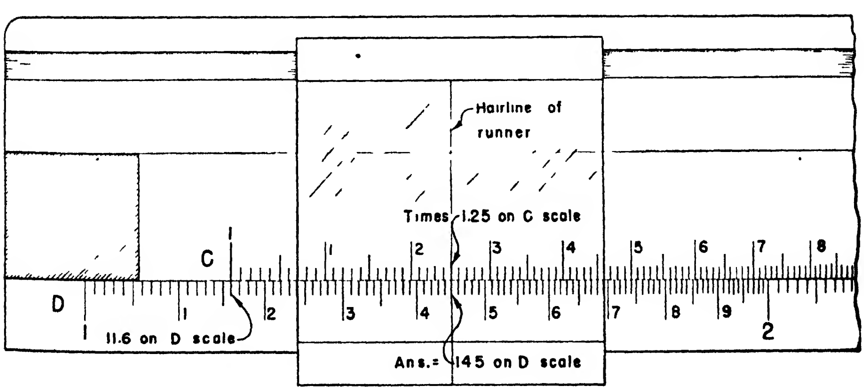


FIG. 285. MULTIPLICATION WITH THE C AND D SCALES

line on the C scale. When you have done this, read the answer (quotient) on the D scale under whichever index of the C scale is between the ends of the D scale. Here is an example to show you the way.

Example

Suppose we wish to divide 756 by 83.5. Figure 287 shows the setting of the slide and runner. First set the hairline of the runner to 756 on the D scale. Next move the slide so that 83.5 on the C scale is under the hairline. Then read the answer as 9.05 on the D scale under the right index of the C scale. In every case, one index of the C scale will be between the ends of the D scale when you are ready to read the answer. Read it on the D scale under this index. Check the answer in each of the following problems.

Problems

1. 1.078 ÷ 5.78 = 0.1863
2. 36.3 ÷ 2.07 = 17.54
3. 533 ÷ 29 = 18.4
4. 72 ÷ 17 = 4.23
5. 4.87 ÷ 1.05 = 4.63
6. 15.22 ÷ 11.73 = 1.296
7. 22.5 ÷ 9.17 = 2.45
8. 0.402 ÷ 6.65 = 0.0605
9. 0.947 ÷ 0.413 = 2.3
10. 11.5 ÷ 3.46 = 3.33
11. 33.3 ÷ 9.04 = 3.69
12. 3.69 ÷ 57.5 = 0.0642
13. 11.6 ÷ 5.85 = 1.982
14. 9.58 ÷ 0.157 = 61
15. 0.757 ÷ 8.83 = 0.0858
16. 376 ÷ 44.1 = 8.53
17. 1.503 ÷ 0.621 = 2.42
18. 14.65 ÷ 7.19 = 2.04
19. 589 ÷ 36.7 = 16.05
20. 9.15 ÷ 7.16 = 1.275

Multiplication with division

Most technical calculations contain more than two numbers. For instance, you will often find that you need the answer to some fraction like this:

$$\frac{16{,}300 \times 242}{12{,}000{,}000 \times 1.38}.$$

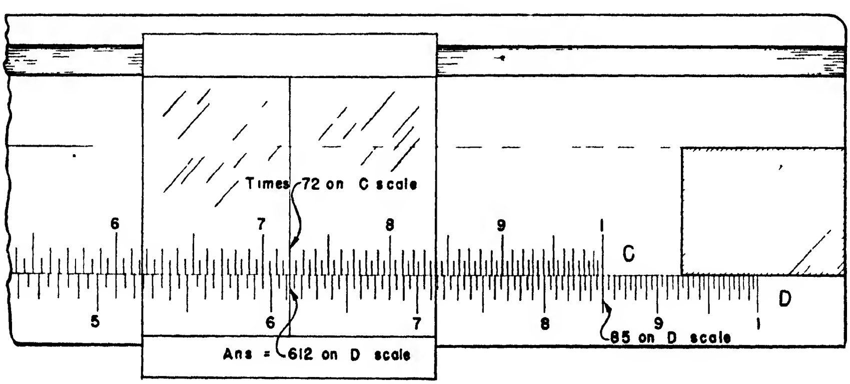


Fig. 286. Multiplication by reversing the slide

This problem includes both multiplication and division but you know how to do both. Just *multiply and divide in any order* until you have the final answer. Then write it down. That is all there is to it. In the example given above, one way to start would be to set the hairline of the runner to 16,300 on the D scale. Then divide by 12,000,000 by bringing 12,000,000 (the same as 12) on the C scale under the hairline. Next, multiply by 242 by setting the hairline of the runner to 242 on the C scale. Continuing, divide by 1.38 by sliding 1.38 on the C scale under the hairline. Finally, read the answer as 0.238 on the D scale under the left index of the C scale. You can solve this type of problem quickly and it is not necessary to write down any intermediate results. Locate the *decimal point* in the answer by "rounding off" the numbers and making the same calculation mentally.

Continuing operation

It is not always quite so easy as this example has seemed. Sometimes when you are ready to multiply by one of the numbers in the numerator, this number on the C scale will be beyond the end of the D scale. When this happens you can reverse the slide. Do so by setting the runner to that index of the C scale that is between the ends of the D scale. Then move the other index of the C scale under the hairline of the runner. An example will illustrate. We want the result of:

Reversing the slide

$$\frac{56.3 \times 0.785}{3.75 \times 13.4}.$$

The normal procedure is to set the hairline of the runner to 56.3 on the D scale. Next, divide by 3.75 by bringing 3.75 on the C scale under the hairline. At this point, you would like to multiply

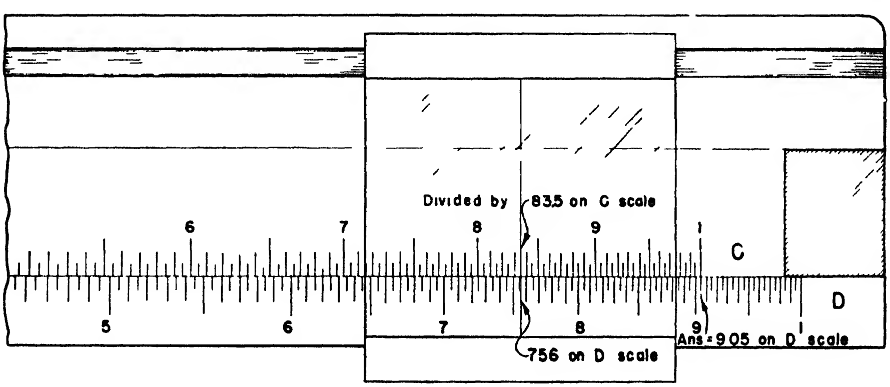


Fig. 287. Division by use of the C and D scales

by 0.785 by setting the runner to 0.785 on the C scale. But 0.785 on the C scale is beyond the end of the D scale so you cannot move the runner to it. Therefore, move the glass runner to the left index of the C scale, which happens to be above 15 on the D scale. Leave the runner in position while you slide the right index of the C scale under the hairline. Now you can multiply by 0.785 by setting the hairline of the runner to 0.785 on the C scale. Then divide by 13.4 by bringing 13.4 on the C scale under the hairline. This completes the manipulation of the slide rule and you read the answer to the problem on the D scale under the right index of the C scale. It is 0.881.

Choice of order

Another method is even better for this example. First divide 56.3 by 3.75, getting 15. Then divide 15 by 1.34 and multiply by 0.785. By this order or sequence there are fewer motions of the slide and runner since the slide does not have to be reversed. However, in some problems this order would require more operations. You can try and see for yourself that this will be true if 0.785 is replaced by 0.925 in the problem above. The problems in the next list are longer than those you have worked before, but they represent a type that engineers often have to solve. Work them by going all the way through before writing down any result. See if the following answers are correct.

Problems

1. $\dfrac{98.3 \times 27.2}{7.95 \times 15.3} = 22$
2. $\dfrac{0.753 \times 1.08}{3.85 \times 3.05} = 0.0693$
3. $\dfrac{433 \times 39}{32.7 \times 683} = 0.757$
4. $\dfrac{13.72 \times 5.68}{10.2 \times 2.14} = 3.57$
5. $\dfrac{6.33 \times 372}{46.7 \times 0.925} = 54.5$
6. $\dfrac{13.01 \times 7.68 \times 0.437}{3.92 \times 12.65 \times 8.60} = 0.1023$
7. $\dfrac{56.5 \times 0.645 \times 11.82}{7.72 \times 8.4 \times 0.754} = 8.82$
8. $\dfrac{0.643 \times 9.48 \times 504}{79 \times 5.30 \times 382} = 0.0192$
9. $\dfrac{5.84 \times 2.58 \times 0.925}{44.1 \times 1.518 \times 0.938} = 0.222$
10. $\dfrac{0.375 \times 11.8 \times 2.72}{6.28 \times 38.7 \times 0.163} = 0.304$

POWERS AND ROOTS

Squaring a number

The A and D scales are ordinarily used together to square a number or to find its square root. The indexes of the two scales are always aligned since they are both on the frame of the rule. To square a number, set the hairline of the glass runner to the number on the D scale and read the square on the A scale under the hairline. You do not need to use the slide at all. For example, the first step in squaring 43.5 is to set the hairline of the runner to 43.5 on

the D scale. Then you read the square of 43.5 on the A scale under the hairline. It is 1890. Try it yourself for practice. Then try the following problems. Find the square of each number.

Problems

1. 18.5	*Ans.* 342	6. 4.27	*Ans.* 18.2
2. 6.32	*Ans.* 40	7. 0.397	*Ans.* 0.158
3. 21.7	*Ans.* 470	8. 1.085	*Ans.* 1.18
4. 0.915	*Ans.* 0.835	9. 2.43	*Ans.* 5.9
5. 51.2	*Ans.* 2620	10. 0.733	*Ans.* 0.537

Square root

To find the square root of a number, set the hairline of the runner to the number on the A scale and read the square root on the D scale under the hairline. This raises the question of *which half of the A scale to use* in starting the process. There is a rule that answers this question:

In taking the square root use the left half of the A scale for numbers having 1, 3, 5, or any uneven number of digits to the left of the decimal point.

This is a fine rule if you remember it. However, you may forget it and besides it is more complicated when you try to apply it to numbers less than unity. For this reason, it is better to use a little mental arithmetic and not to rely upon the rule. Estimate about what the square root should be and move the runner to that number on the D scale. Then adjust the runner to the original number *on the same half of the A scale*. Finally, the square root will be found on the D scale under the hairline. Try this procedure by obtaining the square root of 23.5. You know that the square root of 23.5 is about 5. Therefore move the runner to 5 on the D scale and then adjust it slightly to 23.5 on the A scale. This position will be in the right-hand half of the A scale and the square root of 23.5 is read as 4.85 on the D scale.

Decimal fractions

For large numbers and for decimal fractions such as 0.0693, you may find some difficulty in estimating the value of the square root. It is convenient then to point off pairs of digits from the decimal point. For the number 0.0693 the first pair is .06, and since the square root of 6 is between 2 and 3 the square root of 0.0693 begins with the numeral 2. If you set the runner to 0.0693 in the right half of the A scale, you will read 833 on the D scale, but by setting the runner to 0.0693 in the left half of the A scale you read 263 on the D scale. The latter setting is therefore correct, and by mental arithmetic we place the decimal point at 0.263. Now try a little mental arithmetic in finding the square root of each number below. Then check the exact answer on your slide rule.

Problems

1. 0.643	($\sqrt{64} = 8$)	*Ans.* 0.802	6. 1020	($\sqrt{10} = 3^+$)	*Ans.* 32
2. 8.17	($\sqrt{8} = 2^+$)	*Ans.* 2.86	7. 34.2	($\sqrt{34} = 5^+$)	*Ans.* 5.85
3. 28.2	($\sqrt{28} = 5^+$)	*Ans.* 5.31	8. 0.118	($\sqrt{11} = 3^+$)	*Ans.* 0.344
4. 477	($\sqrt{4} = 2$)	*Ans* 21.8	9. 23,500	($\sqrt{2} = 1^+$)	*Ans.* 153.4
5. 10.2	($\sqrt{10} = 3^+$)	*Ans.* 3.2	10. 0.364	($\sqrt{36} = 6$)	*Ans.* 0.603

Cubing a number

The D and K scales are used together to find the cube or cube root of a number. When the hairline of the runner is set to a given number on the D scale, the cube of the number can be read on the K scale under the hairline. The slide is not used. The cube of 3.12 is found by setting the hairline of the runner to 3.12 on the D scale and then reading the cube on the K scale under the hairline. The cube of 3.12 is 30.3. Here the decimal point is located in the cube by mental arithmetic. Try this method to find the cube of each of the following numbers.

Problems

1. 1.05	*Ans.* 1 16	6. 0.953	*Ans.* 0.866
2. 0.863	*Ans.* 0.643	7. 3.72	*Ans.* 51.5
3. 5.78	*Ans.* 193	8. 6.15	*Ans.* 233
4. 2.27	*Ans* 11.7	9. 0.77	*Ans.* 0.457
5. 12.6	*Ans.* 2000	10. 1.935	*Ans.* 7.25

Cube root

Finding the cube root of a number is just the reverse of finding the cube. The number is located on the K scale by the hairline of the runner and the cube root is read on the D scale under the hairline. You will notice, however, that *a given number can be located in three different places on the K scale.* You can determine which part of the K scale to use by exercising judgment as to what would be a reasonable answer.

For instance, suppose you want the cube root of 14.7. You know that the cube root of 8 is 2 and the cube root of 27 is 3.

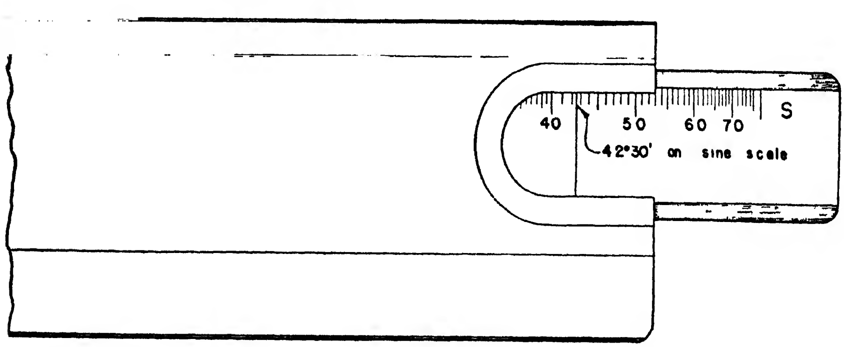


FIG. 288*a*. SETTING THE ANGLE ON THE SINE SCALE

Hence, the cube root of 14.7 must be between 2 and 3 or say 2.5. Therefore, place the runner so that its hairline is at 2.5 on the D scale and then adjust it to 14.7 on the K scale. Read the cube root on the D scale as 2.45. The same method will work for any number although it is easier with large numbers to think in terms of the cube of the approximate answer. For example, we can study the cube root of 93,500. Here you realize that the cube of 40 is 64,000 and the cube of 50 is 125,000. The cube root of 93,500 must, therefore, be between 40 and 50, and you find it to be 45.3.

Example of $\sqrt[3]{14.7}$

Special care must be taken with cube roots of decimal fractions such as 0.00474 and 0.0722. Remember that the cube root of a number less than 1 is always greater than the number itself but cannot be greater than 1. You can find the cube root of 0.00474 this way. The cube of 0.1 is 0.001 and the cube of 0.2 is 0.008. Therefore, the cube root of 0.00474 must be between 0.1 and 0.2. Set the runner between 0.1 and 0.2 on the D scale and adjust it to 0.00474 on the K scale. Then read the cube root on the D scale as 0.168. Now try this method for yourself by finding the cube root of each of the following numbers.

Decimal fractions

Problems

1. 83.4	*Ans.* 4.37	6. 1.87	*Ans.* 1.232
2. 27.9	*Ans.* 3.04	7. 139,500	*Ans.* 51.8
3. 3.37	*Ans.* 1.5	8. 0.068	*Ans.* 0.408
4. 31,300	*Ans.* 31.5	9. 0.0122	*Ans.* 0.230
5. 0.75	*Ans.* 0.909	10. 0.0007	*Ans.* 0.089

TRIGONOMETRIC FUNCTIONS

When you study mechanics and related subjects, you will often need to find the sine or cosine of an angle. On the Mannheim slide rule, the sine is found in the following manner. Turn the en-

Sine of an angle

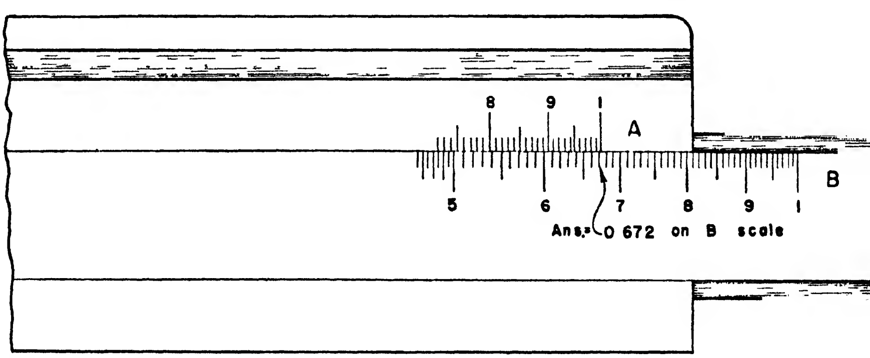


FIG. 288*b*. READING THE SINE OF THE ANGLE ON THE B SCALE

tire slide rule over and pull the slide to the right. Use the sine scale and set the angle for which you want to find the sine under the mark on the celluloid insert. (If there is no mark on the celluloid, use the edge of it instead.) Then turn the slide rule back over and read the sine of the angle on the B scale under the right-hand index of the A scale. As an example we will find the sine of 42° 30′. Turn the slide rule over and set 42° 30′ on the sine scale under the mark on the celluloid insert as in Fig. 288 *a*. Then turn the slide rule back over and read the sine of 42° 30′ as 0.672 on the B scale under the right index of the A scale. The proper setting is shown in Fig. 288 *b*. This procedure is quicker than using a set of tables. To be sure you remember it, find the sine of each angle below.

Setting the angle

Problems

1.	35°	*Ans.* 0.574	6.	47° 40′	*Ans.* 0.739
2.	16° 30′	*Ans.* 0.284	7.	32° 30′	*Ans.* 0.537
3.	22° 20′	*Ans.* 0.380	8.	61° 20′	*Ans.* 0.877
4.	53° 10′	*Ans.* 0.800	9.	7° 15′	*Ans.* 0.126
5.	71°	*Ans.* 0.945	10.	3° 23′	*Ans.* 0.059

Arc sine

To find the *arc sine* of a given number, that is, the angle which has the given number for its sine, just reverse the process of finding the sine. Set the number on the B scale under the right-hand index of the A scale. Then turn the slide rule over and read the angle on the sine scale under the mark on the celluloid insert. The B scale is a double scale so be careful about the decimal point. When you wish to find the sine or the arc sine, think of the left half of the scale as extending from 0.01 to 0.1 and the right half as extending from 0.1 to 1.0. Remember that *the sine of an angle can never be greater than unity.* Find the arc sine of each of the following numbers.

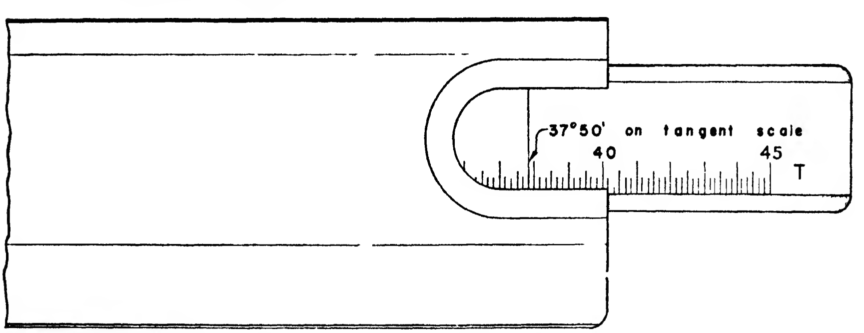


FIG. 289 *a*. SETTING THE ANGLE ON THE TANGENT SCALE

Problems

1. 0.0617 *Ans.* 3° 32′
2. 0.443 *Ans.* 26° 20′
3. 0.718 *Ans.* 46°
4. 0.0823 *Ans.* 4° 43′
5. 0.815 *Ans.* 54° 50′
6. 0.916 *Ans.* 66° 30′
7. 0.547 *Ans.* 33° 15′
8. 0.633 *Ans.* 43° 10′
9. 0.0492 *Ans.* 2° 49′
10. 0.167 *Ans.* 9° 37′

Cosine of an angle

The cosine of an angle cannot be found directly from the slide rule, but you can make use of the fact that the cosine of an angle is equal to the sine of *90° minus the angle*. When you want to find the cosine, just subtract the angle from 90° and find the sine of the difference. For example, to find the sine of 63° 30′, subtract 63° 30′ from 90°. This gives 26° 30′. Then find the sine of 26° 30′, which is 0.443. This is the cosine of 63° 30′. The *arc cosine* of a number can be found by reversing the process. While you have the details in mind, it will be useful to solve the problems below. In problems 1–5, find the cosine of the angle. In the remaining, find the arc cosine of the number.

Problems

1. 32° 50′ *Ans.* 0.840
2. 73° 10′ *Ans.* 0.290
3. 27° 15′ *Ans.* 0.889
4. 84° 45′ *Ans.* 0.0915
5. 43° 30′ *Ans.* 0.725
6. 0.384 *Ans.* 67° 20′
7. 0.0785 *Ans.* 85° 30′
8. 0.850 *Ans.* 32°
9. 0.173 *Ans.* 80° 02′
10. 0.528 *Ans.* 58°

Tangent of an angle

The tangent of an angle is a commonly used trigonometric function and it is convenient to be able to find it from the slide rule. To illustrate, we will find the tangent of 37° 50′. Turn the slide rule over and set 37° 50′ on the tangent scale under the mark on the celluloid. This is shown in Fig. 289 *a*. Then turn the slide rule back over and read the tangent of 37° 50′ as 0.777 on the C scale over the right index of the D scale. The correct position of the slide at this point is shown in Fig. 289 *b*.

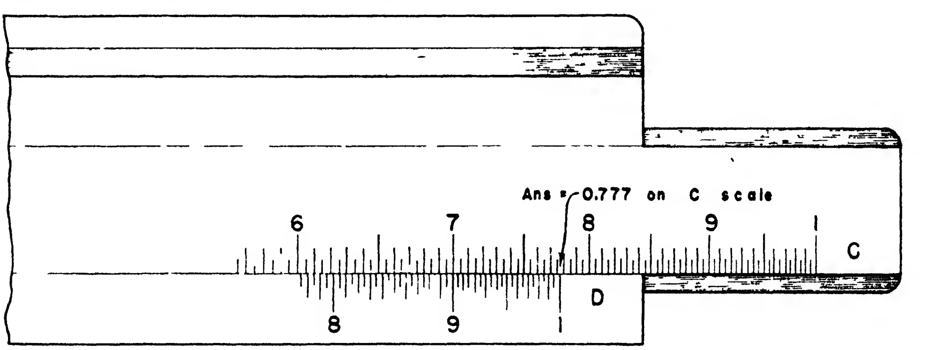


Fig. 289 *b*. Reading the tangent of the angle on the C scale

To generalize, you first turn the slide rule over and pull the slide to the right. Then set the angle on the tangent scale under the mark on the celluloid insert. Next turn the slide rule back over and read the tangent of the angle on the C scale over the right index of the D scale. Think of the C scale as extending from 0.1 at the left end to 1.0 at the right end for this calculation.

Large angles

When the angle is greater than 45°, subtract it from 90°. Set the result on the tangent scale under the mark on the celluloid and read the tangent of the original angle on the D scale under the left index of the C scale. For this purpose, the D scale is to be thought of as extending from 1 at the left end to 10 at the right end.

The *arc tangent* of a given number, that is the angle which has the number for its tangent, is found by reversing the process of finding the tangent. In the first group of problems find the tangent of each angle.

Problems

1. 6° 32′ *Ans.* 0.115
2. 41° 40′ *Ans.* 0.890
3. 22° 15′ *Ans.* 0.409
4. 71° 30′ *Ans.* 2.99
5. 33° 10′ *Ans.* 0.654
6. 52° 40′ *Ans.* 1.311
7. 27° 28′ *Ans.* 0.520
8. 34° 20′ *Ans.* 0.683
9. 67° 15′ *Ans.* 2.38
10. 10° 12′ *Ans.* 0.180

In the following list, find the arc tangent of each number.

1. 0.148 *Ans.* 8° 25′
2. 2.56 *Ans.* 68° 40′
3. 0.456 *Ans.* 24° 30′
4. 0.739 *Ans.* 36° 26′
5. 1.54 *Ans.* 57°
6. 0.238 *Ans.* 13° 23′
7. 0.905 *Ans.* 42° 10′
8. 3.15 *Ans.* 72° 22′
9. 0.673 *Ans.* 33° 58′
10. 0.375 *Ans.* 20° 35′

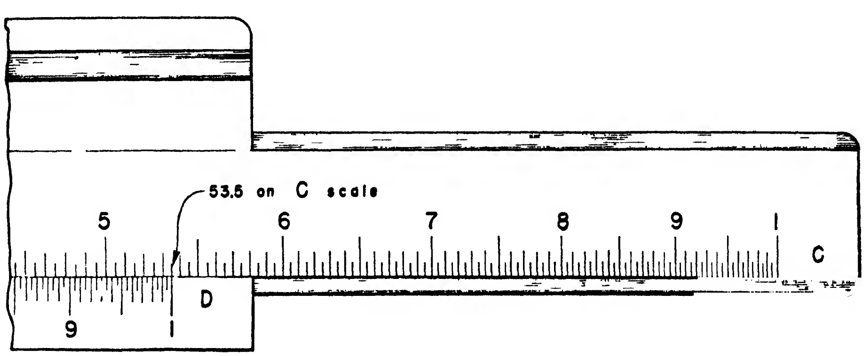


FIG. 290*a*. TO OBTAIN LOGARITHM SET NUMBER ON C SCALE

USE OF SPECIAL SCALES

Logarithms

Many engineering formulas contain logarithms. Of course, you can always look up the logarithm of a number in a table of logarithms, but since you use the slide rule for other calculations, you may save time by getting the logarithms also from it. You remember that there are two parts of a common logarithm, the *characteristic* and the *mantissa.* The characteristic is the part of the logarithm to the left of the decimal point and the mantissa is the part to the right of the decimal point. The common logarithm (to base 10) of 350 is *2.545.* The characteristic is *2* and the mantissa is *545.* The characteristic which is one less than the number of digits to the left of the decimal point of the number itself is determined by inspection. The mantissa is obtained from a set of tables or from the slide rule. It does not depend upon the location of the decimal point, but only upon the sequence of numerals.

How to find the mantissa

To find the mantissa for a given number, you first adjust the slide on the Mannheim rule so that the given number on the C scale is over the right-hand index of the D scale. Then you turn the slide rule over and read the mantissa of the logarithm of the number on the log scale (marked L) under the mark on the celluloid insert. To illustrate this, the first step in finding the mantissa of the logarithm of 53.5 is to set 53.5 on the C scale over the right index of the D scale as shown in Fig. 290 *a.* Then you turn the slide rule over, as in Fig. 290 *b*, and read the mantissa as *728* on the log scale under the mark on the celluloid. This setting is shown in Fig. 290 *b.* There are two digits to the left of the decimal point in the original number 53.5 so the characteristic of its logarithm is $2 - 1 = 1$. Thus the logarithm of 53.5 is *1.728.*

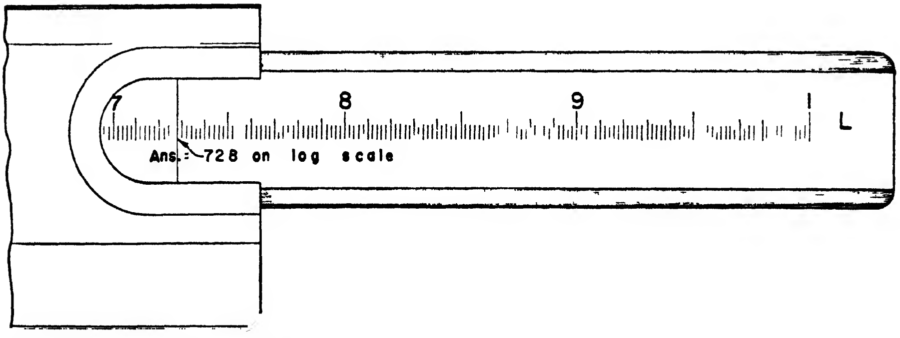


FIG. 290 *b.* READ LOGARITHM ON L SCALE UNDER HAIR LINE

Decimal fraction

There is a special method of writing the characteristic of the logarithm of a decimal fraction. This you probably remember from your study of mathematics. For instance, the logarithm of 0.535 is *9.728 − 10;* the logarithm of 0.0535 is *8.728 − 10;* that of 0.00535 is *7.728 − 10;* and so on. But no matter where the decimal point of a number is located, the mantissa of its logarithm is the same and is found in the same way with the slide rule. In the following problems, find the logarithm of each number.

Problems

1. 34.7	*Ans.* 1.540	6. 627	*Ans.* 2.797
2. 1.952	*Ans.* 0.291	7. 2.74	*Ans.* 0.438
3. 5280	*Ans.* 3.723	8. 125	*Ans.* 2.097
4. 94.5	*Ans.* 1.975	9. 22.8	*Ans.* 1.358
5. 0.872	*Ans.* 9.941 − 10	10. 8.63	*Ans.* 0.936

Each of the following is the logarithm of a number. See if you can find the numbers which are called the antilogarithms or just antilogs.

1. *1.579*	*Ans.* 37.9	6. *3.172*	*Ans.* 1485
2. *0.483*	*Ans.* 3.04	7. *1.172*	*Ans.* 14.85
3. *2.715*	*Ans.* 519	8. *1.950*	*Ans.* 89.1
4. *1.617*	*Ans.* 41.4	9. *2.250*	*Ans.* 178
5. *0.010*	*Ans.* 1.023	10. *8.316 − 10*	*Ans.* 0.0207

Multiplying with the CI scale

The CI scale is also called the *reciprocal scale* because the number on it at a given distance from the left index is the reciprocal of the number on the C scale at the same distance from the left index. It is just a C scale that reads from right to left instead of from left to right. Used properly, it will enable you to avoid

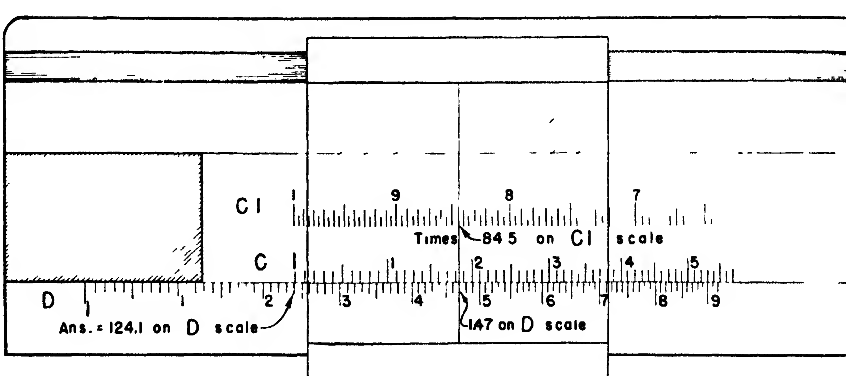


FIG. 291. SETTING THE SLIDE AND RUNNER FOR MULTIPLICATION WITH THE CI SCALE

reversing the slide in problems involving both multiplication and division.

To use the CI scale in multiplication, you use the normal procedure for division. Set the hairline of the runner to the first number on the D scale. Then adjust the slide so that the second number on the CI scale is under the hairline. The answer is read on the D scale under whichever index of the CI scale is between the ends of the D scale. This is illustrated in Fig. 291 where we see the correct setting of the slide and runner to multiply 1.47 by 84.5. The answer is 124.1, read on the D scale at the left end of the slide.

Division with the CI scale

To divide one number by another with the CI scale, follow the procedure you have learned for multiplication with the C and D scales. Fig. 292 shows the correct setting of the slide and runner to divide 93.2 by 108.5. The first step in the problem is to place the right-hand index of the CI scale over 93.2 on the D scale. Then set the hairline of the runner to 108.5 on the CI scale. The answer is read as 0.859 on the D scale under the hairline.

Explanation

If you compare what you have just read with the earlier material on multiplication and division in which you used the C and D scales, you will realize that when you multiply by a number with the CI scale, *you are really dividing by the reciprocal of the number.* This, of course, gives the same result, since $X \times Y = X \div (1/Y)$. Also the process of dividing by a number with the CI scale is equivalent to multiplying by the reciprocal of a number, $X \div Y = X(1/Y)$.

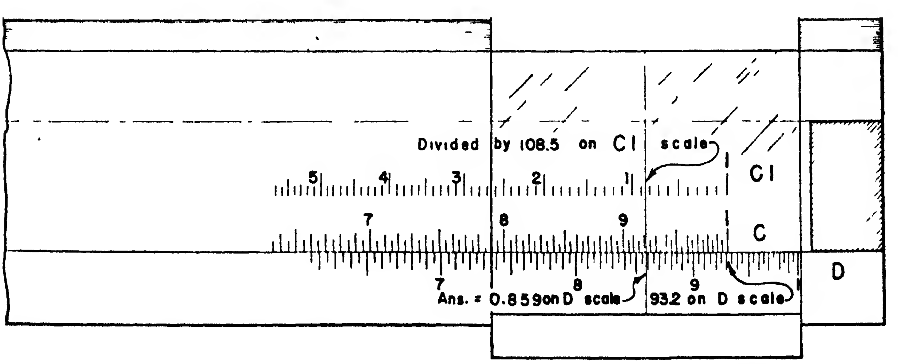


Fig. 292. Setting the slide and runner for division with the CI scale

Other uses for CI scale

The CI scale is especially useful in problems involving more than two numbers; for example, problems such as,

$$19.5 \times 27.3 \times 0.408 = ? \quad \text{or} \quad \frac{5.94 \times 65.7}{0.224 \times 343} = ?$$

These answers can be obtained with fewer movements of the slide and runner, using the CI scale, than with only the C and D scales. To demonstrate this we will solve the first one. Multiply 19.5 by 27.3 in the usual way and then multiply by 0.408 with the CI scale to get 217. In the second example, start by dividing 5.94 by 0.224 using the C and D scales. Then divide by 343 with the CI scale and multiply by 65.7 with the CI scale to obtain 5.08. Use of the CI scale in combination with the C and D scales shortens each of these problems by one movement of the slide or runner. For practice, you can turn back and use the CI scale to work the problems given for multiplication and division.

THE DUPLEX-TYPE SLIDE RULE

Advantages, disadvantages

A duplex slide rule is one with scales on both sides of the stock. One form, known as the "Polyphase Duplex Decitrig" (Keuffel & Esser Co.), is shown in Fig. 293. This type of slide rule has been very popular among engineers and is in wide use at the present time. The stock consists of two pieces of wood, held together at the ends by metal plates. The runner has two pieces of glass so that it can be used on each side of the slide rule. The trigonometric scales and the log scale are placed so that they can be used more conveniently than those on the Mannheim slide rule. Also the folded scales (CF, DF, and CIF) make it possible to multiply very quickly by π and are of advantage in certain other problems in multiplication and division. The duplex-type slide rule is not as convenient to carry with you

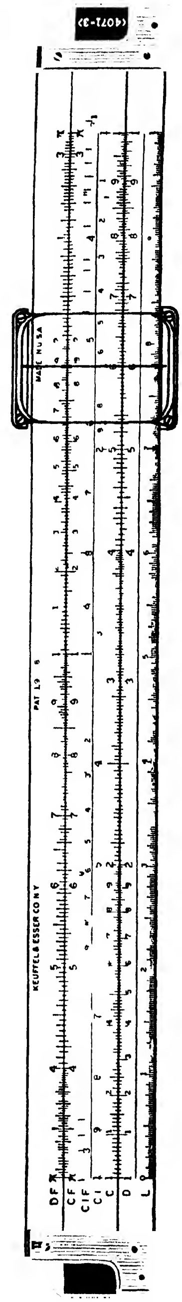


Fig. 293*a*. Front of duplex rule

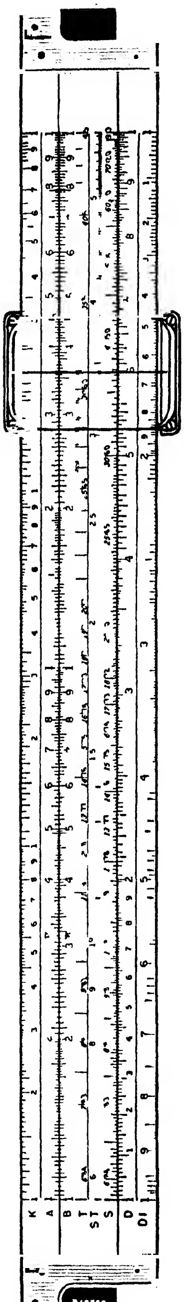


FIG. 293 *b*. BACK OF DUPLEX RULE

as the Mannheim because it is about 2 in. longer than a Mannheim slide rule which has scales of the same length.

Similarity to Mannheim

Most of the calculations you have learned for the Mannheim slide rule are done in the same way with the duplex. For instance, the C and D scales are located on the front of the duplex rule (Fig. 293) and are used to multiply and divide just as on the Mannheim slide rule. On the back of the duplex, you notice the A and D scales which you can use together to find the square or square root of a number. The K scale, on the back of the slide rule, is used with the D scale to find the cube or cube root of a number. The CI scale is located on the front of the duplex-type slide rule and can be used with the C and D scales just as on the Mannheim slide rule.

Sine and tangent

The sine scale on the back of the slide is to be used with the D scale to find the sine of an angle. Just line up the ends of the two scales, set the hairline of the runner to the angle on the sine scale, and read the sine of the angle on the D scale under the hairline. This is simpler than the use of the Mannheim rule. The tangent scale, on the back of the slide, is used with the D scale on the same side to find the tangent of an angle. Be sure to line up the ends of the tangent scale and the D scale before you try this operation. If you look closely at the sine and tangent scales, you can see that they are divided into degrees and decimal fractions of a degree rather than degrees and minutes. This represents a modern step in the development of the slide rule, one which is especially useful when you start with the value of the angle in *radians*. Notice also the numbers which lean to the left on the trigonometric scales in Fig. 293. These numbers give you immediately the difference between the angle and 90°, a figure useful in finding the cosine of an angle or the tangent of an angle greater than 45°.

Small angles The *ST scale* is an addition made especially for finding the sine or tangent of a small angle. Use it, after aligning the indices, by setting the hairline of the runner to the angle on the ST scale and reading the sine or tangent on the D scale under the hairline. (For such small angles, the sine and tangent are so nearly equal that you can take them to be the same.) Think of the D scale as extending from 0.01 at the left index to 0.1 at the right index when you use the ST scale.

Logarithms When the hairline of the runner is set to a given number on the D scale, the mantissa of the common logarithm of the number can be read on the log scale under the hairline. These scales are on the front of the duplex slide rule.

Folded scales The folded scales, CF and DF, match, respectively, the C and D scales. When the hairline of the runner is set to a given number on the C scale, the product of that number and π is on the CF scale under the hairline. This same relation holds for the D and DF scales. The CIF scale is a reciprocal scale for the CF and DF scales. Since each of the folded scales offers a full numerical range, you can use the two together (and also with CIF) for multiplication and division just as you would the C, D, and CI scales. The only difference is that the index is at the mid-point instead of the end of the slide.

Best of all, the CF and DF scales can be used in conjunction with the C and D scales to make certain calculations work out more easily. You remember in a previous problem when we wanted the result of

$$\frac{56.3 \times 0.785}{3.75 \times 13.4}$$

that we divided 56.3 by 3.75 and then had to reverse the slide in

Using C, D, CF, and DF scales order to multiply by 0.785. The reversal of the slide can be avoided by using the folded scales. You can start the calculation in the same way as before by dividing 56.3 by 3.75 with the C and D scales. Then multiply by 0.785 by setting the hairline of the runner to 0.785 on the CF scale. Next divide by 13.4 by sliding 13.4 on the CF scale under the hairline. Finally, read the result (0.88) on the DF scale above the index at the middle of the CF scale. This method can always be used, but to be perfectly safe, follow this rule. If you move up to the folded scales in the middle of a problem and end the calculation there, read the answer there on the DF scale. Do not try to read it on the D scale.

If you have a rather long problem to solve, such as *Mixing scales*

$$\frac{28.5 \times 19.7 \times 0.842}{7.2 \times 1.52 \times 44.8},$$

you may find it desirable to move up to the folded scales for part of the calculation and then back to the C and D scales to complete it. If you do, *the safe procedure is to use the folded scales exactly as many times for division as you do for multiplication.* Otherwise, your answer may be wrong. Now suppose we solve the problem just mentioned. Start by setting the hairline of the runner to 28.5 on the D scale. Then divide by sliding 7.2 on the C scale under the hairline. Next multiply by setting the hairline of the runner to 19.7 on the CF scale. Then divide by bringing 1.52 on the CF scale under the hairline. Now it is safe to return to the lower scales and multiply by setting the hairline of the runner to 0.842 on the C scale. Next divide by sliding 44.8 on the C scale under the hairline. Finally, read the result as 0.963 on the D scale under the right-hand index of the C scale. Rather nice, for you did not have to reverse the slide at all. Now here are problems for you to practice on. Verify each of the following answers. *Example*

Problems

1. $\dfrac{63.5 \times 1.06}{9.25 \times 2.27} = 3.2$
2. $\dfrac{0.707 \times 8.75}{0.313 \times 5.33} = 3.72$
3. $\dfrac{158 \times 1.973}{0.487 \times 7.85} = 81.4$
4. $\dfrac{25.8 \times 0.599}{1.17 \times 2.62} = 5.04$
5. $\dfrac{8.95 \times 6.85}{0.336 \times 1.08} = 168.8$
6. $\dfrac{43.7 \times 0.85 \times 5.5}{2.05 \times 3.62 \times 7.3} = 3.77$
7. $\dfrac{11.8 \times 3.50 \times 92}{6.4 \times 3.46 \times 0.67} = 257$
8. $\dfrac{5.75 \times 1.05 \times 8.93}{72.7 \times 0.042 \times 6.15} = 2.88$
9. $\dfrac{0.195 \times 2.38 \times 0.916}{47.4 \times 7.4 \times 0.45} = 0.0027$
10. $\dfrac{87.2 \times 0.519 \times 15.7}{29.7 \times 3.08 \times 0.611} = 12.73$

THE LOG-LOG SLIDE RULE

Why use the log-log rule

The log-log slide rule is a duplex slide rule in that it has scales on both sides of the stock. One form of it, known as the "Log-Log Duplex Trig" is shown in Fig. 294. It represents a development of the polyphase duplex slide rule and differs from it in that it has five log-log scales in addition to those on the polyphase duplex. Otherwise its construction is the same. Its main advantage is that certain *calculations involving exponentials* are made more easily with it than with simpler slide rules. One disadvantage is that there are so many scales on it that you lose time in picking out the proper

ones to use for the simple calculations such as multiplication, division, and so on. Also, because of its bulk, it is less convenient to carry.

Similarity to other slide rules

Multiplication and division can be performed on the log-log slide rule in the usual manner by using the C and D scales. Also, you can use the A scale for squaring a number and the K scale for cubing a number. Other calculations, such as those which use the trigonometric scales, folded scales, and log scale, are made as with the duplex slide rule.

Log-log scales

The three log-log scales on the front of the slide rule are designated as LL1, LL2, and LL3, respectively They are so arranged that any power of e (e is a number appearing very often in the more advanced phases of engineering and is approximately equal to 2 718) can be obtained directly, and in addition, any power or any root of any number can be found To raise e to any positive power, forget the slide and set the hairline of the runner to that power on the D scale The result can then be read under the hairline on one of the scales LL1, LL2, or LL3. If the power of e is between 0 01 and 0 1, the answer will be found on the LL1 scale; if between 0.1 and 1, on the LL2 scale; and if between 1 and 10, on the LL3 scale.

Any number to any power

You can raise any number to any power by using the log-log scales. Just set one index (whichever one will enable you to complete the problem) of the C scale even with the number on the LL1, LL2, or LL3 scale by use of the hairline. Next move the hairline of the runner to the power on the C scale. Then read the result on the LL1, LL2, or LL3 scale under the hairline. *You can always tell which log-log scale*

Example

to use by estimating what the result should be. For example, let us find the value of 1.17 raised to the power 2.38. Begin by setting the left index of the C scale to 1.17 which occurs on the

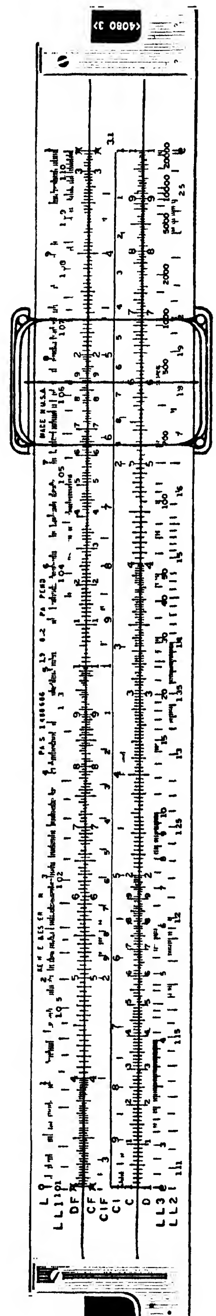


FIG 294*a*. FRONT OF LOG-LOG RULE

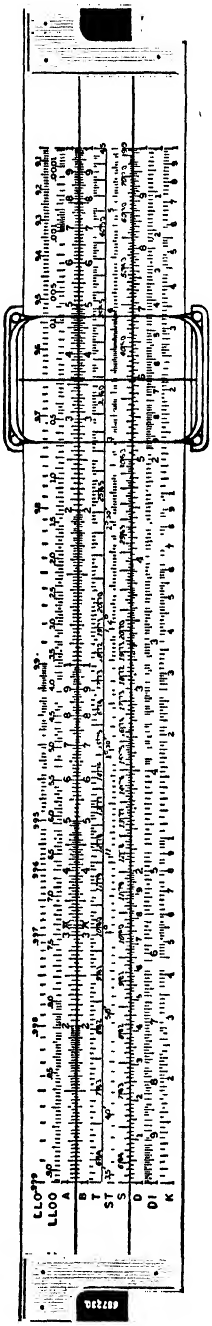


FIG. 294*b*. BACK OF LOG-LOG RULE

LL2 scale. Then set the hairline of the runner to 2.38 on the C scale. Last, read the result as 1.45 on the LL2 scale under the hairline. You can tell that the answer must be read on the LL2 scale because you would read 1.038 on the LL1 scale and 42.0 on the LL3 scale. One of these is much too small for the answer and the other is much too large.

Rule of thumb

In general, when the power is greater than 1, read to the right on the log-log scale with which you start, or read on a log-log scale of higher number. When the power is less than 1, read to the left on the log-log scale with which you start, or read on a log-log scale of lower number.

Decimal fractions

When the number you want to raise to a power is less than 1, use the A scales with the LL0 and LL00 scales. Set one index of the A scale even with the number on the log-log scale. Then set the hairline of the runner to the power on the A scale. The answer is read on one of the log-log scales under the hairline.

Any root

By reversing the process of obtaining a power we can obtain any root of any number. For example, we obtain the seventh root of 152 as follows. Set the hairline on 152 which is found on the LL3 scale. Then bring the root number 7 under the hairline on the C scale. The answer is found under the index of the C scale; it happens to appear on LL2 and is 2.05. Evidently *the root number need not be a whole number* but could have been 7.23 or any other number without complication. If you prefer, *you can always think of roots as fractional powers;* for example, the fifth root is the same as the one-fifth power.

Problems

1. $e^{3.4} = 30.0$
2. $(2.14)^{2.14} = 5.11$
3. $(0.93)^{0.7} = 0.95$
4. $(3.87)^{0.43} = 1.79$
5. $\sqrt[5]{77.2} = 2.38$
6. $(1.07)^{1.07} = 1.075$
7. $(43)^{0.24} = 2.45$
8. $(1.33)^{2.1} = 1.82$
9. $(0.5)^{1.75} = 0.293$
10. $\sqrt[1.28]{4.27} = 3.11$

THE CIRCULAR SLIDE RULE

Scale arrangement

A circular slide rule is just what the name suggests. It is circular in outline with each scale laid off on the circumference of a circle as in Fig. 295. While this type of slide rule has not been as popular as the straight ones, it has much to recommend it. Those who use it find it very convenient and feel that it is better than a straight slide rule. The three outer scales are on a separate piece of metal or plastic which can be rotated with respect to the inner part.

Observe endless scales

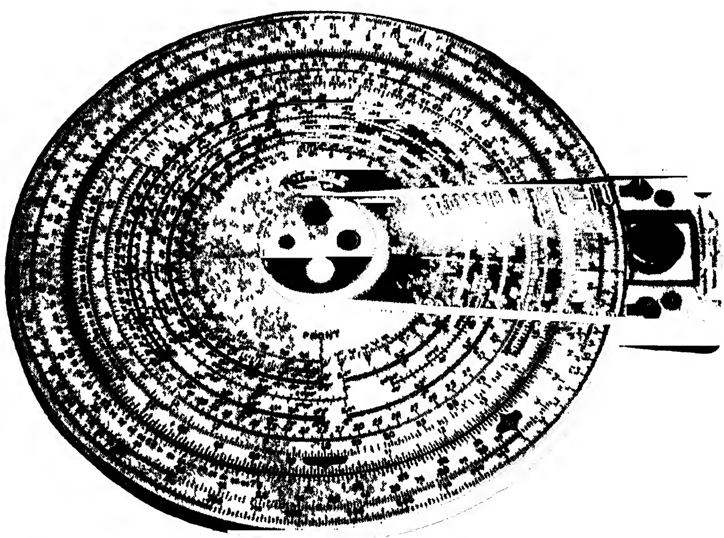


Fig. 295. The circular slide rule

This outer ring you can think of as corresponding to the stock in the straight slide rule. The inner part corresponds to the slide and the movable arm takes the place of the runner. The rule of Fig. 295 is the Dietzgen Rotarule.

Advantages and criticisms

One great advantage of the circular slide rule is its compactness. With a diameter of 4 in., the circumference of the outer scale is about 12.5 in. Thus you secure a scale long enough for precision while the over-all size of the instrument is small. A further advantage is that the scale begins again where it ends; hence, *the runner never runs off the scale in multiplying.* A disadvantage of the circular slide rule is that the scales are not all of the same length. Those on the inside are shorter than the scales of a conventional 10-in. slide rule and, consequently, they do not give such precise results.

In spite of the different shape, the circular slide rule is really quite similar to the straight ones. The fundamental scales are the same and the fundamental calculations are performed in exactly the same manner. Of course, you do not have to think about the left index and the right index, because each scale occupies a full circle and ends where it starts. Hence, there is only one index. *Similarity to straight slide rules*

In using the slide rule shown in Fig. 295, think of the third scale from the outside as the D scale and the fourth as the C scale. To multiply one number by another, turn the center part of the slide rule to set the index of the C scale to the first number on the D scale. Then set the hairline of the runner to the second number on the C scale and read the answer on the D scale under the hairline. You will see that the procedure is practically the same as that used for a straight slide rule. To divide one number by another, just set the hairline of the runner to the first number on the D scale. Then turn the center part of the rule to bring the second number on the C scale under the hairline and read the result on the D scale under the index of the C scale. *Multiplying and dividing*

The second scale from the outside is the same as an A scale and the outside one is the same as a K scale. Move from the D to the A scale to square a number and from the D to the K scale to cube a number. Reverse the order in taking roots. This is enough description to show the similarity of the circular slide rule to the straight ones. If you know how to use a straight slide rule, you can use a circular slide rule because it is actually simpler. *Special scales*

CONCLUSION

By the time you read this you should be able to use the slide rule. Already you know that it is much faster than working arithmetic problems longhand. As you continue to use it, you will develop more speed and proficiency so that it will become a valuable tool in your work. Use the slide rule for all of your problems and get the feel of it while doing the work you have to do anyway. In a surprisingly short time you will find that you can save at least 90 per cent of the time required to do the work longhand. You will appreciate this in your engineering studies because engineering calculations are often long and tedious. You simply will not have time to multiply and divide longhand. When you are working a problem in mechanics, for instance, you need all of *Practice* *Save time*

your time to think about mechanics and will have none to spare for longhand calculations.

Fatigue and eyestrain

There are two other factors closely related to that of saving time. These are fatigue and eyestrain. If using the slide rule makes it possible for you to do your work in fewer hours, then you will undergo less fatigue and less eyestrain. While we are on the subject of eyestrain, we should consider another source of it. Many students use slide rules only 5 in. long. They have to squint to read a short slide rule. Some even use a magnifying glass on such a slide rule with the object of reading more closely which increases the eyestrain. You can get results just as precise with a 10-in. slide rule and your eyes will benefit.

Errors

As a beginner, you may be tempted to check your slide-rule calculations by working the problems longhand. Do so if you wish, but very soon you will realize that you not only work faster with the slide rule but make fewer errors too. After all, the slide rule itself cannot make a mistake; the usual errors come from setting a number wrong or reading one incorrectly. The longer you use your rule, the less chance there will be of your making such errors.

Tricks and short cuts

There are many tricks and special methods that you can learn with the slide rule: methods for solving quadratic equations, making vector calculations, and so on. I have learned many from time to time and forgotten most of them. *Short cuts are worth learning only if you have frequent use for them.* They are not worth learning if you are going to use them only once every year or two because you will forget them in between uses. Concentrate instead on the fundamental operations, which you can use every day and with which you can solve any problem. Be sure to use your slide rule at every opportunity for solving problems. It will soon become a habit.

Light and Electricity

HALLMARKS OF CIVILIZATION

Light and electricity are the products of the electrical engineer. Through their use he has transformed the hard and solitary struggle for existence of early man into a protected life of comfort and convenience, plus many opportunities for communication and social contacts with his fellow men.

Age of electricity

Today we live in an age of electricity. The machines in our factories are driven by electric motors and nearly all of the processes in production depend upon electric control or operation. Our streetcars, buses, railway trains, and most of the ships on the ocean are propelled by electricity taken from power lines or produced by power plants carried on the transporting medium. Our communication contacts via telegraph, telephone, or radio are activated by electricity. Our homes, offices, and stores secure their light and power from the electric power plant.

Morse's telegraph

The numerous applications of electricity have been developed in the past hundred years. These applications have been stimulated or developed directly from five major inventions which occurred within a period of seventy years. About 1838, Morse developed the first successful American telegraph apparatus. Shortly thereafter, telegraph lines were extended between the leading American communities, and railroad transportation and commerce were stimulated by this first form of long-distance communication.

Bell and Edison

In 1876, Alexander Graham Bell completed the first satisfactory telephone instrument. A few years later the system which bears the name of Bell was founded. Since then the extension of communication service for the human voice has grown until today over twenty million telephones are in use in America. In 1881, Thomas Edison built the first successful incandescent electric lamp. The establishment of electric power plants to operate the new incandescent lamps followed rapidly after 1883. The first major use of electric power was for lighting the street, the store, the shop, and the home.

Use of electric power

After about ten years electric power was applied to electric motors and other devices. Today the requirements of electric energy for power far exceed those for lighting. At the present time the United States has about 47 million kilowatts of installed capacity in its power plants. These power plants are generating about 188 billion kilowatt-hours per year. Kilowatts and kilowatt-hours are defined on page 415. Edison's invention had a great stimulative effect in bringing about the electrical age.

Marconi and DeForest

About 1898, Marconi demonstrated that electromagnetic waves (wireless waves) could be used successfully to send messages over the air. In 1907, DeForest added a third electrode (the grid) to Fleming's "electric valve" and called his invention the *audion.* These contributions have given rise to worldwide radio communication.

Electronics

The three-electrode tube has proved to be the most important scientific discovery of the twentieth century. Strange to say, it took six or more years before science perceived the possibilities of this device or could explain its operation. DeForest's audion made possible the amplification of weak signals which permits communication around the world. Recent modifications of the three-electrode tube have revolutionized control and operation of electric power devices. Today electron tubes and their activating circuits constitute a great branch of electrical engineering known as *electronics.*

ELECTRIC LIGHTING

The progress of man has been marked by the methods developed for producing artificial light. These methods have followed the stages of firebrands, candles, oil lamps, gas lamps, arc lamps, incandescent lamps, and the modern gaseous lamps. The important landmark in this progress was Edison's invention of the electric incandescent lamp. Since this landmark was reached, the science of artificial lighting has been developed and applied at an accelerated pace. The excellently lighted rooms of Fig. 296 are in great contrast to the poorly lighted homes of a half-century ago.

Incandescent lamp

Edison's early lamp used a carbonized bamboo fiber for a filament. The efficiency of this lamp was low and its life was short but the lamp did work and thus paved the way for the electrical age which was to follow. This early lamp was improved by a process which consisted of operating the filament in an atmosphere

of gasoline vapor whereby the carbon in the vapor was deposited on the filament in the form of a smooth hard coating which added to the efficiency and life of the lamp

Metal filaments

About 1907, tantalum was tried for incandescent lamp filaments and then a year later the sintered tungsten filament. The latter consisted of particles of tungsten in a binding material. After the lamp was lighted, the binding material became brittle and the

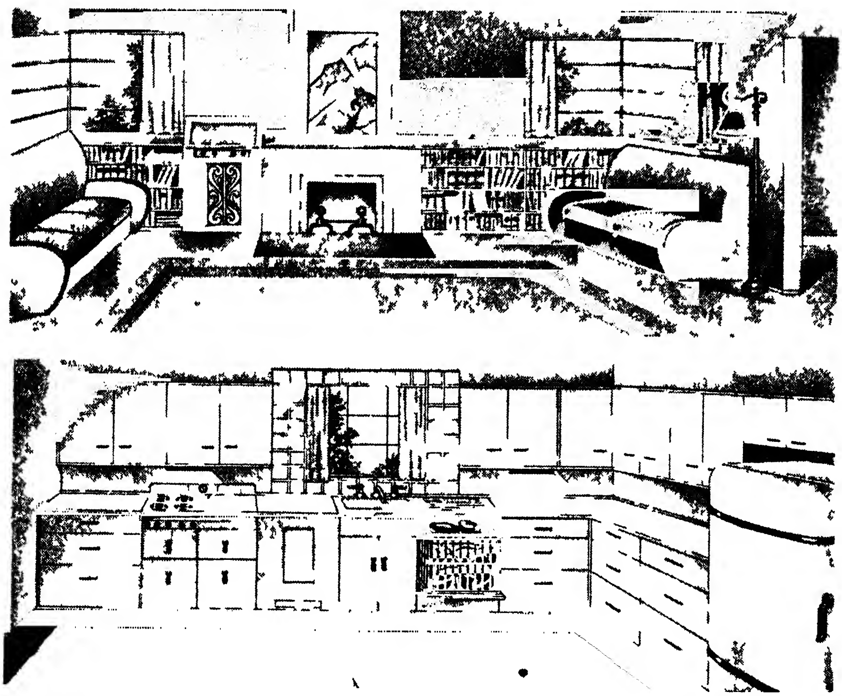


FIG 296 EXCELLENT LIGHTING FOR DAY AND EVENING IS A NECESSARY FEATURE OF MODERN HOMES

sintered tungsten filament was very fragile and broke under the slightest jar. Shortly thereafter, engineering research discovered a heat treatment which made tungsten ductile and permitted it to be drawn into wire. Since that discovery, drawn tungsten wire has been used for incandescent lamps for over a score of years.

Gas-filled lamps

Shortly after the discovery of drawn tungsten wire, a research scientist found that the efficiency of the tungsten lamp could be improved by using an inert gas instead of a vacuum in a bulb The addition of the gas (under pressure) reduced the evaporation of the metal tungsten thus permitting a higher operating tempera-

ture. The higher temperature increased the amount of electric energy transformed into light and improved the *efficiency*. The presence of the inert gas also aided the removal of the heat from the filament.

Improvements

One later improvement in incandescent tungsten lamps resulted from the discovery of a method of etching the glass bulb on its inner surface. This etching reduces the apparent brilliancy or brightness of the source, thus giving a softer light and protection for the human eye. In recent years new mechanical methods of coiling a long fine tungsten wire into a concentrated filament have permitted the use of inert gas and increased the operating temperature and efficiency of low-wattage incandescent lamps. The development of complicated machines for producing lamps on a mass-production basis with little human labor has lowered the cost of light bulbs to the point where the ratio of value to cost is exceedingly high.

Gaseous light sources

Later we will explain how *light is produced in a gas as a by-product of ionization*. Light production in gaseous tubes has been used throughout this century. The first general use was in neon lights for advertising signs. These lights consist of long glass tubes filled with neon gas under low pressure or partial vacuum. A high-voltage alternating current applied to electrodes in the ends of the tubes causes ionization of the gas which develops a glow discharge and gives the characteristic reddish-orange color. The color of this source of light limits its application to signal lights and sign lighting.

Mercury and sodium lamps

Mercury-vapor lamps have been used for some industrial lighting applications for many years. They consist of an evacuated glass tube into which a small quantity of mercury is placed. Part of the mercury evaporates forming mercury vapor; this vapor is ionized and gives out light of the characteristic blue color. This light shows up details particularly well but its color makes it objectionable for many applications.

Metallic sodium has been used instead of mercury in some special light sources designed for highway lighting. The sodium lamp is very efficient but is open to the criticism that it requires several minutes to heat the bulb and vaporize the sodium. The yellow color of the sodium light makes the unit objectionable from the psychological point of view but for the penetration of dust and fog it is considered to have some desirable properties.

The fluorescent lamp is the most important development in electric lighting since Edison's invention of the incandescent lamp. This source is a mercury-vapor lamp which utilizes the fluorescent properties of certain materials placed on the inside of the tube. The construction and circuit of a simple fluorescent lamp is shown in Fig. 297. The tube is first evacuated and then a small quantity of mercury is injected. The mercury evaporates, forming mercury vapor under low pressure. The ends of the glass tube contain *Fluorescent lamp*

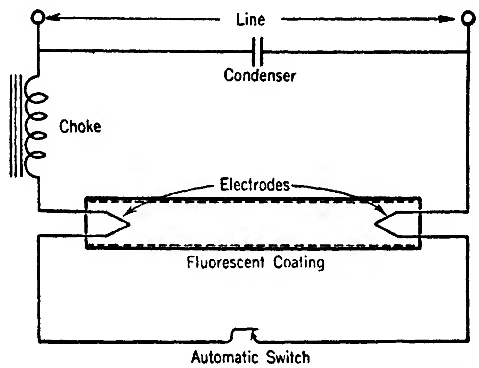


FIG. 297. CIRCUIT FOR A FLUORESCENT LAMP

electrodes which emit electrons when heated. When voltage is applied between the electrodes, *ionization of the mercury vapor* takes place, current flows through the tube, and some light is produced.

A part of this light is the typical bluish mercury-vapor light which is visible to the human eye. Another part of the light produced is ultraviolet light which is invisible and which would be injurious to the human eye. However, this ultraviolet light will not pass through the lead glass used in the tube; hence it cannot injure the eye. The ultraviolet light falls upon a fluorescent coating on the inside of the tube and its energy is transformed into another wave length or color which is visible to the human eye. *Ultraviolet*

The color of the transformed light depends upon the particular fluorescent material or combination of materials used. The common colors of light available are red, pink, gold, green, blue, white, and daylight. The efficiency of the fluorescent lamp depends on the color of the light, but in general the efficiency is from two to three times as high as for the corresponding incandescent lamp. In addition to a higher efficiency, the fluorescent source has a lower brightness and gives a better diffusion of light. The disad- *Efficiency*

vantages of the fluorescent light are the higher cost for bulbs and the auxiliary equipment required.

Circuit of fluorescent lamp

The circuit and operation of the fluorescent lamp are interesting. On starting, as shown in Fig. 297, the current flows through a series circuit consisting of a choke, one electrode, automatic switch, second electrode, and line. After a few seconds during which the *electrodes become hot and emit electrons*, the automatic switch opens the circuit between the two electrodes. The opening of this switch places the full line voltage across the tube from electrode to electrode. This voltage is high enough to produce *ionization and resultant current conduction* through the mercury vapor. After conduction starts, *the bombardment of the electrodes by the ions keeps them hot.* The function of the choke is to protect the tube by limiting the flow of current after ionization takes place. As ionization increases, the resistance of the gas column decreases, and the current will rise to damaging values without the limitation produced by the choke. The choke tends to give the fluorescent lamp a low power factor. The low power factor can be improved by the use of the condenser C shunted across the line. Such technical expressions as *power factor* and *condenser* will be defined later. In fact, you can now turn to pages 432 and 433 for these definitions if you wish.

THE PHYSIOLOGY OF LIGHTING

The science of artificial lighting covers the control of light in intensity, direction, and color, and involves parts of the sciences of physiology, psychology, and physics.

The camera

The human eye is analogous to a camera. In the camera (Fig. 298) light reflected from an image passes through a lens

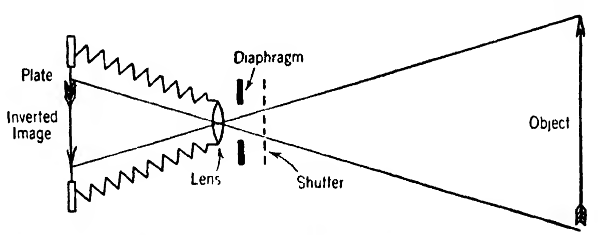


FIG. 298. ACTION OF LIGHT IN A CAMERA

and is focused on a plate or film. The lens must be in the proper position or proper shape to make a sharp focus on the plate. The amount of light admitted to the camera is controlled by the open-

ing of a diaphragm before the lens. The opening to the camera is exposed or closed by a carefully timed shutter.

The eye

In a similar manner your eye has a lens called the *crystalline lens* (Fig. 299). The shape of this lens is controlled by muscles which serve to bring the image into focus on the retina. The retina is filled with nerve ends which are sensitive to light in a way analogous to the sensitivity of the chemical surface on the plate or film

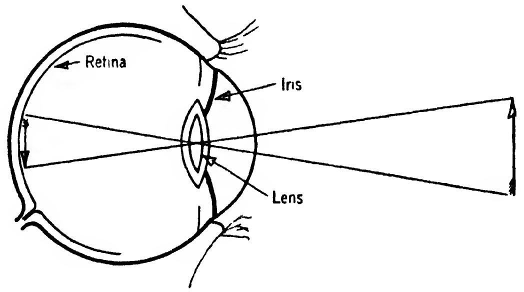


FIG. 299 ACTION OF LIGHT IN THE HUMAN EYE

of the camera. The amount of light admitted to the retina is governed by the iris, which is a curtain in front of the crystalline lens. The iris is controlled by muscles which vary the diameter of the pupil of the eye. Lastly, light is admitted to or shut off from the eye by the eyelids.

Light intensity

In order to obtain a satisfactory picture a sufficient amount of light must be permitted to fall on the film in the camera. Too little light makes a dim picture. In like manner, a certain level of light intensity is necessary on your retina. Too little light gives poor vision and also strains and fatigues the nerve centers in the retina (eyestrain). A camera pointed toward the sun will result in an overexposed film. An eye not shaded from the sun or from a bright lamp in the field of vision will be blinded by the brilliant light. Here the iris of the eye cannot act quickly enough to limit the incoming light. And even if it could act quickly enough, there would be too much *contrast* between the source and the surroundings for any satisfactory image on the retina. Hence the retina is injured and eyestrain results.

Sources of eyestrain

If the light is flickering, the muscles controlling the iris will attempt to change the size of the pupil to follow these changes and naturally these muscles will become tired — another case of eyestrain. The surface of the ground and the floors of our homes

have always been relatively dark. Hence, reflected light of a rather low intensity comes into the eye from angles below the horizontal. The part of the retina receiving this low intensity has become adapted to it and gives good vision. But if you walk about on a ground covered with white snow in bright sunshine, you will suffer great discomfort because an unaccustomed part of the retina of your eyes is receiving a relatively brilliant light.

Contrast

The human eye is uncomfortable when too large a contrast exists in the field of vision. If a light-colored scene occurs under a dark background, the iris of the eye attempts to admit the proper amount of light for the retina to envision both the lighter and the darker part of the view. The result will be that part of the retina receives too much light and another part too little. Too much contrast may be produced by the light reflection from the objects viewed or it may result from a faulty distribution of light.

THE PSYCHOLOGY OF LIGHTING

Psychology of seeing

We humans are very sensitive to the color of light. In general, we prefer a red apple to a yellow one though both may be equally palatable and luscious. The green color of foliage and landscapes seems to have a soothing effect upon human nerves and souls. Blue seems to give a feeling of coldness and sometimes sadness. Red arouses a feeling of warmth and passion. Yellow-orange, the color of a flame, gives rise to a feeling of coziness and comfort.

Daylight color may be defined as the light which is reflected from the sky and enters a north window. This light has more blue than either direct sunlight or the incandescent lamp bulb. This color is very desirable where color matching is important. Hence daylight lamps are used in stores and many factories.

Mercury-vapor lamp

The mercury-vapor lamp gives a light strong in yellow, green, and blue. It is deficient in red and orange, so these colors appear black. People look ghastly and cold under mercury-vapor light, and naturally we would object strenuously to lighting our homes with this light. The advantages of the mercury-vapor lamp are that its color spectrum makes it well adapted for taking photographs and it shows up details on drawings and machines better than daylight or the normal incandescent lamp.

For ornamental lighting at night, for floodlighting of buildings, trees, fountains, or other displays, color may be used to produce artistic and beautiful effects.

Much of the preceding discussion of the principles of lighting may be summarized into seven important suggestions which might be termed the *golden rules of lighting.*

Golden rules of lighting

1. Use sufficient light — an illumination suited to the work to be accomplished.
2. Keep bright-light sources out of the field of normal vision.
3. The light source should be large in area so that brightness, which will be defined later in candles per square inch, is less than 0.5.
4. Avoid direct reflections from polished surfaces on table tops, glossy paper, or working surfaces.
5. Avoid too much contrast — light the walls and ceilings as well as the plane of work.
6. Avoid light from any low angle other than normal reflections from floors.
7. Use a color of light suited to its function or to the work being performed.

UNITS OF LIGHT

Candlepower

The measurement of light makes use of units developed from our early source of the candle. The particular candle is one burning sperm oil at a specified rate per hour. If you pass a horizontal plane through a lighted candle as shown in Fig. 300, the intensity of light in that plane is assumed to be uniform in all directions. The intensity of light in this horizontal plane is said to be *one candle* (1 c.) *or one candlepower* (1 cp.). The intensity of light refers to direction only, and hence the intensity is 1 c. at all points whether 1 ft., 1 yd., 1 rod, or 1 mile away from the source. The candlepower of a 100-watt incandescent lamp is about 120 in the horizontal plane.

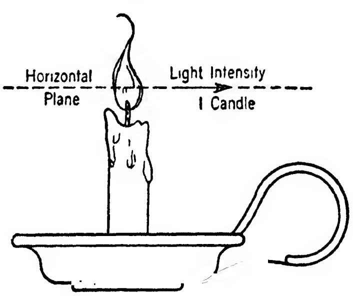


FIG. 300. ILLUSTRATING THE UNIT OF LIGHT INTENSITY

Illumination

Illumination is the measure of the amount of light falling on a surface and the unit of illumination is called the *foot-candle.* The derivation of the foot-candle unit is illustrated in Fig. 301. Thus if you place a board 1 ft. from a burning candle and at right angles to the direction of the rays, the illumination on the board is said to be *one foot-candle* (1 ft-c.). The direct illumination from the sun at midday under a clear sky may be of the order of 8000 ft-c., whereas the illumination under a full moon is about $\frac{1}{10}$ ft-c. On your study table you should have an illumination of 25 to 50 ft-c.

The intensity or candlepower of the light from a candle is not the same in all directions. Thus it will be less at a point directly above the candle than in a horizontal direction from the flame, and at a point underneath the candle it may be zero. In order to

Point source of light

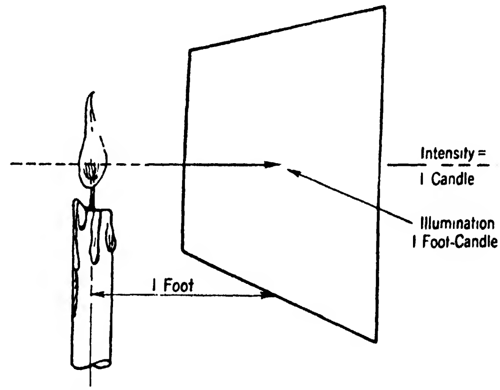


FIG. 301. ILLUSTRATING THE FOOT-CANDLE

define a unit for the quantity of light given forth in all directions or in some given sector, the concept of a point source of light is necessary.

A point source of 1 c. is one which exists at a point and gives forth an intensity of 1 cp. in all directions from the point. If you place a point source of 1 c. at the center of a sphere (Fig. 302) having a radius of 1 ft., the illumination at every point on the inside surface of the sphere will be 1 ft-c.

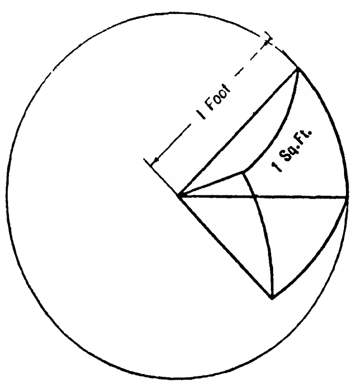


FIG. 302. THE UNIT SOLID ANGLE

Next form on the surface of the sphere a region whose area is 1 sq. ft. This may be done, for instance, by drawing a suitable spherical polygon bounded by arcs of great circles (Fig. 302). Planes drawn from the point source through the sides of this polygon enclose a solid angle. Since the area enclosed is 1 sq. ft., this is called a *unit solid angle*. The amount of light emerging from the point source of 1 c. into the unit solid angle is defined as *one lumen* of light. Three units of illumination are in this concept. A source of 1 c. or 1 cp. produces a uniform illumination of 1 ft-c. at all

The lumen

points on the inner surface of the sphere and this same source gives forth 1 lumen into the unit solid angle. Since there are 4π solid angles (spherical surface $= 4\pi r^2$) around a point, 4π lumens are given forth from the point source of light.

Law of inverse squares

If you construct a sphere having a radius of 2 ft. around the point source of the preceding paragraph, its surface area will be four times that of the one having a radius of 1 ft. (Fig. 303). And

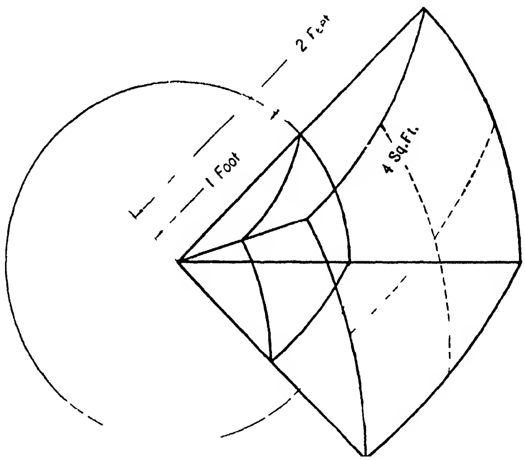


FIG. 303. ILLUSTRATING THE LAW OF INVERSE SQUARES

it will follow that if the boundary lines of the unit solid angle are extended to the larger sphere, the area encompassed will be 4 sq. ft. The lumen of light from the point source will now be spread uniformly over 4 sq. ft. instead of 1 sq. ft., giving an illumination of $\frac{1}{4}$ ft-c. Since the surface area of a sphere varies as the square of the radius, it follows that the intensity of illumination from a point source varies *inversely as the square of the distance from the source.* The preceding statement, called the *law of the inverse squares*, forms the basis of light measurements and calculations.

PHYSICS OF LIGHT

Control of light

The distribution of light is controlled by reflection, refraction, transmission, and absorption. Light falling on a mirrored or highly polished surface is reflected so that the angle of reflection is equal to the angle of incidence as shown in part *A* of Fig. 304. The mirrored surface may be curved so as to direct the incident rays in

some desired pattern. An excellent example of this is the parabolic reflector illustrated in Fig. 305. The cross section of this reflector is a parabolic curve which has the property of reflecting all light rays in parallel lines if a point source of light is placed at the focus

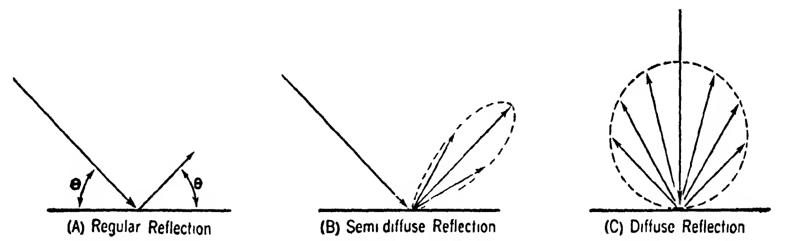


FIG. 304. TYPES OF LIGHT REFLECTIONS

of the parabola. Parabolic reflectors have been used in automobile headlights and in searchlights for years.

Diffuse reflection

If a surface is not mirrored but slightly roughened, the light will be reflected at different angles since all units of the reflecting surface do not lie in the same plane. A perfect matte (roughened) surface will reflect light in all directions. Such reflection is known as "diffuse reflection" and is illustrated in parts *B* and *C* of Fig. 304. Porcelain enamel and aluminum paint are examples of diffuse reflectors.

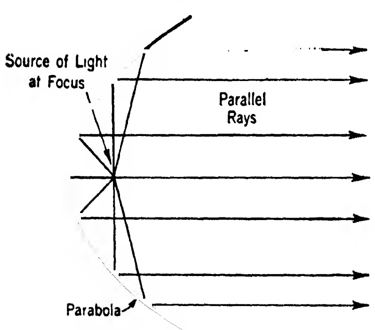


FIG. 305. LIGHT REFLECTION BY PARABOLA

Light rays passing from one medium to another at an angle other than a right angle are bent or *refracted*. Figure 306 illustrates the action of a prism in bending rays of light. Prisms are molded into glass disks and reflectors for directing and diffusing light. Street-lighting units often use a pris-

Refraction control

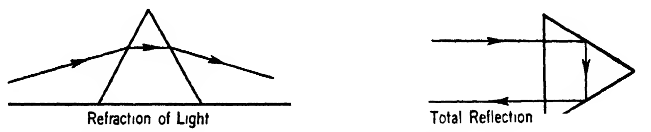


FIG. 306. CONTROL OF LIGHT BY REFRACTION

matic reflector which directs nearly all of the light into the street. That part of the light coming from the source toward the building or trees is directed back (total reflection) into the street where it is

useful. Pressed prismatic glass lenses are also used on automobile headlights to aid in the distribution of the light upon the highway.

Transmission control

Clear smooth glass transmits light with little loss or change of direction. If one side of the glass is ground or etched, this roughened side will spread or diffuse the light. If fine white particles are mixed in the molten glass, it will be milk-white in color and multiple reflections will take place on the inside of the glass. This will give a diffusion of the light transmitted. If glass is colored, some colors of light will be absorbed and the color of the transmitted light may be controlled.

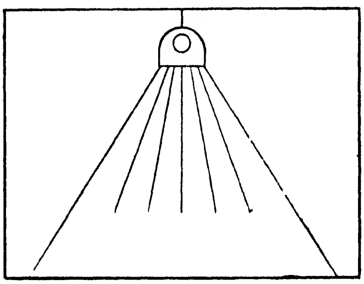


FIG. 307. DIRECT LIGHTING

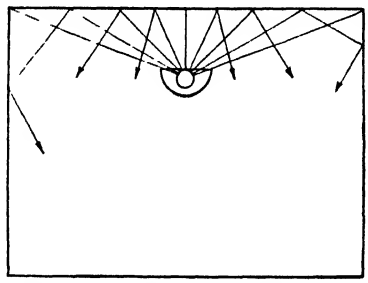


FIG. 308. INDIRECT LIGHTING

Direct lighting

Electric lighting is controlled by three different systems. These systems are termed *direct*, *indirect*, and *semi-indirect*. In direct lighting an opaque reflector is used to direct all light upon the plane where it is needed, as illustrated in Fig. 307. This is the most efficient system of lighting but it is open to the objection that the walls and ceiling will appear dark and may offer too much contrast. Generally, the light is not well diffused and shadows are heavy unless fluorescent lighting units are used.

Indirect lighting

The indirect system of lighting makes use of an opaque reflector to direct all of the light to the ceiling and upper walls from whence it is reflected to the working plane (Fig. 308). This system gives a soft light with excellent diffusion and distribution and with zero or near zero shadow effect. With sufficient foot-candles of illumination this system meets all of the rules of good lighting suggested on page 399. Yet to many people indirect lighting alone is unsatisfactory for two psychological reasons. The first reason is that they cannot see the source of light, and hence it makes them feel uncomfortable. The second reason is that they say the light is cold because of that inherent human desire for the soft flame-colored rays of light. If some flame-colored ornamental sources of

light are added to the indirect system, most persons will be well pleased with the lighting effect.

Semi-indirect lighting

The semi-indirect system of lighting is a combination of the direct and indirect systems. It is effected by using an inverted translucent reflector or bowl as illustrated in Fig. 309. This system gives good diffusion with little shadow, yet it has the desirable direct component of light added to the reflection from the ceiling. An important modification of the semi-indirect system is the I.E.S. unit which is really a portable semi-indirect light source (see Fig. 310). The portable feature makes it possible to apply the light locally and get the necessary intensity at a place for studying, reading, writing, or sewing.

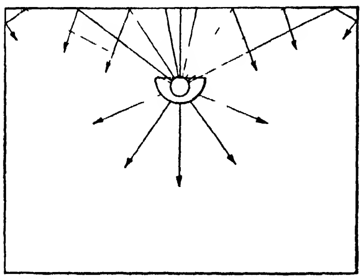


FIG. 309. SEMI-INDIRECT LIGHTING

I.E.S. lamps

The letters *I.E.S.* signify that the lamp has been built to meet certain specifications drafted several years ago by a committee of the Illuminating Engineering Society. These I.E.S. units are made in table-lamp and floor-lamp models. Since the translucent bowl is brought down to the level of the eye (standing) it is necessary to surround it with a shade to protect the eye from the brightness of the bowl. This shade gives the unit an ornamental touch with the possibility of using color.

Multiple units

The use of two or more I.E.S. units in a room permits a localized or nonuniform distribution of light which adds character to a room, whereas the more uniform distribution from an indirect unit tends to give a flat, colorless effect.

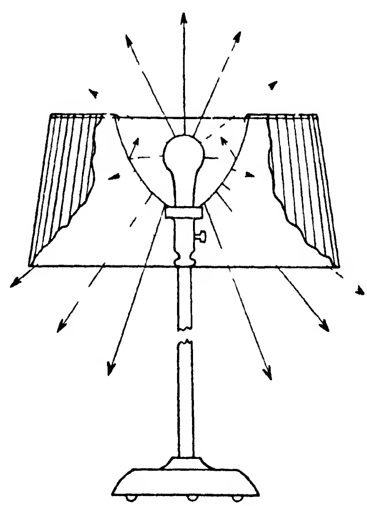


FIG. 310. CONTROL OF LIGHT BY I.E.S. PORTABLE UNIT

APPLICATIONS OF LIGHT

Proper lighting for the study table is important to the student. The method or unit to be used for lighting your study table may

be analyzed by using the golden rules of lighting. First, there should be sufficient light, and that means 25 or more foot-candles on the top of the study table. Some points of the table may have 75 and 50 ft-c. but any parts of the table where you read or write should have 25 ft-c. or more. Second, the source of light must be kept out of the field of your vision. This means that the source and its reflector should be higher than your head when seated and to the left to avoid shadow of your right hand and pencil. A good position for the source is over your left shoulder. Third, a light source of large area is desirable since it gives a better diffusion of light on the table (light coming from many angles). This good diffusion minimizes direct reflection from glossy paper or table top. Fourth, the wall that you face when writing and the ceiling above should have a fair degree of illumination to *avoid strong contrasts* as your eye shifts from table to wall and back again.

Study table lighting

Any color of light such as the regular incandescent or daylight lamp or the white or daylight fluorescent lamp should be satisfactory. However, the color and surface of the table top is important. One or two large blotters should be used because they have a matte finish and will not give a direct harmful reflection. The color of the blotter should be gray, tan, or yellow, but not red, and preferably not too dark or too light (avoid contrast).

Effect of color

The I.E.S. desk lamp will meet the foregoing requirements if it is placed in the proper position on the table — usually the left center of the table. A fluorescent lamp, either table type or one suspended from the ceiling over the left center of the table, will give excellent distribution and little or no glare. Two examples of good lighting of the study table are shown in Fig. 311.

The proper consideration of the applications of electric lighting in the home would require an entire book, and space will permit only a few helpful hints on this subject. The kitchen is a workshop. The tops of the stove, table, and shelves are workbenches and should be well lighted. Some form of shaded or concealed units should light these workbenches. In addition, there should be a central lighting unit at the ceiling to furnish general illumination throughout the room.

Lighting the home

The important consideration in the dining room is the top of the dining table and the food that is placed thereon. This table should be lighted by a fixture suspended from the ceiling. The unit should be one which gives a soft general illumination to the

Fig 311 Examples of good lighting for the study table

walls, ceiling, and faces of the diners, but a good white illumination on the top of the table.

Lighting combinations

In the living room a wide range of units and types may be employed. First, there should be some general lighting unit or units for the room which are controlled from a wall switch at two or more points about the room. These general units may be cove lights, ceiling lights, or portables. Localized portable lighting units of the table or floor type should be used at tables, writing desk, radio, reading chairs, davenport, piano, and so on. These portables may be used to spotlight pictures on the wall and in other ways to give color and character to the lighted room.

In the bathroom fluorescent or tubular lights are highly desirable. They may be placed one above and one below the mirror or one on each side of the mirror used for shaving. For the bedroom, lights should be provided on both sides of the mirrors on dressing tables and bureaus. A ceiling unit for general illumination is recommended. Lastly, a unit for giving a soft dim light on the floor is highly desirable to permit movement about the room for anyone awakened in the night or for attending someone who is ill.

Lighting a drafting room

The plans for lighting public buildings should be made by illumination engineers. Good designs require lighting experience and judgment plus the employment of the principles of lighting previously considered. Calculations and selections of units require the use of data compiled from many laboratory experiments. The actual calculations for the lighting of large rooms in stores, offices, and factories are not difficult. As an example, let us take the problem of lighting a drafting room 36 ft. by 60 ft. in which we desire to have an illumination of approximately 50 ft-c. on the tables. Such a room can be lighted satisfactorily by an indirect system using *incandescent* lamps or by a direct system using *fluorescent* lamps.

Design for indirect lighting

For an indirect system the room should be divided into sections small enough to be lighted by one indirect source. Experience shows that such a section should be approximately square, having a range of 10 to 16 ft. on each side. The floor area of 36 by 60 will divide into square sections 12 ft. by 12 ft. Each section has a floor area of 144 sq. ft. It has been shown that when 1 lumen of light is spread uniformly over 1 sq. ft. of area, the illumination is 1 ft-c. Hence to get an illumination of 50 ft-c., 50 lumens would

be required for each square foot or $144 \times 50 = 7200$ lumens for each section of the drafting room.

Losses

In the indirect-lighting system a part of the light from the source is lost in the numerous reflections which take place. For a room of the size being considered and for light walls and ceiling, tests have shown that only one-third of the light coming from the source will be effective on the working plane. Therefore, to give 50 ft-c. of *effective illumination*, $\frac{7200}{0.333}$ lumens or 21,600 ft-c. should be produced in each lighting unit.

Bulb size

If you refer to the table below for incandescent lamp bulbs, you will find that a 1000-watt lamp gives 21,000 lumens. This is slightly under the required amount, but the next size, 1500 watts,

OUTPUT OF COMMERCIAL BULBS AND TUBES

INCANDESCENT LAMPS		FLUORESCENT LAMPS				
					Lumens	
Watts	*Lumens output*	*Watts*	*Length*	*Diameter*	*Daylight*	*White*
40	410	14	15	$1\frac{1}{2}''$	390	475
100	1600	15	18	$1''$	525	585
150	2600	15	18	$1\frac{1}{2}''$	450	525
300	5850	20	24	$1\frac{1}{2}''$	760	860
500	9850	30	36	$1''$	1300	1450
750	15000	40	48	$1\frac{1}{2}''$	1800	2100
1000	21000	100	60	$2\frac{1}{8}''$	3700	4200

would give nearly 50 per cent excess illumination. Hence the 1000-watt lamp will be chosen to light each 12-ft. by 12-ft. floor section. There are $3 \times 5 = 15$ floor sections which also is the total number of indirect units required. The indirect units should preferably be hung at a height of 10 ft. above floor level, but they must be far enough below the ceiling (2 to 4 ft.) to give an even distribution of light over the ceiling. The kilowatt load of this installation will be $(15 \times 1000)/1000$, or 15.0 kilowatts (kw.).

Design for fluorescent lighting

In considering an installation of fluorescent lights for a drafting room, a number of sizes of bulbs and colors of light are available. In general, the daylight-colored light is preferable for drafting and *the 40-watt fluorescent tube is the most efficient.* This bulb gives 1800 lumens of light and when used in a direct reflector unit 60 per cent of the light will be useful. Thus, $0.6 \times 1800 = 1080$ lumens will be effective from each bulb. Since 50 lumens are required per square foot to produce 50 ft-c., one 40-watt bulb will give that

illumination over $\frac{1080}{50}$, or 21.6 sq. ft. The bulb is 4 ft. long so that one bulb would light an area 4 ft. by 21.6/4, or 5.4 ft. wide.

Fluorescent layout

The calculated dimension indicates the spacing of the fluorescent strip reflectors. Two tubes side by side are desired; hence for a width of 2 ft. $\times$ 5.4 ft. or 10.8 ft., there must be six strips in the 60-ft length of the room. Each strip will require $(\frac{36}{4}) \times 2 = 18$ bulbs. Total bulbs required will be $6 \times 18 = 108$, and, at 40 watts each, we have a load of $(40 \times 108)/1000 = 4.32$ kw.

Comparison of systems

A comparison of the two systems just analyzed shows a 4.32-kw load for fluorescent lighting and a 15-kw. load for the indirect system using incandescent lamps. A more careful analysis will show that there are some losses in the auxiliaries required with fluorescent bulbs, which lowers their apparent advantage in power consumption, but the power saving will be at least 50 per cent. Also the fluorescent unit will have a higher initial cost than the indirect unit.

Deterioration of illumination

The calculations just completed give the illumination (foot-candles) when the installation is new. The accumulation of dust on the bulbs, reflectors, and walls, the discoloration of walls with time, and the *decreased efficiency of the lamps with age* will reduce the illumination. This reduction may be as much as 20 to 35 per cent.

Problems

1. The candlepower of a certain 50-watt incandescent lamp is 50 in the horizontal plane What is the intensity in candles (a) 2 ft from the bulb? (b) 5 ft? (c) 10 ft? (d) 100 ft? *Ans* (a) 50, (b) 50.

2 A 100-watt bare lamp is hung above a study table If the lamp has a candlepower of 150 on the vertical line (through the bulb), what will be the illumination in foot-candles directly under the bulb when it is (a) 1 ft. above the desk? (b) 2 ft. above? (c) 3 ft above? (d) 6 ft above?

Hint. Apply law of inverse squares.

Ans. (a) 150; (b) 37.5; (c) 16.67; (d) 4.17 ft-c.

3. The average candlepower of a certain lamp is 25 c. How many lumens will be given forth by the lamp? (Same as point source of 25 c) *Ans* 314 lumens.

4. A shop for woodworking has floor dimensions of 20 ft by 40 ft It is to be lighted by a direct system using eight 300-watt lamps placed in porcelain reflectors. Assume one-third of all the light is useful on the workbenches Draw a floor plan showing how the lighting units should be placed and calculate the average illumination in foot-candles on top of the workbenches.

Hint One lumen of light spread uniformly over 1 sq ft. gives 1 ft-c of illumination Use the table on p. 408 for lumen output *Ans.* 19.5 ft-c.

5. A single study table is to be lighted by fluorescent tubes placed $2\frac{1}{2}$ ft. above its center line. If the table size is 3 ft. by 4 ft, what size and number of tubes would be needed to produce an average illumination of approximately 30 ft-c? Assume that one-fourth of the light given off by the lamps is effective on the table top.

Ans. Two 20-watt, 24-in white or daylight fluorescent lamps.

ELECTRON THEORY OF ELECTRICITY

Electrons

Prior to the twentieth century men knew little of the nature of electricity, though some theories of a speculative nature had been advanced. Scientists understood something of the behavior of electricity but admitted that they knew little concerning its true nature. About 1900, J. J. Thomson performed some important experiments and suggested to the scientific world the presence of the fundamental unit of electric charge — the *electron.* A few years later Robert A. Millikan performed many experiments and developed an exact and satisfactory method for determining the charge on the electron. The results of his experiments, announced to the world in 1913, confirmed the work of Thomson and gave scientists a new basis for explaining observed phenomena of chemistry, physics, and electricity.

Proton

Under the new theory the electron becomes the fundamental unit of electric charge and at the same time a particle which is one of the building blocks of matter. The other building block of matter is the *proton* which carries a positive charge, equal in magnitude to the negative charge on the electron, and having a mass about 1830 times that of the electron.

Bohr's atom

In 1913, Bohr offered a theory for atomic structure utilizing the new conception of electrons and protons. Bohr pictured the atom as consisting of a small dense core or nucleus about which one or more electrons revolve (Fig. 312).

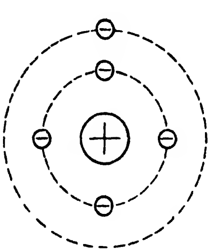


FIG. 312. BOHR ATOM

This structure is analogous to our solar system wherein the sun corresponds to the nucleus and the earth and planets to the revolving electrons. The nucleus contains all the protons and usually a part of the electrons. Chemical elements are determined by the total number of protons and electrons in the atom (always equal in number).

Hydrogen atom

For the simple hydrogen atom a single proton constitutes the nucleus and a single electron serves as the lone planet. You may obtain a conception of the relative magnitude of this hydrogen atomic system by expanding the nucleus to the size of a baseball located at the geographical center of the United States (near Manhattan, Kansas). The revolving electron will then pass through New York City and San Francisco and will be a sphere 300 ft. in

diameter — big enough to fill an average-size stadium or baseball park. Actually the radius of the hydrogen atom is of the infinitesimal size of only one hundred millionth of a centimeter. This new picture of the atom shows it to be a miniature hollow nebulous sphere; its parts form a swarm of whirling specks occupying a small part of space.

Neutron, positron

In more recent years some other particles and charges such as the *neutron* and the *positron* have been discovered. These and other discoveries have led to later theories defining the structure of the atom. However, for the general purpose of the electrical engineer and for satisfactory explanation of the action taking place in the electron tube, the Bohr atom continues to serve a useful purpose.

Atoms are combined to form the molecules of elements and compounds. Identical atoms may unite to form a molecule of an element and different atoms may unite to form a molecule of a compound. However, in the case of mercury vapor and in argon and neon gas the molecule is a single atom.

Electric charges

The Bohr atom is held together by the natural attraction of the negatively charged electrons and the positively charged protons which constitute its structure. Additional knowledge of the natural *repulsion of like charges* (negative repulsion of negative and positive versus positive) completes the basic theory which may be utilized for explaining many phenomena of electricity. The necessary concepts are relatively simple. Take the example of the electric charge. A neutral or uncharged body is one having an equal number of electrons and protons within its form and on its surface. If a body has an excess of electrons (more than protons), it has a negative charge. Conversely, if it possesses a deficiency of electrons, it is said to be positively charged.

Static electricity

The charge on a glass rod rubbed with silk is positive. This means that the friction of the silk rubbed on glass removes electrons from the glass, leaving it with a positive charge. In the process the silk becomes negatively charged owing to the acquired electrons. It should be noted that the removal of electrons from outer orbits of atoms does not necessarily change the chemical constitution of a substance.

Electric potential

Electric potential depends on the density or concentration of the excess (or deficiency) of the electrons on a body. The difference in potential between two points depends on the difference in the concentration of the electrons at those points. In contrast to

this concept it should be noted that the electric charge on a body depends on the total excess or deficiency of electrons on that body irrespective of the *area* of the body or the relative density of the electrons.

Charge on a body

To illustrate this point, let two neutral spheres, each 1 cm. in diameter, be placed 10 cm. apart and let a billion or so electrons be taken from one and added to the other. Now these spheres will have charges of opposite sign and will attract each other. Also there will be a difference in potential between these spheres determined by the concentration of their charges. If these spheres are shrunk to 1 mm. in diameter, leaving the distance between centers unchanged and without any exchange of electrons, they will still have the same charges and attract each other with the same force. However, the difference of potential between the spheres would now be increased because of the increase of *concentration* of electrons on one sphere and the *deficiency* on the other.

Electric current

An electric current is merely a coordinated flow or transfer of electrons. A random movement of electrons does not constitute an electric current since the result of the motion is not a coordinated action in some direction. The movement of electrons may be continuous and in the same direction as produced by a battery or direct-current generator, or the movement may be in a rapid back-and-forth (alternating) direction as produced by an alternating-current generator.

Electron velocity

The velocity of the individual electrons which constitute a current may vary through wide limits. At one extreme, electrons in a vacuum tube under the influence of a high potential may move with a velocity approaching that of light. Again, they may "drift" at a snail's pace of a fraction of a centimeter per second through a solid conductor.

Magnitude of current

The magnitude of the electric current is determined by the number of electrons which move past a point in a second. Since the charge on the electron is very small, the number of electrons required to equal 1 ampere (amp.) is beyond the power of human conception. The number is 6,300,000,000,000,000,000 (6.3×10^{18}) electrons per second. Moving electrons, instead of protons, constitute the current in an electric circuit because of their small mass, $\frac{1}{1830}$ times the proton.

An electric current will not pass through pure distilled water but if an *acid* base, or *salt* is added to the water, a transfer of electrons

takes place. For example, ordinary table salt (sodium chloride) added to distilled water disassociates into fragments or ions. The sodium atom separates from the chlorine atom and lacks one electron, so it has a positive charge and is called a *positive ion.* The chlorine atom carries an extra electron and is called a negative ion.

Conduction in liquids

If two electrodes are now placed in the solution as in Fig. 313, the positive electrode attracts the negative ion which on reaching it gives up its extra electron and becomes a neutral chlorine atom. In like manner the positive sodium ion is attracted to the negative electrode where it seizes an electron and becomes a neutral sodium atom which promptly reacts with water to form hydrogen gas and sodium hydroxide. This simple process has served to remove an electron from the negative electrode and to deposit an electron on the positive electrode. This action repeated continuously by millions of ions serving as carriers constitutes an electric current through an electrolyte.

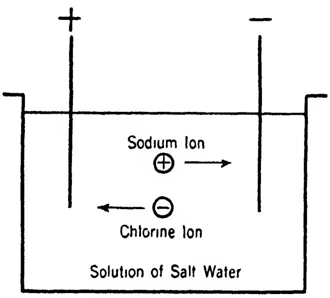


FIG. 313. CURRENT IN AN ELECTROLYTE

Conduction in solids

In solids there are some "free" electrons. Free electrons are those which are not closely bound to the atoms of which they are a part, and hence are susceptible to being moved along the conductor from atom to atom. This transfer or movement may be imagined as a relay race of electrons and it constitutes an electric current in solids.

Electrical resistance

Electrical resistance is that property which opposes an electric current or movement of electrons. It depends upon the opposition to a free movement of ions in liquids and gases and upon the resistance to removal of electrons from atoms in solids.

UNITS OF ELECTRICITY AND POWER

The electric circuit

Electricity which is used for lighting and power moves in closed circuits or paths. A simple electric circuit is shown in Fig. 314. The flow of *current* in this circuit consists of a movement of electrons. The movement of these electrons is caused by an electrical pressure or *difference of potential* which is created in the battery. The magnitude of the electric current for this circuit is limited by

the load or *resistance* which is inserted in the circuit. The three factors which enter into this circuit are related to each other in a simple manner. Thus,

Law of flow

$$(1) \qquad \text{Electric current} = \frac{\text{electric difference of potential}}{\text{electric resistance}}.$$

This relationship is known as Ohm's Law and it is the basic equation for all electric circuits. Each of the three quantities used in Ohm's Law must be represented by some unit of measure. While there are several systems of units, just as there are British units and metric units, for our purposes only the commercial units or practical units will be considered.

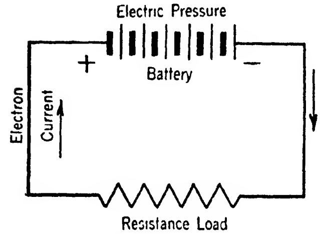


Fig. 314. Simple electric circuit

Ampere

It would seem logical to measure electric current by the number of electrons passing a point in a given unit of time, but electrons were unknown at the time our practical system of units was adopted. Hence the practical unit of electric current is the *ampere*, defined as that current which in flowing through a conductor 1 centimeter long placed at right angles to a magnetic field of unit strength will be urged out of the field with a force of $\frac{1}{10}$ dyne. This definition does not give us a simple method of measuring electric current but it does give a basis for determining a standard ampere anywhere in the world. Actually, we measure current by use of indicating meters which are calibrated to read current in amperes.

Volt

Electric pressure or difference of electric potential is measured in *volts*. The volt is defined as the difference in potential produced by cutting magnetic flux at the uniform rate of 10^8 lines per second. Difference in potential in volts is measured by the use of indicating meters calibrated in accord with the definition given above.

Ohm

Electric resistance is measured in *ohms* and is defined as the resistance which produces a difference of potential of 1 volt when a current of 1 ampere flows through it. In practice, resistance in ohms is calculated from measurements of voltage (*volts*) and current (*amperes*) which are substituted in the equation of Ohm's Law.

The common form for Ohm's Law using the units just defined becomes

(2) $$\text{Amperes} = \frac{\text{volts}}{\text{ohms}}, \quad \text{or} \quad I = \frac{E}{R}.$$ *Ohm's Law*

When an electric current moves through a resistance under the influence of an electric difference of potential, energy is expended. The *rate at which this energy is expended is* electric power which is expressed in *watts* and equals the product of volts times amperes.

Thus, we may write

(3) $$\textit{Electric power} = \textit{watts} = \textit{volts} \times \textit{amperes}.$$

Electric power

The watt is a rather small unit of power, so 1000 watts, or the *kilowatt* (kw.), becomes the more common unit. The common unit of mechanical power is the horsepower which is equivalent to 746 watts, or about $\frac{3}{4}$ kw. Work is done when power is expended over a period of time. Thus watts $\times$ hours = watthours, and 1000 watthours is the *kilowatt-hour* (kw-hr.) which is the accepted unit of electric work or energy. When you buy electricity, you pay for it at so many cents per kilowatt-hour.

Electric energy

An electric current passing through a resistance expends energy and this consumption of energy causes the temperature of the resistance to rise. Thus electric energy is transformed into heat. This action is utilized in all kinds of electric heating devices. The electric hot plate of Fig. 315 is an example. This hot plate is rated at 110 volts and 800 watts. Since watts = volts $\times$ amperes, this unit must have a normal current consumption of

FIG. 315. ELECTRIC HOT PLATE

(4) $$\text{Amperes} = \frac{\text{watts}}{\text{volts}} = \frac{800}{110} = 7.27.$$

Units of heat and electricity

Heat may be expressed in a number of units. One common unit is the British thermal unit (B.t.u.) which is *the energy required to raise the temperature of 1 lb. of water 1° F.* One kilowatt-hour is equivalent to 3415 B.t.u. If 10 lb. of water are placed on the electric hot plate referred to for 15 min., how many degrees will the temperature of the water be raised if all of the energy of the electric plate is transferred to the water without loss? The hot plate transforms 800 watts into heat for 15 min. The watthours equal

$800 \times \frac{15}{60} = 200$, or 0.2 kw-hr. This energy is equivalent to $0.2 \times 3415 = 683$ B.t.u. But 683 B.t.u. by definition above would raise 1 lb. of water 683° F. Hence, for 10 lb. of water the temperature rise will be $\frac{683}{10} = 68.3°$ F.

Electrical calculations

In using electric circuits it becomes necessary to determine the values of current, resistance, potential, or watts when only two of these quantities are known. Calculations can be made readily through the use of rearrangements of the equation of Ohm's Law. Thus we write

$$\text{Amperes} = \frac{\text{volts}}{\text{ohms}}. \tag{5}$$

$$\text{Volts} = \text{amperes} \times \text{ohms}. \tag{6}$$

$$\text{Ohms} = \frac{\text{volts}}{\text{amperes}}. \tag{7}$$

$$\text{Watts} = \text{volts} \times \text{amperes}. \tag{8}$$

$$\text{Watts} = \text{amperes} \times \text{amperes} \times \text{ohms}. \tag{9}$$

$$\text{Watts} = \frac{\text{volts} \times \text{volts}}{\text{ohms}} = \frac{(\text{volts})^2}{\text{ohms}}. \tag{10}$$

Examples

We may use these equations to solve practical problems. For example, how many amperes pass through a 110-volt, 200-watt lamp? Substituting in equation 8, we obtain

$$200 = 110 \times \text{amperes}$$
$$\text{Amperes} = 1.815.$$

How many volts are required to cause a current of 5 amp. to flow through a resistance of 10 ohms? Using equation 6, we write

$$\text{Volts} = 5 \times 10 = 50.$$

Resistances in series

Resistances may be connected in *series* or in *parallel*. The series connection is illustrated in Fig. 316 where the current through

FIG. 316. RESISTANCES IN SERIES

resistances R_A and R_B is the same. The combined resistance of the circuit is the sum of the resistance R_A and the resistance R_B.

Resistances in parallel

The parallel connection is illustrated in Fig. 317 where the current will divide in passing through the parallel paths R_C and R_D. To find the combined resistance of R_C and R_D in parallel (R_p), we first note that the voltage across both R_C and R_D is the same voltage V. Then the current through the resistance R_C is V/R_C and

through R_D it is V/R_D. The combined current is $V/R_C + V/R_D$. Again the combined current should be V divided by the combined resistance R_p, or V/R_p.

Current division

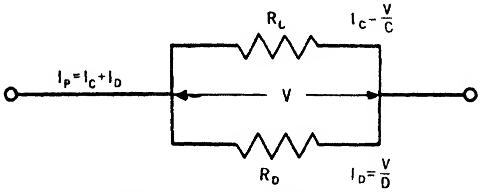


FIG. 317. RESISTANCES IN PARALLEL

Hence $V/R_p = V/R_C + V/R_D$, and dividing both sides of the equation by V gives the general equation for resistance of parallel circuits.

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots. \tag{11}$$

Problems

1. A 110-volt flatiron has a resistance of 18.35 ohms. What current does it use? *Ans.* 6 amp.

2. A current of 3.2 amp. is passed through a resistance of 10 ohms. What voltage drop exists across the resistor? *Ans.* 32 volts.

3. How many watts are consumed in problems 1 and 2? *Ans.* 660 watts; 102 watts.

4. A modern heavy-duty flatiron takes 9.1 amp. at 110 volts. What is its rating in watts? *Ans.* 1000 watts.

5. An electric water heater is rated at 220 volts and has a resistance of 11 ohms. What is its rated load in kilowatts? *Ans.* 4.4 kw.

6. Two resistances of 20 ohms each are connected (a) in series; (b) in parallel. What is the combined resistance? *Ans.* (a) 40 ohms; (b) 10 ohms.

7. What is the combined resistance of 20 ohms and 10 ohms connected (a) in series; (b) in parallel? *Ans.* (a) 30 ohms; (b) 6.67 ohms.

8. A flatiron having a resistance of 22 ohms and a toaster rated at 660 watts (110 volts) are connected in parallel across a 110-volt supply. Calculate the current through each unit, watts of each unit, total current, and total kilowatts.

Ans.	*Current*	*Watts*
Flatiron	5	550
Toaster	6	660
Total	11	1.210 kw.

Cost of power

9. A washing-machine motor requires 250 watts for operation. If the washing machine is used 3 hours per week and electricity costs 5 cents per kw-hr., what does it cost per month (4 weeks) for electricity for the family washing? *Ans.* 15 cents.

10. Assume ironing is performed with a 1000-watt iron used with current on for 4 hours per week. What is the cost of electricity for ironing per month (4 weeks) at 5 cents per kw-hr.? *Ans.* 80 cents.

MAGNETISM

Magnets and the compass

The term *magnetism* is associated with anything which has the property of attracting iron. We think of magnetism in connection with the permanent bar magnet of Fig. 318. Such a magnet will attract iron at regions near its ends which are termed *poles*. If this magnet is suspended freely, one end will swing until it points toward the north magnetic pole of the earth. This north-seeking pole is termed the *north pole* of the magnet, while the other end becomes the *south pole*. Compasses consisting of little bar magnets have served for centuries to guide ships at sea and explorers in the wilds.

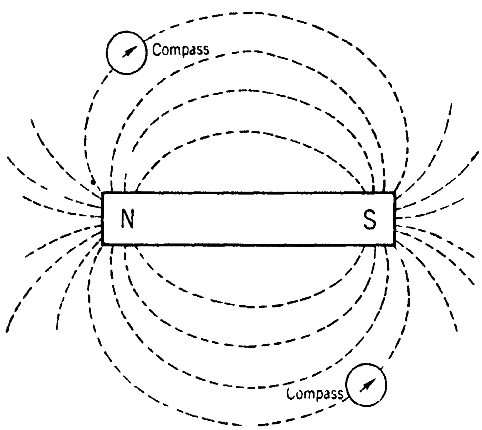


FIG. 318. MAGNETIC FIELD SURROUNDING A PERMANENT BAR MAGNET

Magnetic field

The bar magnet has a magnetic field of stress which exists along the lines shown in Fig. 318. A tiny compass needle may be used to indicate the direction of this magnetic field by the way its north pole is urged. The paths of the field form a closed circuit leading through the bar magnet and out of the north pole through the air and reentering at the south pole. This path constitutes a magnetic circuit and is analogous to the electric circuit previously discussed. Here a magnetic potential or pressure arises in the bar, a magnetic flux flows around the circuit meeting a magnetic resistance or *reluctance* throughout the path. Calculations for this circuit are somewhat complicated and will not be considered here.

Permanent magnets

In order for the bar magnet to retain its magnetism it must be constructed of a hard form of steel. In recent years many alloys of metals have been discovered which give superior magnetic properties — greater permanence, and extremely strong magnets.

For many purposes the usefulness of the permanent magnet is increased by bending it into the form of a horseshoe. This brings both poles close together and reduces the reluctance of the path through air. The electrical engineer uses permanent magnets in

many places. Some of the more common applications are for telephone receivers, loud-speakers, telephone bells, electric meters,

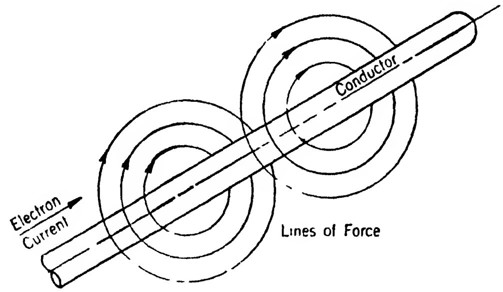


Fig. 319. Magnetic field surrounding a conductor carrying an electric current

Visualize circles as perpendicular to conductor

and for the fields of magnetos in telephones, gas engines, and speed-indicating devices.

The unit of magnetic flux or flow is the line of force. This unit is called the *maxwell*. Flux density is the number of lines of force or maxwells per unit of area.

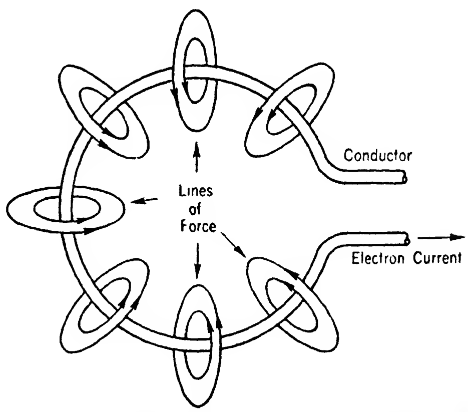


Fig. 320. Magnetic field surrounding a turn of wire carrying current

Strong field at center

An electric current passing through a straight conductor creates a magnetic field in the surrounding medium. The lines of force of this field exist as indicated in Fig. 319. If this straight wire is bent into a circle, the field produced by the conductor on the inside of the circle has a common direction, and hence it is greatly increased in strength. (See Fig. 320.) Winding a long conductor into a circular spool consisting of many turns will further increase the magnetic field along the center of this spool. Such a spool is called a *solenoid* and is illustrated in Fig. 321.

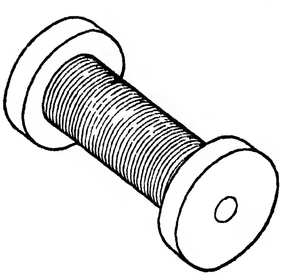


Fig. 321. Solenoid spool

Solenoids

The magnetism created by the solenoid can be further increased (perhaps 1000 times) by inserting a soft iron core within it.

This iron offers a much lower reluctance than air to the passage of magnetic flux and thus increases the magnetism. The combination of a solenoid on an iron core constitutes an electromagnet and is one of the most important basic parts of electric devices which utilize magnetism. The usefulness of this device arises from (1) the fact that the strength of the magnetism or *pull* of the electromagnet can be controlled by varying the current in the solenoid, and (2) the fact that the magnetism can be reduced quickly to zero or near zero by stopping the current.

Common uses of electromagnets

One of the earliest applications of the electromagnet was in the telegraph sounder, shown in Fig. 322. Here a horseshoe type of electromagnet pulls down an iron armature and gives audible

Fig. 322. Telegraph key

Fig. 323. Doorbell

sounds for the dots and dashes of the Morse code. Every schoolboy is familiar with the electric bell or buzzer like Fig. 323 which announces the guest at his home. In this device the electromagnet pulls the striking gong against the force of a spring and in doing so breaks the circuit which initiated the pull. The deenergizing of the electromagnet permits the spring to pull the armature back and reestablish a closed circuit so that the operation may be repeated. Electromagnets form a vital part of all types of electric power and communication equipment.

PRODUCING VOLTAGE

Voltage by chemical action

A difference of electric potential (voltage) can be produced (1) by *chemical action*, (2) with *heat*, (3) through *light*, or (4) *mechanically* through magnetic influence. The chemical battery was the earliest source of voltage and power. Today the dry cell is used for operating flashlights and portable radios, whereas the storage battery is an essential part of every automobile and truck. In both of these batteries chemical action takes electrons from the positive electrode and concentrates them on the negative electrode. The

resulting concentrations give rise to the difference of potential at the battery terminals.

Thermal electricity

When two pieces of dissimilar metal such as iron and copper are joined together at one end with the other ends left free, a difference of potential will exist at the free ends if they are at a different temperature than the united ends. This unit is called a *thermocouple* and its use in conjunction with a voltmeter is one method used for measuring temperatures.

Voltage from light

Light falling upon a combination of certain dissimilar materials will produce a difference of potential between them. Such a combination may consist of a thin layer of gray selenium upon a base of iron. When light falls upon the selenium side of this unit it penetrates to the barrier layer and gives up its energy in such a way as to remove electrons from the selenium and carry them into the iron. This action causes the iron to assume a negative potential and the selenium becomes positive. A sensitive meter connected to the two sides, as shown in Fig. 324, will give a deflection which is proportional to the voltage and to the intensity of the light falling on the selenium. Many exposure meters used by photographers operate on this principle.

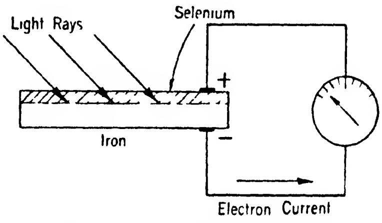


FIG 324. LIGHT PRODUCING AN ELECTRIC POTENTIAL

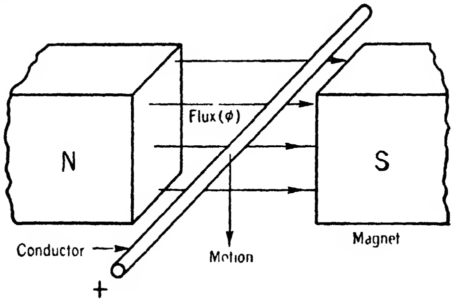


FIG. 325. VOLTAGE INDUCED BY CUTTING FLUX

Voltage by electromagnetic action

The common method of producing a voltage for electric power utilizes the *electromagnetic principle*. When a conductor is moved across a magnetic field, a difference of electric potential appears between the ends of the conductor. This action is illustrated in Fig. 325. This phenomenon of a production of a difference of poten-

tial may be explained by the electron theory. The conductor is made up of protons and electrons and when it moves it carries the electrons with it. These moving electrons constitute an electric current. It will be explained in an early paragraph how a conductor bearing an electric current is pushed aside when placed in a magnetic field. Thus the electrons in the moving conductor are pushed aside and urged in one direction along that conductor. This transfer of electrons along the conductor produces a difference of concentration of electrons (difference of potential) between its ends.

Voltage

In Fig. 325, the conductor may remain at rest and the field move, or both may move, because as long as a relative motion exists between them a difference of potential or a voltage will be induced. One volt is produced when the conductor cuts the magnetic field at the rate of 100 million (10^8) lines of force or of magnetism per second.

POWER GENERATION

Generators

In order to generate large voltages, it is customary to connect many moving conductors in series; the machine for doing this is called a *generator*. Two conductors may be connected in series and revolved between the two poles of a magnet as shown in Fig. 326.

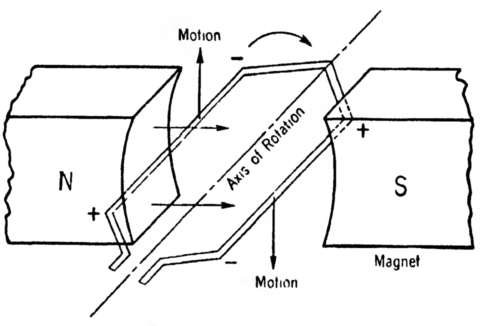


FIG. 326. VOLTAGE INDUCED IN A ROTATING TURN OF WIRE

Here the conductors cut the magnetic-field flux in opposite directions which causes *reversed polarities* of the voltages induced per conductor. This reversed polarity permits the conductors to be connected in series, plus to minus, so that the induced voltages add.

Increasing voltage

The *induced voltage* may be increased manyfold, first, by increasing the number of turns of the single coil of Fig. 326, second, by

using many coils placed around the periphery of a circular cylinder of iron and, third, by connecting all or part of these coils in series. The practical unit embodying these features is called an *armature* and is illustrated in Fig. 327.

Armature

In the armature the coils consist of several turns of wire and several similar coils are distributed throughout the circle of rotation. These coils are placed in slots in a laminated iron cylinder, and the ends of the coils are connected to copper commutator bars.

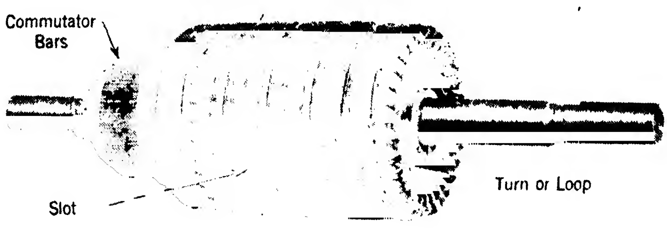


FIG 327. AN ARMATURE, THE REVOLVING PART OF A MOTOR

The generated voltage and the resulting current is collected from the commutator bars or terminals of the coils by stationary carbon brushes bearing on these bars. The rotation of the armature serves to reverse the connection from coil to brush at the proper time to cause the external voltage to be of the same sign (polarity) and to cause the current flow to be unidirectional (direct current).

Alternating voltage

It should be noted that if the connection to coils on the armature is not reversed by commutator action, the resulting voltage will change its polarity and the output voltage will be alternating. Thus, inherently, the voltage induced in an electric generator is alternating.

Wind-driven generator

The theory of electric-generator action may be applied to a practical case as follows. Assume that you desire a small *direct-current generator* to be driven by a wind propeller to charge a 12-volt storage-battery plant. For a small plant a current output of about 20 amp. and a potential of 15 volts is desirable. Experiment has shown that a propeller approximately 7 ft. in diameter will develop sufficient power for wind velocities of 12 miles per hour and higher. This propeller will run at about 300 r.p.m. Since this is a low speed for a generator, we will use a higher speed, say 900 r.p.m., and attain this speed by a "step-up" V-belt drive.

Checking an automobile generator

To charge the 12-volt batteries we will make a calculation based on the 12-volt generator used on the old 4-cylinder Dodge automobile. Figure 328 gives the dimensions of the generator suggested. From this figure we see that the area under each field pole equals $2.7 \times 3.7 = 10$ sq. in. A reasonable flux density in the air gap between the pole face and the armature is 25,000 lines per square

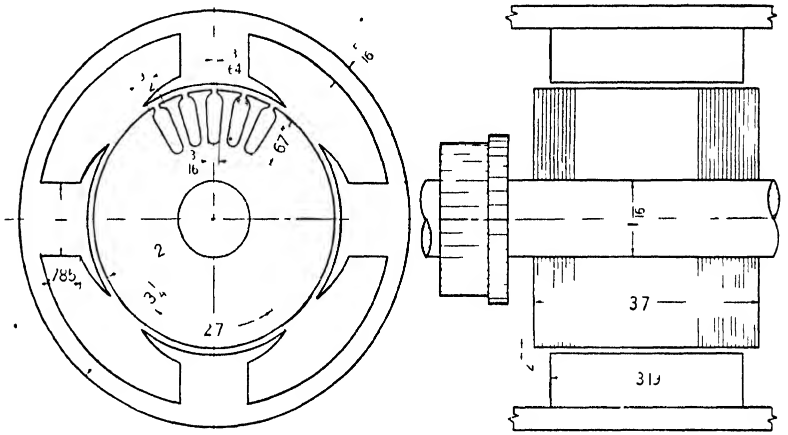


FIG. 328 DIMENSIONS OF A SMALL DIRECT-CURRENT GENERATOR

Flux cut

inch. This flux density would give $10 \times 25{,}000 = 250{,}000$ lines or maxwells for the entire face of one magnetic pole. For the four poles on this machine there would be a total flux of $4 \times 250{,}000 = 1{,}000{,}000$ maxwells crossing the air gap and being cut by the conductors on the armature during each revolution.

Conductors

The armature on the Dodge generator has 25 slots and eight conductors were placed in each slot. A conductor may consist of a single wire or two or more wires in parallel. In any case the conductor acts as one wire so far as cutting flux and generating a voltage are concerned. With 8 conductors per slot and 25 slots, there are $8 \times 25 = 200$ conductors on the armature. The voltage output of a generator depends on the way the conductors are connected together at the commutator. For all small machines and many others the wave or series connection, called the *winding*, is used. For this connection there are only two paths in the armature regardless of the number of poles. Thus for a wave winding on the machine under consideration there will be $\frac{200}{2}$ or 100 conductors per path for generating voltage.

A speed of 900 r.p.m. corresponds to $\frac{900}{60}$ or 15 revolutions per

second. Since a single conductor cuts 1,000,000 lines in one revolution, it will cut 15 × 1,000,000 or 15,000,000 lines per second. One hundred conductors in series will cut 100 × 15,000,000 = 1,500,000,000 lines per second. One volt is produced when 10^8 maxwells are cut per second at a uniform rate. Hence the generated voltage of our direct-current generator will be 1,500,000,000 ÷ 10^8 = 15 volts.

Voltage calculation

Windings

The armature of this machine should be wound with 25 coils of four turns or loops each and connected to 25 commutator bars. Each turn consists of 2 conductors. The sides of each coil are placed in slots in the armature separated by the distance between the centers of the field poles. Each conductor will consist of two No. 16 (American Wire Gauge) wires in parallel. Each No. 16 wire will carry 5 amp. continuously without overheating. Two wires in parallel will carry 2 × 5 = 10 amp. per path. Since there are two paths in the winding in parallel, the total current output will equal 2 × 10 or 20 amp., thus fulfilling the specifications for our direct-current, wind-driven generator.

Problems

1. A conductor cuts the flux under a field pole of a dynamo in $\frac{1}{100}$ sec. If the flux coming from the pole is 1,000,000 maxwells, what average voltage is induced between the ends of the conductor? *Ans.* 1 volt.

2. A coil of wire has 10 turns and revolves between two poles in a magnetic field of 1,000,000 maxwells ten times per second. What is the average voltage induced in the coil? *Ans.* 4 volts.

3. A 10-kw. d-c generator has a speed of 1800 r.p.m. If each path of the winding on the armature has 100 conductors in series and the flux per pole is 1,050,000 maxwells, what is the generated voltage? There are 4 poles. *Ans.* 126 volts.

ELECTRIC MOTORS

Force in magnetic field

A conductor carrying an electric current is surrounded by a *magnetic field* as shown in Fig. 319. The magnetic field passes around the conductor in circular paths. If this conductor is placed in the magnetic field existing between two poles of a permanent magnet or an electromagnet, there will be an interaction between the magnetic field surrounding the conductor and the second magnetic field. This interaction will be such as to *push* the conductor at a right angle to the direction of the main field, thereby developing a *force* as illustrated in Fig. 329.

You will note that the field surrounding the conductor is in the same direction as the main field above the conductor and in oppo-

site direction beneath the conductor. The result of these directions of field is to increase the strength of the field above the conductor and to reduce its strength beneath. It is convenient for us to think of the resulting field above the conductor acting like stretched elastic bands displaced upward, which force the conductor downward. This *force* acting upon a current-bearing conductor placed in a magnetic field is the fundamental principle of all electric motor action. This statement applies to both direct-current and alternating-current motors.

Why a force

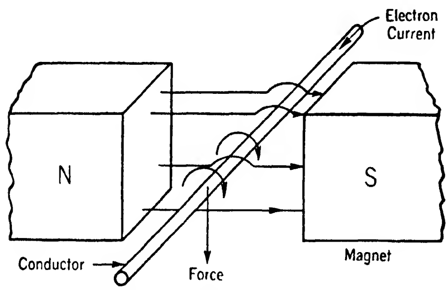


FIG. 329. FORCE ACTING ON A CONDUCTOR CARRYING CURRENT IN A MAGNETIC FIELD

It generally happens that the conductors are partially or completely surrounded by iron so as to increase the field produced by the conductor. Also, the practical motor may hold the conductor stationary and permit the main magnetic field to move. The force or the power developed is the same either way.

Force developed

The force acting upon the current-bearing conductor in a magnetic field is proportional to the strength of the main field, the length of the conductor, and the magnitude of the current in the conductor. This may be expressed by the equation

$$F = \frac{BIL}{10} \tag{12}$$

where F is the *force* in dynes, B is the *flux density* in maxwells per square centimeter, L is the *length* in centimeters, and I is the *current* in amperes. The ordinary concept of a force as a *push* or *pull* is sufficient for an understanding of motor action. However, for a scientific concept of force and rotation you will read a later section of this book on mechanics which covers the effect of forces. An illustration of the effect of a common force (an uppercut to the chin) is shown in Fig. 373, page 471. Since we are familiar with weights and forces in pounds, it is well to remember that 444,823 dynes equal one pound of force.

Dynes per lb.

The length of conductor L may be effectively increased by increasing the number of conductors. Thus to produce an elementary motor, two conductors may be arranged into a loop or coil and

mounted so as to revolve in a magnetic field as shown in Fig. 330. The right conductor is urged downward and the left upward. The *turning forces* will continue as the loop revolves in a clockwise direction until the plane of the loop is vertical. At this point the

Motor action

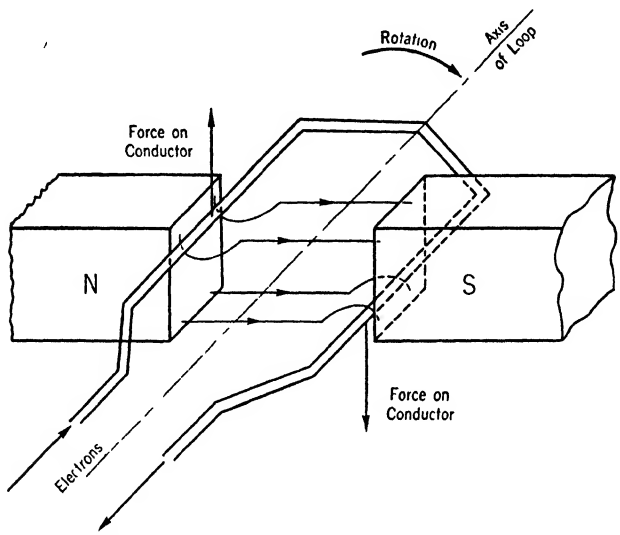


FIG. 330. FORCES ACTING ON A LOOP OF WIRE CARRYING CURRENT IN A MAGNETIC FIELD

motion will cease unless the direction of the current through the loop is reversed.

If the current is reversed and the momentum of the moving coil carries it past the vertical position, the forces now acting upon the coil will continue the clockwise rotation until the plane of the coil reaches the next vertical position, where it will be necessary again for a reversal of current to occur.

Torque

Each force acts at a constant radius from the center of rotation. The product of the force F and the radius R is called the torque, T. The total torque produced by a motor is the sum of the individual torques produced by the current flowing through each conductor or coil.

Motor construction

In practical motors the loop of wire just considered is expanded into a coil of many turns of wire, and several other similar coils are distributed throughout the circle of rotation. The resulting construction is the armature illustrated in Fig. 327. In the motor armature, the current is led to the coils and conductors via stationary brushes bearing on the commutator bars. The rotation of

the armature serves to change the direction of current through the active conductors at the proper time so that the turning effort is always in the same direction.

Motor fields

A motor or generator could be constructed with permanent magnets for producing the magnetic poles. Some small machines are constructed in this manner. However, the more common practice is to use electromagnets consisting of soft iron cores with solenoid coils (Fig. 321). The electromagnet can produce a much stronger field (higher flux density) and it is more economical in the use of material for large machines having four or more poles. The current for magnetizing the field structure is generated by the generator itself or taken from the supply line in the case of the motor.

Automobile starting motor

The active parts of an automobile starting motor are shown in Fig. 331. The motor operates for a few seconds at a time but dur-

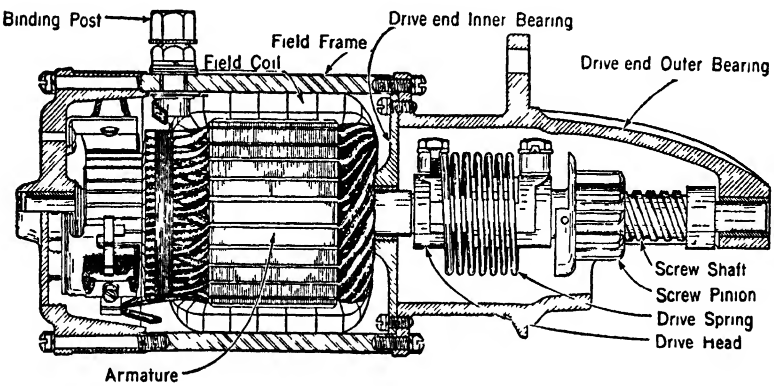


FIG. 331. CROSS SECTION OF AN AUTOMOBILE STARTING MOTOR

ing that time it must produce a *large starting torque* to overcome the inertia and friction of the engine during cold weather when the oil in the crankcase is very "stiff." The voltage to drive this motor may be as low as 4 volts at the motor terminals; hence the current must be large and of the order of 150 to 300 amp. To carry this large current the conductors on the armature are large rectangular copper bars (of low resistance) and only one conductor or bar is placed in an armature slot.

Series winding

In order to obtain a strong magnetic field, the designer connects the solenoids or coils on the field magnets in series with the armature circuit. The complete circuit for the starting motor

operation is shown in Fig. 332. The auto-starting motor is called a *series motor* because the armature and the field coils are connected in series.

Starting torque

The series type of motor develops a large torque on starting because the same large current passes through the armature conductors and the field coils. This large current gives a strong magnetic field acting upon conductors carrying large currents and results in a strong force and torque (torque is force times radius).

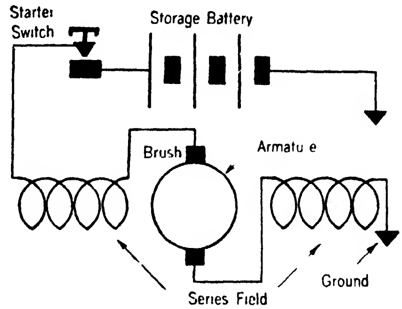


Fig. 332. Circuit for an automobile starting motor

Series motors are used wherever heavy torque must be developed for starting an operation. This requirement applies to electric railways, buses, elevators, hoists, and most fans. A modern electric railway motor of the series type is illustrated in Fig. 333.

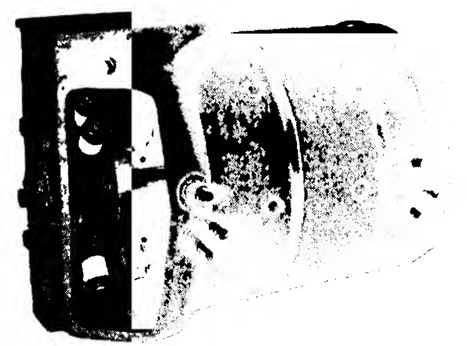


Fig. 333. Railway motor of series type produces large starting torque

Many other types and connections are used for the millions of electric motors in use today, yet all of them apply the *basic principle of the force acting upon a current-bearing conductor placed in a magnetic field.*

Problems

1. A rectangular copper conductor on the armature of an automobile starting motor carries 100 amp. of current. If the conductor is 10 cm. long and the flux density under the field pole is 10,000 maxwells per square centimeter, what is the force on each conductor in dynes? *Ans.* 1,000,000 dynes or 2.25 lb.

2. A force of 1,000,000 dynes acting on a radius of 3.5 cm. will produce what torque in foot-pounds? *Ans.* 0.258 ft-lb.

Hint. Torque is force times radius and there are 444,823 dynes per pound.

3. If there are 14 (out of 20 total) copper conductors within the field flux and the armature makes 850 r.p.m., what horsepower is being developed by the starting motor of problems 1 and 2? *Ans.* 0.585 hp.

Hint. One horsepower equals 33,000 ft-lb. per min. Foot-pounds developed is the force in pounds times the distance traveled by any conductor.

4. If the voltage at the starting motor terminals is 5.0 and the current is 200 amp. (2 × 100 amp. per path), what is the motor's efficiency under the conditions of the problems above? *Ans.* 43.6%

Hint. Efficiency equals output divided by input.

ALTERNATING CURRENT

Alternating voltage

The circuits and electrical equipment that we have considered in this section have been adapted to direct current which flows in one direction. However, the voltage produced by mechanical rotation is fundamentally of a changing or *alternating character.* By referring to the loop of Fig. 326, you will remember that each time the

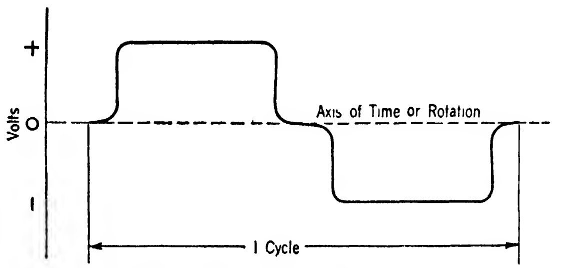


Fig. 334. Rectangular a-c voltage wave

plane of the coil passes the vertical position it cuts the magnetic field in an opposite direction, and hence the polarity of the voltage is reversed. If the generated voltage is brought out from the machine without a reversal of connections, an alternating voltage results.

Voltage wave

If the magnetic flux is distributed uniformly underneath the pole faces, the generated voltage would present a rectangular wave

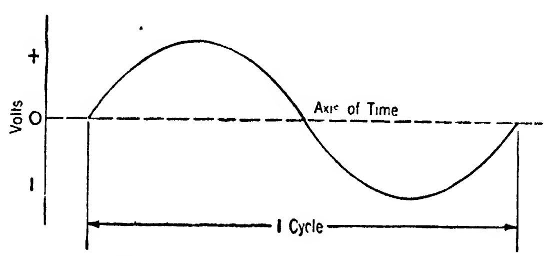


Fig. 335. Sine wave

as shown in Fig. 334. Here the height of the curve indicates the magnitude of the voltage and the distance from zero along the horizontal axis represents the degrees of rotation or the time since the armature rotation started. The curve represents the voltage change while the coil or loop turns through a complete circle and

each conductor cuts past two field poles. This period of one revolution is called *one cycle.*

Sine wave

In general, we know that the rectangular voltage wave is objectionable and a smoother form known as a "sine wave" is desired. In the sine wave the magnitude of the voltage or current varies with time as the trigonometric sine of an angle changes through a complete revolution from zero to 360 degrees. This wave form, illustrated in Fig. 335, can be attained either by the proper distribution of the field flux or by the proper distribution of the conductors in which the voltage is generated.

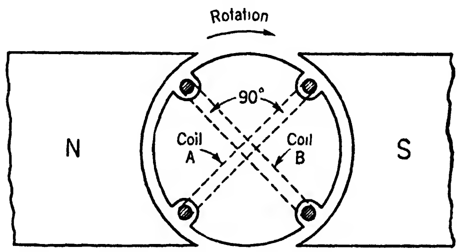


FIG. 336. SIMPLE TWO-PHASE GENERATOR

Phase shift

Each of two like coils placed at right angles to each other and rotated between the poles of the magnetic field of Fig. 336 will generate the same voltage wave. But their voltage waves will be out of step or out of time because the coils do not pass the same part of the magnetic field at the same time. A picture of these voltage waves is shown in Fig. 337 and the displacement of the

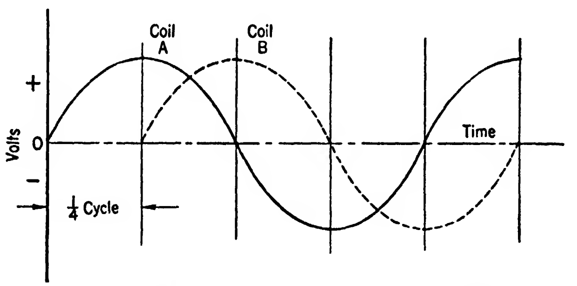


FIG. 337. TWO A-C VOLTAGES 90° OUT OF PHASE

waves is known as "difference in phase" or *phase shift.* For the case assumed, the phase shift will be 90 degrees since the coils are at right angles to each other.

Alternating current

An alternating voltage applied to a resistance will produce an alternating current and the resulting current will be in step or in phase with the voltage as shown in Fig. 338. If the same voltage is applied to a coil of wire wound on an iron core (*choke coil*), the resulting current will lag behind the impressed voltage. (See Fig. 339.) This lag of current is due to a property of choke coils

which opposes any change of current. By change we mean either increase or decrease of current. This opposition to current change acts like the inertia or momentum of a moving body.

Condenser capacity

A condenser is a sandwich consisting of two plates or *conductors* separated by an *insulator*. When a voltage is applied to a condenser a charge is built up on the condenser. In producing a charge on the condenser, electrons must be withdrawn from one plate and concentrated on the other. Strangely enough, if an alternating voltage is applied to a condenser, an alternating current will flow in the circuit though no electrons can pass through the insulator of the condenser. This action is a result of the electric storage *capacity* of the condenser.

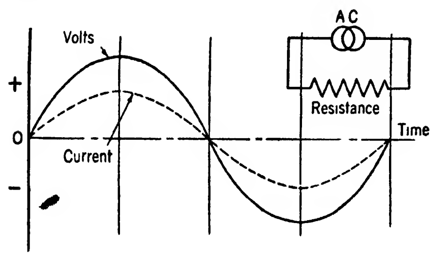


FIG. 338. A-C VOLTAGE AND CURRENT IN PHASE

Thus when the impressed voltage assumes one polarity, one plate of the condenser absorbs a large number of electrons while the other plate gives up an equal number. Then when the polarity of the impressed voltage reverses, the crowded plate gives up all

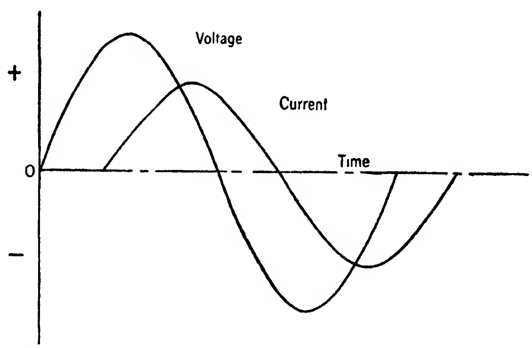


FIG. 339. A-C VOLTAGE AND CURRENT OUT OF PHASE

of its excess electrons plus an equal number more, so that the other plate now assumes or acquires negative charge. By this process a condenser is said to *conduct* an alternating current. Its ability to store electricity and produce this current is its capacity.

Leading current

The current resulting from an alternating voltage applied to a condenser is out of step or phase with the voltage. The current

through the condenser leads the voltage by one-fourth of a cycle or by 90 degrees.

Effective values

You are likely to wonder what value should be assigned to a voltage or current which is constantly changing. Your first suggestion might be to take the average of all values as being the solution. However, engineers have long since agreed that the proper value is that of an equivalent direct current or voltage and is called the *effective value*. Since the heating value of a direct current in watts is the current squared times the resistance, the effective value depends upon the square of the instantaneous values. Thus *the effective value is the square root of the average of a large number of the squares of instantaneous values.* Effective values are indicated by a-c voltmeters and ammeters and all alternating voltages and currents should be interpreted as effective values unless otherwise specified.

Impedance

The shift in phase of the current where an alternating voltage is applied to a circuit containing chokes or condensers introduces complications in calculations of electric power and other factors of the circuit. Ohm's law must be modified to read

$$\text{(13)} \qquad \text{Amperes} = \frac{\text{volts}}{\text{impedance}} \quad \text{or} \quad I = \frac{E}{\tilde{\lambda}}.$$

Impedance ($\tilde{\lambda}$) is the effective sum of the resistance and another opposing factor called reactance. *Reactance* represents the effects of a choke or condenser or both in the circuit.

Power factor

In the direct-current circuit the product of volts times amperes is equal to the electric power in watts. But in the alternating-current circuit the product of effective volts and effective amperes is called *apparent power* and generally is greater than the true power in watts. This difference between direct current and alternating current is due to the phase shift between voltage and current in the alternating-current circuit. True power in watts is measured by an instrument called a wattmeter.

The ratio between true power in watts and the apparent power in *effective volt-amperes* is called *power factor* (p.f.). The ratio may be expressed thus:

$$\text{(14)} \qquad \text{Power factor} = \frac{\text{true power}}{\text{apparent power}} = \frac{\text{watts}}{\text{volts} \times \text{amperes}}.$$

The maximum value for power factor is 1, and is called unity power factor. Unity power factor exists when the alternating voltage and current are in phase. Unity power factor exists on

loads consisting of incandescent lamps and most electric heating devices. The power factor falls below unity as the alternating voltage and current get out of phase. Hence, for a constant alternating voltage and wattage, the current rises in value as the power factor decreases (equation 14). The increase of current reduces the power which may be obtained from the apparatus because the heating losses in generators, transmission lines and motors vary as the square of the current (I^2R). Low power factor can be corrected through the use of auxiliary devices but only with increase in the cost of electric service. Hence, power factor is a very important consideration in the loads placed on electric power and distribution lines.

Problems

1. A 100-watt, 110-volt incandescent lamp takes 0.91 amp. on a 110-volt alternating-current circuit. (a) What is the impedance Z of the lamp? (b) What is the power factor of the lamp? *Ans.* (a) 121 ohms; (b) 1.0.

2. A 100-watt, 110-volt fluorescent lamp and its circuit (without any condenser) takes a current of 1.3 amp. on a 110-volt, a-c circuit. (a) What is the impedance of the circuit? (b) What is the power factor of the circuit? *Ans.* (a) 84.7 ohms; (b) 0.7.

3. A 110-volt, $\frac{1}{4}$-hp., a-c motor on an electric washing machine takes 2.5 amp. and 60 watts when running without load. Under load the same motor takes 4.8 amp. and 375 watts. What is the impedance and the power factor: (a) without load; (b) under load? *Ans.* (a) 44 ohms, 0.218 p.f.; (b) 22.9 ohms, 0.71 p.f.

SPECIAL USES FOR A.C. AND D.C.

Importance of voltage transformation

Alternating current is almost universally used as the medium of generation and distribution of electric power. Its great value lies in the ease of transforming from one voltage to another by means of a simple stationary device called a *transformer*. To illustrate the importance of transforming voltage, let us assume that 5 kw. of power are to be transferred over a line half a mile long. If 1000 volts are used for transmission, the current will be 5000 watts/1000 volts = 5 amperes. For a line consisting of No. 10 copper wire, there will be a resistance of approximately 1 ohm per 1000 ft., or 5.28 ohms for one mile. Under these conditions the voltage drop in the line will be 5 × 5.28, or 26.4 volts, which is 2.64 per cent of the voltage of transmission (receiving end).

Next, suppose the same power is to be transmitted at 200 volts. Then the current will be 5000 watts/200 volts = 25 amperes, and the voltage drop would be 25 × 5.28, or 132 volts. Here 66 per cent of the voltage would be absorbed in the line.

To avoid such a large drop in line voltage we could use a wire of larger cross section, but that means a greater cost for copper. We could show that for the same power loss in the line *the weight of copper required varies inversely as the square of the voltage.* Hence, for transmitting large blocks of power, high voltages are used. For distribution about a town or city, a voltage from 2200 to 2400 is frequently used, while on the line from Boulder Dam to Los Angeles 287,000 volts has been chosen. Between these extremes a wide range of line voltages is used, determined by the length of line and the amount of power to be transmitted. *Saving copper*

Alternating current has two disadvantages. One disadvantage due to the possibility of low power factor has been suggested. The second disadvantage arises from the fact that *alternating-current motors do not have good starting characteristics and also that they usually have a fixed speed.* Hence, if we need a high starting torque or must be able to control the motor speed, direct-current motors are used and some means must be employed for transforming alternating current into direct current. For power purpose this transformation is made by use of a rotary convertor, a motor-generator set, or an electronic rectifier. *Objections to alternating-current motors*

The preceding discussion suggests some of the characteristics and problems involved in the use of alternating-current circuits and equipment. A more complete discussion is beyond the scope of this introduction to the problems of electrical engineering.

THE TRANSFORMER

The theory of action of a transformer requires a concept of flux linkages. You understand how one link in a chain connects or links with the next one. Likewise you are familiar with the way several keys may be linked or threaded on a key ring. In a similar way, lines of flux may link or thread through an electric circuit as shown in Fig. 340. These unions or interlinks are called *flux linkages.* In Fig. 340, a single turn of wire is shown linking with one line of force. If there were two lines of force and two turns of wire, there would be 2 × 2, or 4 flux linkages. Now you cannot pull a key through a key ring without breaking it, but you can pull lines of force out of their link with an electric circuit. *Flux linkages*

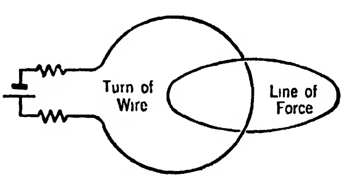


FIG. 340. FLUX LINKAGE

Induced voltage It was shown in an earlier paragraph that a voltage is induced in a conductor that moves at right angles to a magnetic field. A similar induction of voltage exists when there is a change in the lines of flux linking with or threading through an electric circuit.

If one pole of a permanent magnet be moved into a coil of wire as shown in Fig. 341, there will be a deflection of the voltmeter indicating that a voltage is being induced. Here the voltage is

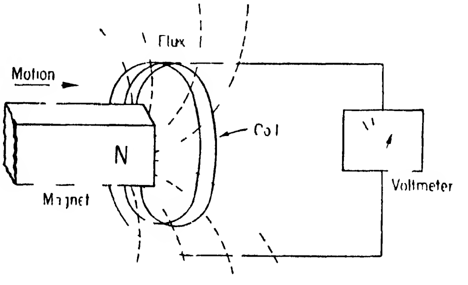


FIG. 341 VOLTAGE PRODUCED BY FLUX CUTTING A COIL OF WIRE

created by a change of the flux linking with the electric circuit. A uniform change of flux linkages at the rate of 10^8 per second will induce 1 volt in the electric circuit. This same factor termed "cutting magnetic flux at the uniform rate of 10^8 maxwells per second" was used when we were discussing the electric generator.

Voltage ratios The transformer is a simple arrangement of two or more electric circuits *interlinked with a magnetic circuit.* The electric and magnetic

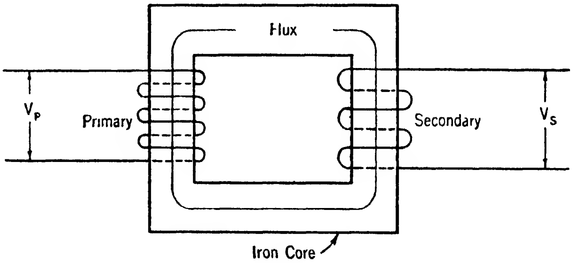


FIG. 342. ELECTRIC AND MAGNETIC CIRCUITS OF A TRANSFORMER

circuits of a simple transformer are illustrated in Fig. 342. An alternating voltage applied to the primary winding on the left will induce an alternating magnetic flux to circulate through the iron core. This alternating (changing) flux will link both the primary and secondary windings and induce an alternating voltage in each one.

The magnitude of the voltage induced at any instant will be equal to the rate of change in flux linkages divided by 10^8. If the same flux circulates through both windings, the voltage induced in each winding will be proportional to the number of turns on each coil. Hence the voltage ratio between the primary and secondary windings is approximately equal to the ratio of the number of turns on each.

Transformer calculation

Let us assume that we wish to transform 2300 volts to 115 for distribution in a city. If the magnetic iron core of the transformer has a cross-section of 9 sq. in. and we use a flux density of 70,000 lines per sq. in., the flux will rise to a maximum of $9 \times 70{,}000$, or 630,000 maxwells. When the current reverses, the flux will reverse its direction to a value of 630,000, thus making a change of $2 \times 630{,}000$, or 1,260,000 for one-half a cycle. For a full cycle the change is $4 \times 630{,}000 = 2{,}520{,}000$ maxwells. With a frequency of 60 cycles per second the total flux change will be $60 \times 2{,}520{,}000 = 151{,}200{,}000$. One turn of wire cut or linked by this flux change would have induced in it an average of $151{,}200{,}000 \div 10^8 = 1.51$ volts.

Since the same voltage will be induced in each turn, the low-voltage winding will have $115/1.51 = 76$ turns of wire, and the high-voltage winding $2300/1.51 = 1522$ turns.

Form factor

The answer attained in the calculation above is only approximate since it applies to average values of voltage rather than to effective values. For a sine wave of voltage and flux, the number of turns should be divided by the ratio of "effective value to average value" which is 1.11 and is called the form factor. Applying the form factor correction gives us 68.5 turns for the low-voltage winding and 1372 turns for the high-voltage winding.

Transformer action

You may be interested in the reactions taking place inside a transformer. An alternating voltage impressed across the primary winding of the transformer in Fig. 342 causes an alternating current to flow in this winding. This current in turn sets up an *alternating flux* through the iron core. This alternating flux induces an alternating voltage in both the primary and secondary windings. In the case of the primary winding this induced voltage is directly opposed to the impressed voltage and acts to limit the current in the primary circuit. If the secondary circuit is open (that is, without load), the induced voltage in the primary will be nearly equal and also opposite in direction to the applied voltage so that very little current

flows in the primary. This primary current is just sufficient to maintain the flux in the iron core and supply the losses. This current is limited by the resistance and the reactance of the primary winding.

Control of current needs

When a load (resistance) is placed on the secondary coil, a current flows having a magnitude determined by the load and the voltage across the secondary coil. The action of this load current upon the primary side is of special interest. The current flowing in the secondary side will also tend to produce a magnetic flux in the iron core, this time in a direction opposite to the flux produced by the primary coil. The effect of this opposing action will be some reduction in the *resulting flux*.

This reduction in flux lowers the induced voltage in both the primary and secondary windings. On the secondary side it lowers the load voltage slightly, while on the primary side it permits more current to flow until the power input in the primary rises to compensate for the secondary load plus the losses in the transformer. Thus the change in magnetic flux causes the transformer to act like a *valve* to permit electric energy to flow through it.

Transformer efficiency

The efficiency of a well-designed transformer is inherently high because the *losses are small*. The principal losses are heat losses due to the current through windings and certain losses in the iron which appear as heat. Naturally there are no friction or air-resistance losses such as occur in rotating machines. Efficiencies of power transformers vary from 90 to 99 per cent.

Everyday use of transformers

The transformer is one of the most common and useful electrical devices known. It has made possible the transmission and distribution of alternating currents at relatively high voltages (up to 287,000) and their use at low safe voltages in the office, factory, store, and home.

Problems

1. The flux in the iron core of a transformer changes from 0 to 1,000,000 in $\frac{1}{240}$ sec. (60 cycles). Assuming that the rate of change is constant, what average voltage will be induced in each turn of wire on transformer coils? *Ans.* 2.4 volts.

2. If 1000 turns of wire are placed in series on one coil (primary) of the core in problem 1 what voltage should be applied? *Ans.* 2400 volts.

3. How many turns should be placed on a secondary coil to deliver 120 volts to a residence? *Ans.* 50 turns.

4. If the input to the primary of a transformer is 5210 watts and the delivered output is 5000 watts, what is the efficiency of the device (efficiency is watts output divided by watts input)? *Ans.* 0.96, or 96%.

ELECTRONICS

In 1883, Thomas Edison discovered that the region surrounding a hot body is a conductor of electricity. Edison placed a metallic plate in a carbon lamp bulb and connected it in series with a sensitive meter, as shown in Fig. 343. He observed that when the plate was connected to the positive side of the lamp filament a current was indicated, but when connected to the negative side there was no deflection of the sensitive meter. This phenomenon came to be called the *Edison effect.* Edison did not attempt to explain this discovery, but other scientists studied the phenomenon a few years later and developed theories and made further discoveries which have culminated in the great modern achievements in the field of electronics. *Edison effect*

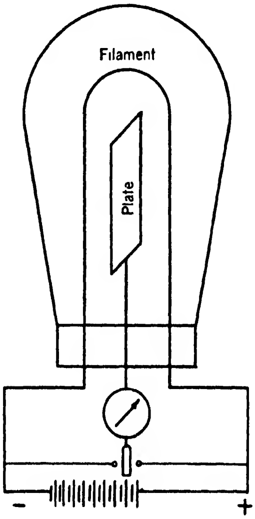


Fig. 343. Edison effect

Electrons are held to the substance of which they are a part by a force. This force or affinity is so large that few if any electrons ever escape under ordinary conditions. The work which must be done in removing an electron is called the *work function* of a substance. It is also known as the electron affinity of a substance. *Electron affinity*

Electron affinity may be overcome and electrons removed from solids in five different ways. These ways are termed: (1) high field emission, (2) secondary emission, (3) photoelectric emission, (4) radioactive disintegration, and (5) thermionic emission. When a metal is cold or at ordinary room temperature it is very difficult to overcome electron affinity. If an extremely high electric field or voltage is produced around a cold metal, it is possible to pull electrons out. The field required is of the order of 1,000,000 volts per centimeter. This extremely high field is difficult to produce with the result that *high field emission* is limited to rare experimental use. If an electron is attracted through space to a plate by a voltage, it usually acquires a velocity sufficient to splash out electrons when it strikes the plate. This splashing-out process is called *secondary emission.* *Electrons from solids*

Electron affinity varies with different substances and is low for

some elements like sodium, barium, cesium, and strontium. Under suitable conditions the energy in light may be sufficient to remove a few electrons from cold substances. This form of removal is called *photoelectric emission* and is the basis of action of photoelectric tubes. Radioactive materials such as radium go through a slow process of disintegration during which the highly complex compound or element breaks down into simpler atomic structures. In the course of this process of *radioactive disintegration*, electrons called *beta rays* are ejected into space.

Brownian movement

In 1827, Robert Brown discovered that minute particles of dead matter in suspension in liquids can be seen in a high-powered microscope to perform irregular wiggling motions suggesting "life." This phenomenon came to be known as the *Brownian movement* and was explained fifty years later. These movements are due to the continued bombardment of the inanimate particles by the thermal agitation of the molecules of the liquid. The Brownian movement is proof that constant thermal agitation of molecules exists in all gases, liquids, and solids when above the temperature of absolute zero. This thermal agitation increases with the rise in temperature and serves to explain many physical phenomena including *thermionic emission* of electrons.

Evaporation

The thermal agitation or molecular movement at the surface of liquids causes a bombardment of the surrounding medium. If the medium happens to be a gas, many of the molecules will "jump" out of the liquid into the gas. Some of these ejected molecules return to the liquid while others remain in the gas. As the temperature of the liquid is raised, the *thermal agitation* increases, the number of emitted molecules rises, and the distance which they jump becomes greater. Thus more of the molecules remain in the gas. This is the phenomenon of evaporation of liquids.

Metallic vapors

In solids, and especially in metals, the molecules are very close together and their thermal movement is greatly restricted. However, they do move about positions of equilibrium. As the temperature of the solid is raised, the thermal agitation of its molecules rises and as the substance approaches the melting point, some molecules do leave the metal never to return. This is illustrated by the gradual blackening of the bulb of an incandescent lamp throughout its life. This blackening of the inside of the bulb is due to a deposit of tungsten that has evaporated from the tungsten filament.

In a manner analogous to the preceding explanation, the electrons in a solid may be ejected from that substance due to thermal agitation. The electrons are closely bound to the atoms of which they are a part by electron affinity at ordinary temperatures. But with the rise in temperature the molecules will move more rapidly and through a greater range. The electrons in turn move faster in their orbital motion with the rise in temperature. And with this rise in temperature, a point will be found where a few electrons have a sufficient velocity to escape from the surface. This process may be thought of as a "boiling out" or evaporation of electrons.

Thermionic emission

At the instant of escape, the electron has a certain initial velocity. As the electron (negative charge) emerges from the surface it leaves behind its mate (positive charge) which *attracts the electron back toward the surface*. Then as the electron moves out into space, the force of attraction of the positive charge remaining in the hot solid overcomes its initial velocity and it is pulled back into the solid. Thus, unless some other forces enter into the picture, the electron never gets very far from the heated solid or cathode.

Recapture of electrons

ELECTRON-TUBE OPERATION

Let it be assumed that a filament or cylindrical conductor of tungsten is placed in a vacuum or in any space filled with an inert gas so that oxidization cannot take place. With a rise in the temperature of the filament (produced by an electric current), a point will be found where the thermal motion of the molecules and electrons becomes great enough so that a number of electrons are thrown out into space, as illustrated in part *a* of Fig. 344.

Filament emission

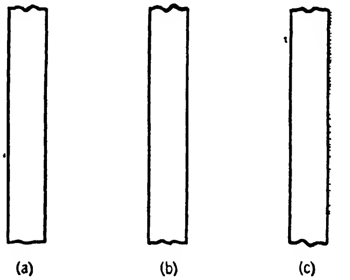


FIG. 344. ELECTRON EMISSION SURROUNDING A HOT FILAMENT

A further rise in temperature of the filament will be accompanied by an increase in the *kinetic energy* of the electrons and an increase in the number and the initial velocity of those emitted. This will result in an atmosphere of electrons around the filament as depicted in part *b* of Fig. 344. A still higher temperature of the filament will result in a greater agitation and velocity of electrons so that the cloud will thicken and move out to the condition of part *c*, Fig. 344.

Electron activity

It should be understood that Fig. 344 represents an isolated filament and the surrounding space magnified several times. Many electrons emitted from the filament never travel over 0.01 mm. before returning and very few go farther than 1.5 mm. from the filament. The change of emission with temperature is illustrated by Fig. 345. This shows that the rise in emission is cumulative and becomes very rapid as the melting point of a metal is approached. But, obviously, there is some safe temperature which must be observed in the operation of any emitter.

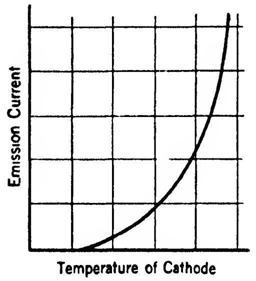


FIG. 345. VARIATION OF EMISSION TEMPERATURE

Cathodes

The electron-emitting element of the electron tube is called the *cathode*. The *hot cathode type* uses the principle of thermal emission described in the preceding discussion. Hot cathodes are constructed in a vast number of shapes, sizes, and materials. Two general types are illustrated schematically in Fig. 346. Parts *a* and *b* illustrate two forms of the directly heated cathodes. Type *a* will be used as a symbol for the cathode in all drawings. Part *c* shows the indirectly heated type where a cylinder or sleeve covered by some emitting material is heated by a tungsten resistance or heater on the inside of the sleeve. The latter form permits the use of alternating current for heating the cathode, yet serves to insulate the alternating current from the vital circuits of the electron tube and thus prevents any "hum" in the output.

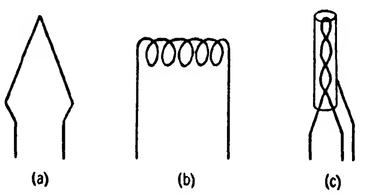


FIG 346. TYPES OF HOT CATHODES

Cathode materials

The three active materials used in the construction of hot cathode emitters are tungsten, thoriated tungsten, and oxide coating. Tungsten has a high melting point and is tough and durable so it withstands bombardment of ions at high voltages. It is used for large tubes where a voltage of 4000 and higher is employed. The operating temperature of tungsten cathodes is about 2400° K. (brilliant white heat). The thoriated tungsten consists of tungsten in which thorium oxide is embedded. When heated, the thorium oxide is reduced and the thorium atoms diffuse and come to the surface forming a skin *one atom thick*. Since thorium has a low

electron affinity, the electron emission is increased about 100 times that for pure tungsten. This material is used in medium-capacity tubes for voltages in the approximate range of 600 to 4000 and is operated at a temperature of about 2100° K. (yellow heat).

Coated cathode

Oxide-coated cathodes consist of tungsten or some other metal base carrying a coating of barium or strontium oxide. These cathodes operate under a temperature range of 900 to 1200° K. (dull red heat). This low temperature is adapted to the heater type of cathode and requires a low energy input for heating the cathode. The oxide-coated cathode is used for nearly all low-capacity and low-voltage tubes, such as those in receiving sets.

GAS-FILLED TUBES

Gas ions

Many electron tubes contain gas and part of the conduction in these tubes is by means of *ions* of gas serving as carriers. A positive gas ion is an atom which has lost one electron, and hence carries a positive charge. A negative ion can be an atom which has gained one electron and has a negative charge. A negative ion may also be an electron. In general, negative ions *are* electrons.

Gaseous conduction

Gaseous conduction is a very simple process. A single positive ion and a negative ion are shown in the tube of Fig. 347. When a difference of potential is applied to the two electrodes as shown,

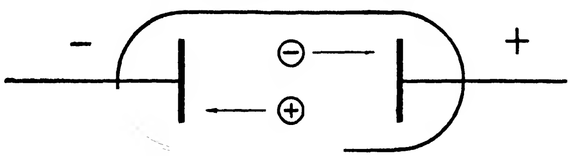


FIG. 347. GASEOUS CONDUCTION WITH IONS

a logical thing occurs. The positive ion is attracted to the negative electrode where it takes on an electron and becomes a neutral atom. The neutral atom then drifts back into space. In the meantime the negative ion (an electron) is attracted to the positive electrode where it gives itself up. The foregoing process has served to remove an electron from the negative plate and to deposit an electron on the positive plate. This is equivalent to a transfer of an electron from one plate to the other, and a movement of an electron is an electric current. Multiply this simple process millions of times per second and gaseous conduction is the result.

Gas ions may be produced in three ways. *Heat* applied to a gas may cause some of the atoms to separate into ions. *X-rays* passed

through a gas will cause the formation of ions. The more common process of producing ions where gaseous conduction takes place is by *collision*. In the process of gaseous conduction considered in the preceding paragraph, the positive ion and the electron in moving to the electrodes will travel at an accelerated pace. While en route to the electrodes either one is likely to collide with neutral molecules and atoms of gas which are present. (See Fig. 348.) If any

Ionization by collision

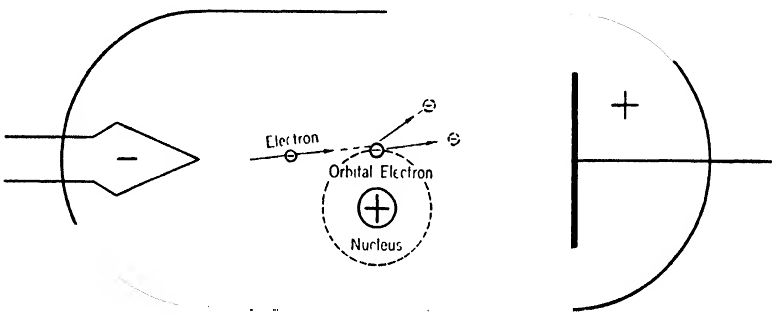


FIG. 348. IONIZATION BY COLLISION

of these collisions is sufficiently violent, an electron in the outer orbit of some atom will be knocked free from it. Thus a new positive and a new negative ion have been formed. Again, multiply these ionizing collisions millions of times and a condition exists where ionization by collision and gaseous conduction may be self-sustaining.

Light by ion bombardment

During the process of ionization by collision, many collisions will occur between ions and molecules wherein the velocities and energies involved are insufficient to overcome electron affinity and produce ionization. Many of these collisions may have sufficient force to remove an electron from its orbit around an atom and set it in an unstable condition. This process of temporary removal from the orbit gives energy to the electron. Shortly after this happening, this electron will give up its unstable condition and return to its original orbit or status. In order to make this return, the energy absorbed from the collision must be released.

The released energy takes the form of a wave motion and may appear as *light visible to the eye*. The color of the light depends on the gas involved. This principle of light production as a by-product of gas ionization is utilized in many electric-lighting units such as neon, mercury-vapor, and sodium lamps.

THE DIODE OR TWO-ELECTRODE TUBE

One-way conduction

The early experiment of Edison known as the "Edison effect" showed that electrons would pass from a hot filament or *cathode* to a cold plate whenever the plate was at a higher potential than the filament. This property of a device which permits current to flow in one direction only is called *unilateral conductivity.* The unilateral conductivity of a tube having two electrodes, one hot and the other cold, was utilized by Fleming for the detection of high-frequency radio waves. This device, known as the "Fleming valve," was patented by him in 1905. The Fleming valve was important in the early application of radio telegraphy and it was one of the important developments in the history of electronics.

Diodes

A diode is a two-electrode tube containing a cathode and a second electrode called an *anode.* The anode is always the electron-collecting electrode in a tube. In the first tubes built the anode was constructed in the form of a plate or sometimes two little plates connected in parallel and placed on each side of a filament type of cathode. We still use the term *plate* for the anode even though it is often of a cylindrical form as shown in Fig. 349. Anodes may be made of metal or graphite, the latter being common for rectifying tubes.

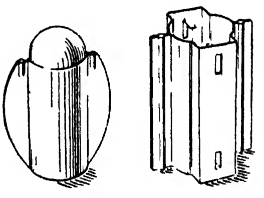


FIG. 349. ANODES

Rectification

The diode utilizes the principle of unilateral conductivity in its operation. In the high-vacuum tube of Fig. 350, the cathode is heated to its normal operating temperature so that a cloud of electrons is being emitted. The anode is maintained at a positive potential by the battery *B.* This positive potential attracts the electrons being emitted by the cathode, and a large part of them pass to the anode continuously and then return to the cathode via the battery *B.* If the connections to the battery *B* are reversed so that the negative terminal of the battery is connected to the anode of the diode, no action will take place in the tube. Now the anode is negatively

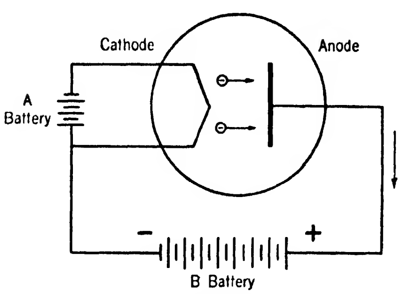


FIG. 350. UNILATERAL CONDUCTIVITY IN A VACUUM

Changing alternating current to direct current

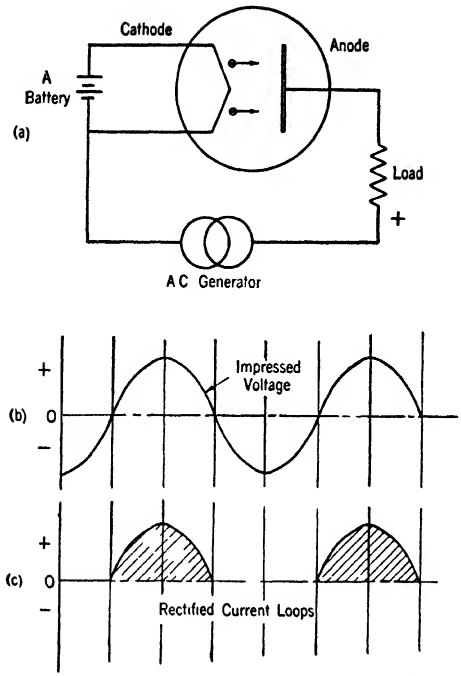


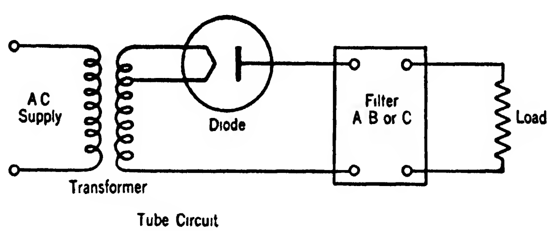
FIG 351 RECTIFICATION OF AN ALTERNATING CURRENT

charged and it will repel the negative electrons; hence the electron movement is blocked.

If we substitute an alternating current having a sine form of voltage wave for the battery as in Fig. 351, we obtain current flowing in only one direction. This is known as "rectification." Thus during the period when the anode of the diode is made positive, electron current flows in the tube, but when it is negative the tube blocks any electron movement. The resulting one-directional current loops are shown in the lower part of Fig. 351.

Diode for rectification

This *rectifying action* of a diode is a very valuable property. The unidirectional pulses of current from a rectifier are satisfactory for charging a storage battery. However, this pulsating current and



SIMPLE RECTIFIER CIRCUIT WITH FILTER

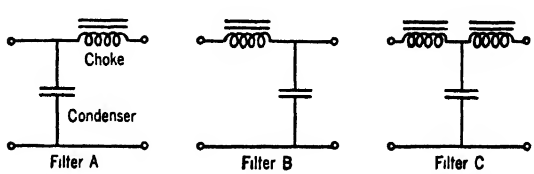


FIG. 352. THREE TYPES OF FILTERS

voltage is not suitable where a steady direct current is desired. The addition of a *filter* will smooth out these pulses.

Filters

Filters are made up of combinations of condensers and chokes. Some simple filters are shown in Fig. 352. The condenser serves to store and release a charge, while the choke opposes changes of current. The proper combination of condensers and chokes will give the desired degree of smoothness.

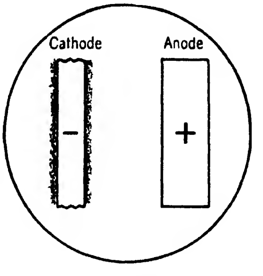


FIG. 353. NEGATIVE SPACE CHARGE

The number of electrons attracted to the anode depends upon a number of factors, one of which is the negative space charge. A consideration of the region between the cathode and anode (Fig. 353) shows a cloud of electrons around the cathode with a very high concentration close to the filament. This intense concentration of electrons (negative charges) produces a strong negative charge in the space surrounding the cathode. This *negative space charge repels the electrons* which are ejected from the cathode until they have broken through it; this charge also opposes the attraction of the positively charged anode. Thus the negative space charge is an important factor in controlling the number of electrons passing to the anode.

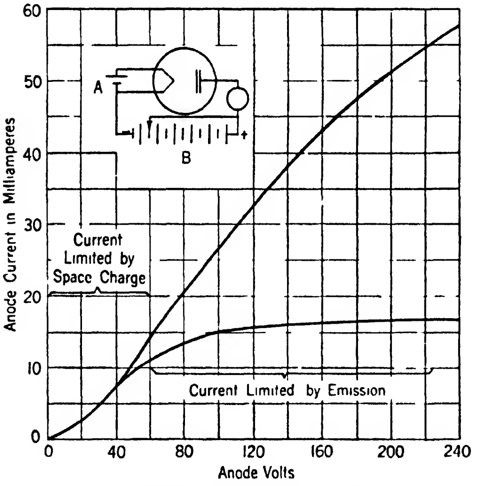


FIG. 354. CHARACTERISTIC OF A VACUUM DIODE

Chart of tube characteristics

The effect of the anode potential upon the electron current flowing to the anode of a vacuum tube is shown in Fig. 354. At low anode voltage the negative space charge limits the electron movement as just explained. Higher positive plate potentials will reach farther back into the electron cloud, thus overcoming the negative space charge. At still higher anode potentials nearly all of the electrons emitted may be drawn to the anode and the curve bends over. This curve

represents the important characteristic of the vacuum diode. Anode current is expressed in milliamperes ($\frac{1}{1000}$ amp.). The vacuum diode is used as a detector in radio receivers and as a rectifier for high-voltage applications.

Gaseous diodes

The addition of an inert gas under low pressure to a vacuum diode greatly increases the current flow in the device. This increase is due to two factors. First, the gas becomes ionized and the electron transfer due to *gaseous conduction* is added to the pure electron flow. Second, the presence of the positive gas ions serves to neutralize partially the negative space charge. This action greatly increases the number of *emitted electrons* which are attracted by the anode. Thus the addition of gas reduces the resistance of the tube and lowers the voltage drop from cathode to anode.

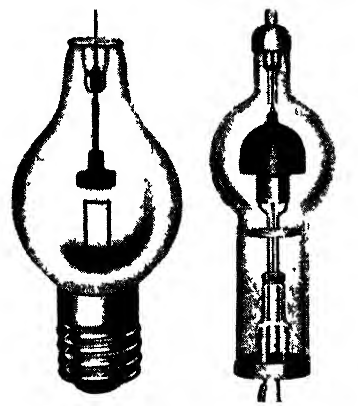


Fig. 355. Rectifier tubes used for charging storage batteries

Practical uses of gas tubes

Two gaseous rectifier diodes are shown in Fig. 355. The one on the left is filled with argon under 2 lb. of pressure. It is used for charging storage batteries and is made in current-carrying capacities varying from $\frac{1}{2}$ to 10 amp. The diode on the right of Fig. 355 contains a small quantity of mercury, and hence operates as a mercury-vapor rectifier. It is used for charging storage batteries and for small direct-current applications. It is built in ratings up to 30 amp.

THE TRIODE OR THREE-ELECTRODE TUBE

Grid added to tube

In 1907, DeForest added a third electrode to the Fleming valve (diode) and called the new device the *audion*. The third electrode, called the *grid*, was placed between the cathode and anode as illustrated in Fig. 356. The value of the grid lay in its ability to control the current between the cathode and the anode. This invention constituted one of the most important developments of the twentieth century.

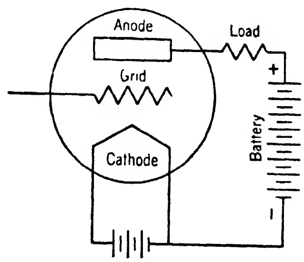


Fig. 356. Circuit of a triode

It has served to extend the field of communication by wire across continents, and it has made possible radio communication around the world. And now it is revolutionizing the uses of electric power in the industrial world.

Grid construction

The modern three-electrode tube, known as the "triode," uses cathodes and anodes like the diode. *The grid is usually a helically wound coil of fine wire interposed between the cathode and the anode.* The grid of a triode is illustrated in Fig. 357. The grid of the three-electrode vacuum tube functions by a change of the charge residing on the grid. This change in charge (and potential) serves to control the electron stream between the cathode and the anode. Since the grid is very close to the cathode, its changes in potential are more effective than like changes on the anode. When the grid is negative it repels those electrons which pass through the negative space charge, and when it is positive it helps electrons to get through the negative space charge and thus produces a large increase in electron flow.

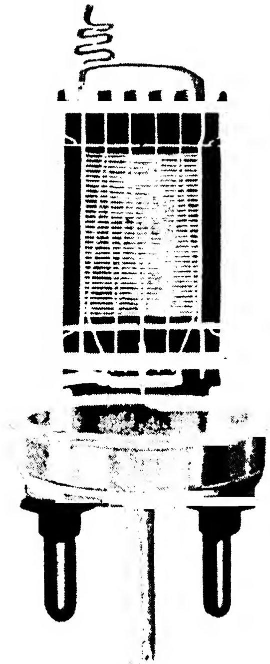


FIG. 357. TRIODE GRID

The triode requires 3 sources of potential difference for operation

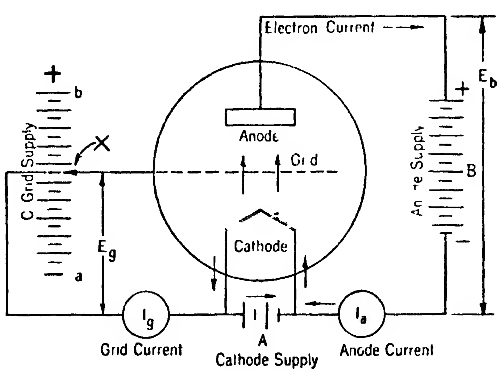


FIG. 358. CIRCUIT FOR VARYING VOLTAGE ON GRID OF A TRIODE

Grid action

The action of the grid is illustrated by the circuit and curve of Fig. 358 and Fig. 359. The potential supplied to the anode is called the *B supply*, that for the cathode the *A supply*, and that for the grid the *C supply*. As the point X is moved from a to b, the grid goes through a succession of values from strongly negative through zero to strongly positive with respect to the cathode. This variation of grid voltage is indicated by the range of a to b on the horizontal base of the curve (Fig. 359).

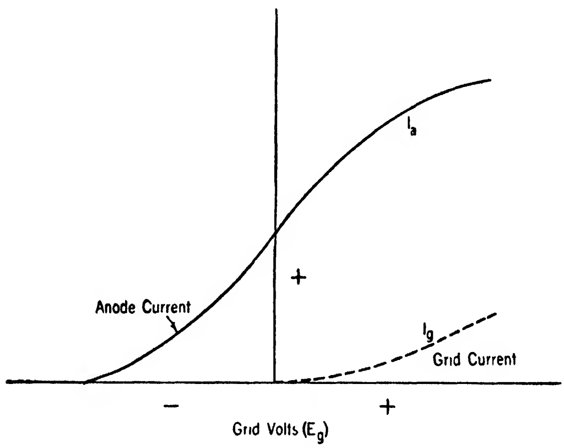


FIG. 359. CHARACTERISTIC OF A VACUUM TRIODE

Cutoff point

With anode-to-cathode potential held constant, the cathode-to-anode electron movement increases as shown by the curve marked I_a. At a certain negative grid voltage, say at a, the grid prevents any electrons from reaching the anode, and that status is called the *cutoff point*. Then, as the grid is made less negative, some electrons pass the grid; after the grid becomes positive it helps the electrons pass to the anode or plate. After the grid becomes positive it will attract a few electrons to itself, and this grid current is shown by the dotted curve I_g of Fig. 359.

Uses of triodes

The uses of the vacuum triode in communication may be classified as follows: (1) *amplifier;* (2) *oscillator* or alternating-current *generator;* (3) *modulator;* and (4) *detector*. The particular class of work performed by the triode does not depend on the tube itself but upon the circuit to which it is connected. Hence, a single tube might be connected in circuits for serving as amplifier, oscillator, modulator, or detector. In practice certain types of tubes are chosen for the different applications, but this selection is not

determined by any difference in their inherent theory of operation.

A simple amplifier circuit using a triode is given in Fig. 360. *Amplifier* The anode is given a positive, *B*, supply and the grid is made

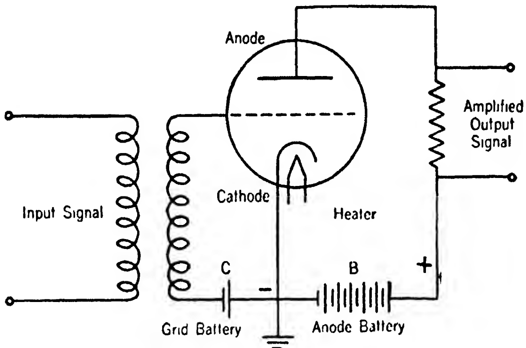
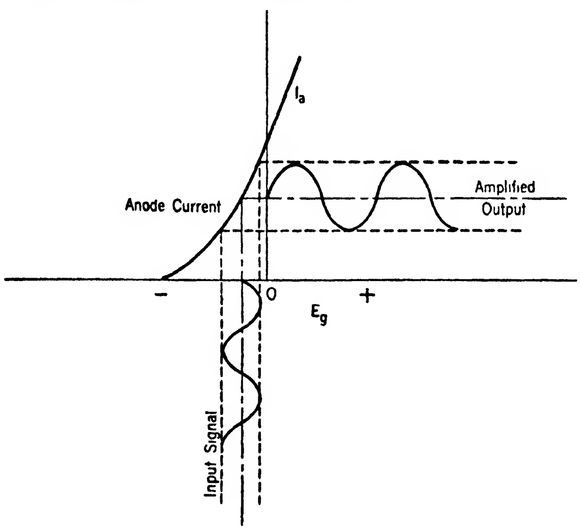


FIG. 360. SIMPLE AMPLIFIER CIRCUIT

normally negative by the battery, *C*. An alternating-current input voltage (sine wave) of low value is applied across the transformer and between the cathode and grid. This is called the *input* to the tube. The resulting changes in the potential on the grid cause similar



Input signal is a voltage whereas the amplified output is a current

FIG. 361. AMPLIFYING ACTION IN A TRIODE

variations in the current flowing in the anode or output circuit. This action is illustrated in the curves of Fig. 361.

It should be noted that the input signal is a small change in *Volts to* *voltage*, whereas the amplified *output* is a *current*. The input is a volt- *amperes*

age change without current and represents no power. The amplified current output may be utilized in different ways. For example, if you desire an amplified voltage to apply to a second tube, the current is passed through a high resistance giving a large alternating voltage (RI) drop. If you desire an amplified power, the current may be passed directly (or via a transformer) into a loudspeaker or other form of load where the current squared times the resistance of the load (I^2R) equals watts or power.

Oscillator

The preceding paragraph explained how a small signal voltage change applied to the grid of a triode would cause an amplification of that signal in the anode circuit. Under normal conditions the amplified signal will have the same frequency and wave form as the input signal. A little consideration will show that if a part of the amplified signal voltage could be applied or "fed back" to the grid with the proper time and phase relationship, the tube would "tickle itself" or stimulate itself so as to give a continuous *self-generated signal* whose frequency would depend upon the characteristics of the tube and of the circuit.

Oscillator circuit

A simple circuit for accomplishing this result is given in Fig. 362. It should be noted that any changes in the anode circuit are fed back to the grid (input) through the transformer. The frequency at which this tube and circuit will oscillate depends on the capacity of the condenser, C, and the choking action of the transformer winding, S. Any change in the value of the condenser, C, will vary the frequency of the oscillation or of the generated signal. While the power produced by an oscillator circuit is small, the alternating voltage generated may be increased in magnitude by the introduction of a triode amplifier plus a suitable amplifying circuit. Oscillators are used for producing high-frequency currents called *carriers* which will be discussed below.

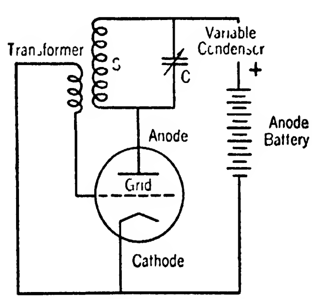


FIG. 362. SIMPLE OSCILLATOR CIRCUIT

TRANSMISSION AND RECEPTION

Carriers

Radio communication is a process of sending electromagnetic waves out into space and receiving them at some distant point. If a 60-cycle alternating voltage is applied to a conductor or antenna,

it will set up a changing electric field around the conductor, and this field will move out in waves as shown in Fig. 363. However, for 60 cycles the waves would not carry much energy, and the signal would not have enough strength to travel far.

Waves are 500 or more kilocycles

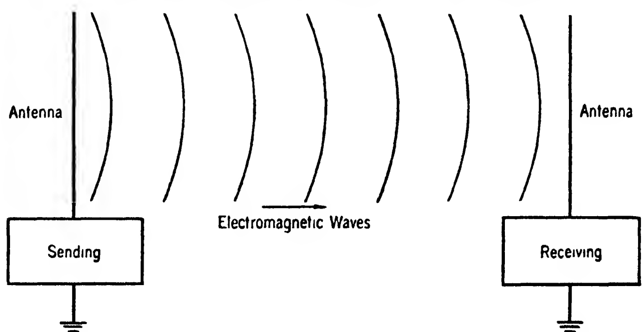


FIG. 363. ILLUSTRATING COMMUNICATION VIA RADIO WAVES

If an alternating current of 500,000 cycles or higher is applied to a conductor of suitable position, length, and so on, electromagnetic waves having relatively large energy are produced, and they will travel far with sufficient signal strength for satisfactory reception. Hence, if it is desired to transmit a signal of 60 cycles via radio waves, it is necessary to use a *high-frequency carrier* and then to impress the 60-cycle signal into the carrier in some manner.

Modulation

This process of impressing a signal upon a carrier is known as "modulation" and is illustrated graphically in Fig. 364. Part *a* represents the high-frequency carrier, part *b* the low-frequency signal (a sine wave), and *c* the modulated carrier. The modulated carrier has the necessary high frequency for radio transmission, and its amplitude has been modified by the signal being carried.

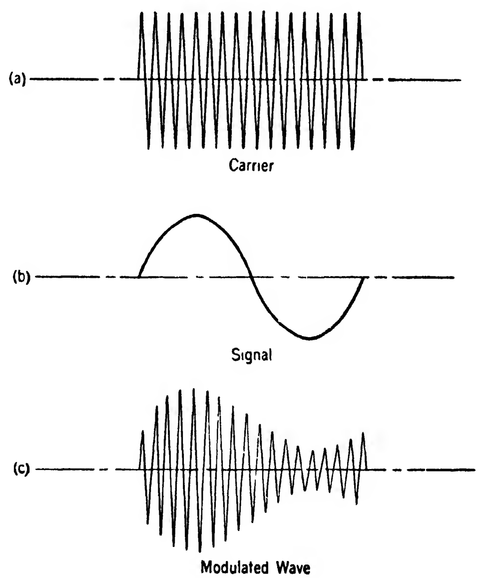


FIG. 364. ILLUSTRATING MODULATION OF A CARRIER BY A SIGNAL WAVE

A simple circuit showing the use of a triode for modulation is given in Fig. 365. The signal is a voice current produced by a microphone. The carrier current is of high frequency produced by an oscillator (not shown). The carrier and the signal are connected in series and fed into the grid-to-cathode circuit. The output current appears in the anode-cathode circuit and is the carrier which has been molded or modulated by the voice signal.

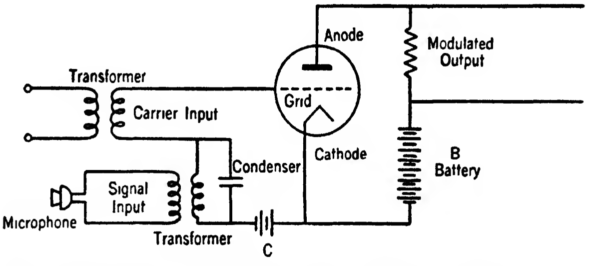


FIG. 365. CIRCUIT FOR MODULATING CARRIER WAVES

Detection

Detection or demodulation is the reverse process of modulation. It involves the detection or separation of a transmitted signal from the modulated carrier. Triodes are seldom used for detection, but they will serve this purpose if desired. A simple detector circuit using a triode is shown in Fig. 366. The modulated carrier is fed into the grid-cathode circuit. This modulated carrier consists of two parts, a high-frequency carrier and a low-frequency signal.

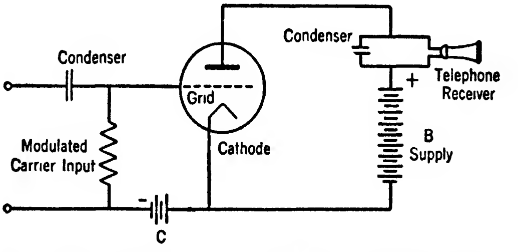


FIG. 366. CIRCUIT FOR DETECTION USING A TRIODE

Filters

When the modulated carrier is amplified in the anode circuit, it is passed through a parallel circuit consisting of a condenser and a telephone receiver. This parallel circuit, when properly designed, acts as a *filter device* in such a way as to pass the high-frequency part of the output through the condenser and to permit the low-frequency signal to pass through the receiver. The receiver delivers a sound that is a close replica of the original sound which produced the voice signal current.

The pentode is a five-electrode tube. It is a triode with two additional grids placed between the regular or control grid and the anode or plate. The pentode does the same things as the triode but performs them better. The first additional grid in the pentode is a *screen grid.* It carries a potential somewhat lower than the

Pentode

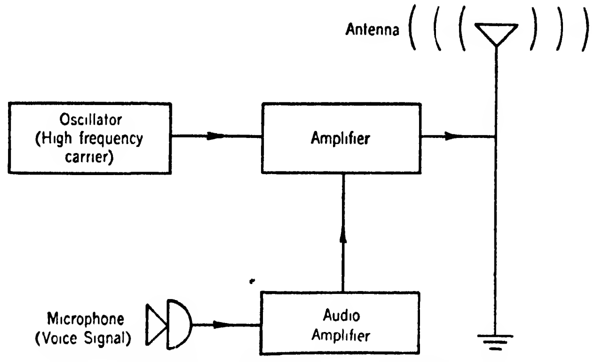


FIG. 367. SCHEMATIC DIAGRAM OF TRANSMISSION VIA RADIO

anode and its function is to set up a constant potential in the space it occupies. This constant space potential assures a constant attraction for the electrons from the cathode, independent of any fluctuations in the anode voltage resulting from changes in load. The second grid added in the pentode is the *suppressor grid.* It is connected to the cathode and ground and it serves to prevent electrons of secondary emission from the anode reaching the screen grid. Pentodes are generally used as amplifiers.

Radio transmitter

Radio communication is conducted by electromagnetic waves propagated through space from a transmitting station to a receiving station as suggested in Fig. 363. The transmitting or sending station embodies certain electronic units connected together as shown in the block diagram of Fig. 367. A high-frequency carrier current is generated in the oscillator, a voice current or signal is produced by a microphone and amplified in the audio amplifier. The amplified signal and the carrier are fed into an amplifier where both modulation and amplification take place. The modulated output or carrier is fed into an antenna. A very simple circuit using triodes for producing the steps as outlined is given in Fig. 368. The anode battery supply for all tubes is shown at B, and the general operation for the component parts of this complete radio transmitter station follows the theory given in the preceding discussion.

Radio receiving The receiving station of Fig. 363 must pick up the transmitted radio wave and transform it into sound. The steps for producing the desired result are indicated in the block diagram of Fig. 369.

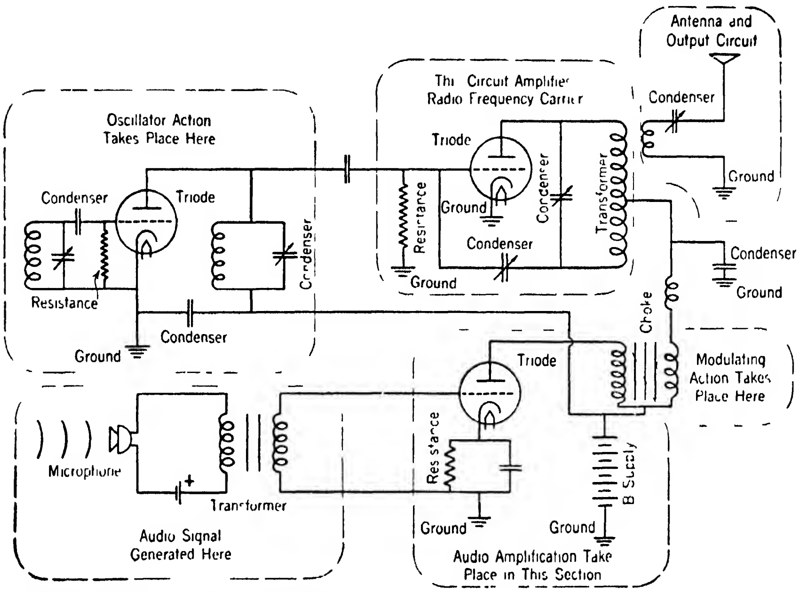


FIG. 368. SIMPLE RADIO TRANSMITTER CIRCUIT

The weak incoming modulated carrier arriving at the receiving antenna is first passed through an amplifier. Then the amplified signal is passed through a detector which extracts the *audio signal* from the modulated carrier. The detected signal is next amplified and then fed into a loud-speaker or receiver which reproduces the

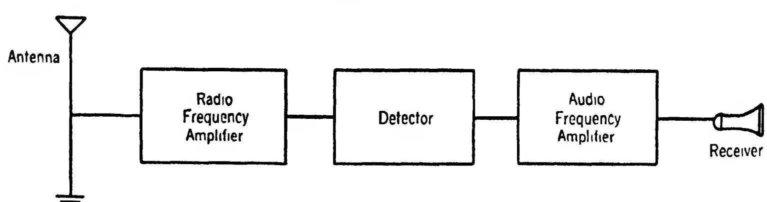


FIG. 369. SCHEMATIC DIAGRAM FOR RADIO RECEPTION

sound originally impressed on the microphone in the transmitting or sending station.

Communication engineering A simple receiver circuit using two triodes and one diode (detector) is given in Fig. 370. The function of each part of the circuit is given on the circuit diagram, and the action of the tubes and circuits of each part follows the general theory of the preced-

ing discussion. Space does not permit a complete discussion of the theory of action of the transmitter and receiving circuits. A more thorough treatment of mathematics, physics, electronics, and alternating current is necessary for the proper conception of these and

Science background needed

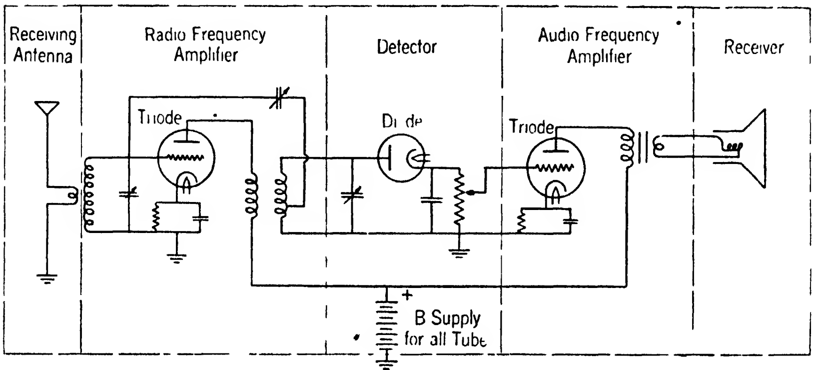


FIG. 370. SIMPLE RADIO RECEIVER CIRCUIT

the more intricate devices and circuits used in radio communication. This knowledge is acquired in the various studies leading to the work of the professional electrical engineer.

PHOTOTUBE

Phototube action

It was suggested previously that electrons may be removed from cold solids by the energy of light rays. Devices which utilize this property of light are called *phototubes*. The construction and simple circuit of a phototube are shown in Fig. 371. The cathode or emitter is made of some light-sensitive substance such as sodium, potassium, barium, or cesium on a metallic subbase and enclosed in a vacuum tube or a tube with inert gas under low pressure. When light impinges upon such a cathode, it gives up its energy to a few electrons in such a way that they are splashed off the surface. These

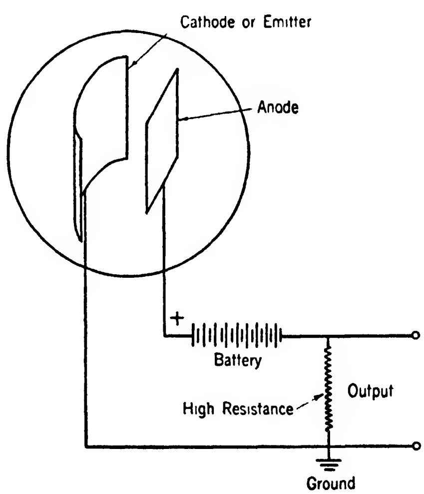


FIG. 371. CIRCUIT FOR A PHOTOTUBE

emitted electrons are then attracted by the positive potential (usually 90 volts) on the anode. After reaching the anode these electrons return via the battery and resistance to the cathode.

Electric eye

The current produced by the phototube is *directly proportional to the intensity of the light* falling on the cathode. Hence the device gives an accurate measurement of illumination. The magnitude of the current is very small, being only a few millionths of an ampere. This minute current will produce a small drop in voltage across the series resistor R. This voltage change can be impressed across the grid of a triode and thus produce an amplified signal of sufficient strength to operate electromagnets (relays), etc. Through its ability to operate electromagnets, the phototube becomes a very useful control device. A phototube is shown in Fig. 372.

FIG. 372. PHOTOTUBE

The phototube is often called the *electric eye* because of its ability to transform light changes into electric current. Among the applications of the phototube are (1) measurement of light intensity, (2) analysis of color of fabrics, (3) turning on and off artificial lights with changes in daylight outdoors, (4) counting the passing of people, cars, or packages, and (5) detecting the sound on film for sound motion pictures as will be explained below.

Sound on film

In motion-picture making the voice sounds on the movie set are picked up by microphones and transformed into voice currents. These voice currents control the dark and light area on a small sound tract on the film. In reproducing the scene on the screen in the movie theater, a light shines through the narrow sound strip on the film upon a phototube. This light varies (changes) with the frequency of the voice and the phototube changes the light variation back into an electric current variation. This variation is amplified and passed into the loud-speakers behind the projection screens. Thus phototubes play a vital part in the dual transformation chain of *sound — electricity — light — shadow on film* — thence *shadow on film — light — electricity — sound.* Many other electron tubes play a vital part in the magic chain just cited.

SPECIAL ELECTRONIC DEVICES

Industrial tubes

In addition to the electron tubes that we have discussed, there exists a large number of special tubes and devices which are exceed-

ingly useful and very interesting in theory and application. The *thyratron* is a three-electrode tube containing gas and is probably the most valuable of all control devices for commercial and industrial applications of electricity. The *ignitron* is a three-electrode mercury-vapor tube which has wide application in the control of electric welding and for controlling electric-power rectification. The *magnetron* is capable of generating ultrahigh frequencies needed for radio aircraft detection through distance, darkness, smoke, and fog.

The *cathode-ray tube* is capable of producing pictures of alternating currents (cathode-ray oscillograph) and of reproducing pictures transmitted via television. X-ray tubes produce penetrating rays which permit man to see through opaque materials; they can also be used for the treatment of cancerous growth. Newly developed electron microscopes magnify fine particles 50,000 times and offer man the opportunity of new studies into the unknowns of nature. These and many other devices and applications make the study of electronics an interesting excursion into the latest developments in the scientific world. *Oscillograph*

Mechanics

THE EFFECTS OF FORCES

This section takes up a study technically called *analytic mechanics*. Of course, you need not be too concerned with this technical name unless perchance you wish to impress your friends with the extent of your vocabulary. No matter what you call it, this subject is a study of forces, the kind that makes an airplane fly, the action which a bump in the road produces on your automobile, the blow which gave you that black eye. Forces may be applied to an object, gently at first, then with gradually increasing intensity, they may act constantly without change, or they may be applied with a bump. In this study, we shall think of forces as unchanging, although the forces may change the instant after we analyze them. While we work a problem, we think of an object as being in a suspended state of change, if change is taking place, much as the diver who is held momentarily in mid-air in the movies.

How forces are applied

When an engineer sets out to design an airplane, a machine, or the steel framework of an office building, he must first *estimate the magnitudes of the forces* on each part of the structure or machine. How big is the force on the bolts which attach the engine to the body (fuselage) of the plane? Before the size of the bolt can be decided on, this force must be known. What forces act on the framework of an office building because of the action of the wind, the weight of the building, or the snow on the roof? What is the maximum force on the bearing of the connecting rod in your automobile?

Why materials fail in service

These forces must be known, or approximately known, else the engineer is working in the dark. If he is too much in the dark, or if he makes a serious error, a bridge may fall, a balcony collapse, or a connecting rod may fly through the crankcase of your car. Now do not get the notion that these questions are going to be answered completely in this book. That is too big an order. However, we can learn some of the principles which control the design of structures, airplanes, and machines, even if the actual design must be studied later.

HOW A FORCE IS REPRESENTED

Definition of force

As an illustration of a force, imagine a boxer delivering an uppercut to the chin. Since this is not your chin, you can make a scientific and objective analysis of the force. First, you note that this force will be directed sharply upward, that is, you see that the force has *direction.* Then you notice that it landed squarely on the chin, that is, you find that a force has a *point of application.* Finally, you may imagine the force to be a gentle tap or a heavy blow. In other words, you observe that a force has *magnitude.*

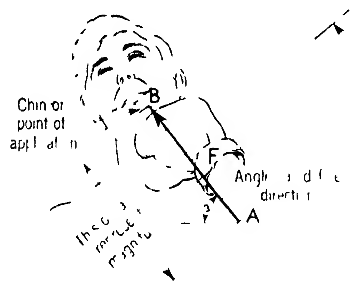


FIG. 373. FORCE OF A FRIENDLY UPPERCUT

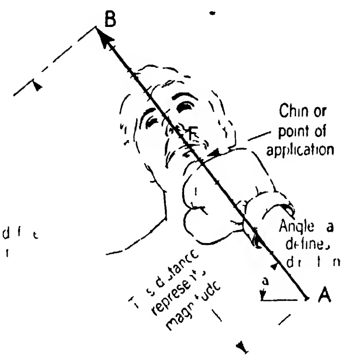


FIG. 374. FORCE OF PUGILIST'S UPPERCUT

Force arrow

Asking ourselves how we can represent something that has magnitude, point of application, and direction, we answer immediately, "By an arrow." Thus, in Fig. 373, the arrow *AB* represents a force completely. Its length is proportional to the magnitude of the force. The angle *a*, which the line of the arrow makes with the horizontal, determines the direction of the force. The point of application is designated, and may be any point on the line of the arrow, depending of course upon the position of the arrow. That is, the point of the arrow is not necessarily the point of application. Hence, we are actually concerned here only with the location of the direction line of the arrow, called the *line of action of the force.* As we see in Fig. 374, the arrow *AB* passes beyond the point of application, but it still is an accurate representation of the force.

Line of action

The arrow in Fig. 373 is 5 units long. Each unit of length represents a predetermined number of pounds of force. If a unit of length represents 15 lb., the force in Fig. 373 is 15 × 5 = 75 lb. Similarly, the force arrow in Fig. 374, being 12 units long, represents 15 × 12 = 180 lb. to the same scale.

WHAT CAUSES A FORCE?

Piston pressure

Forces are always caused by the action of one body on another. The explosion of the fuel in the engine of your car results in a large pressure or force on the top of the piston exerted by the burned gases. This gas pressure exerting a force on the piston causes the piston to move downward and makes the engine run. (See Fig. 375 and its caption.) Observe that the force described in this paragraph exists because two bodies (the piston and the hot gases) are in actual contact. The location of the point of application of the force may be determined from the area over which the force acts, which is the top of the piston in this instance.

Force of gravity

Another force which is always present is the *force of gravity*, commonly called the weight of the body. This force exists, not because of the physical contact of the earth and the body, but because of the attraction of the earth and the body for each other.

What is the location of the arrow that represents the force of gravity (weight)? The arrow which represents a force, by the way, is technically known as a "vector," and we might as well call a vector a *vector.** A force may act in any one of all directions, but one force, a vector quantity, can act only in one direction. But back to the question. The vector of the force of gravity passes through the *center of gravity*. Those readers who have some familiarity with the airplane may know of the center of gravity as the *c.g.* The location of the center of gravity or c.g. in an airplane is an important factor in its design. If the weight and control surfaces are not properly located with reference to the c.g., the airplane will be unstable and dangerous, if not impossible, to fly. Automobiles are now safer machines, for one reason, because the c.g. is lower in modern automobiles than in older ones.

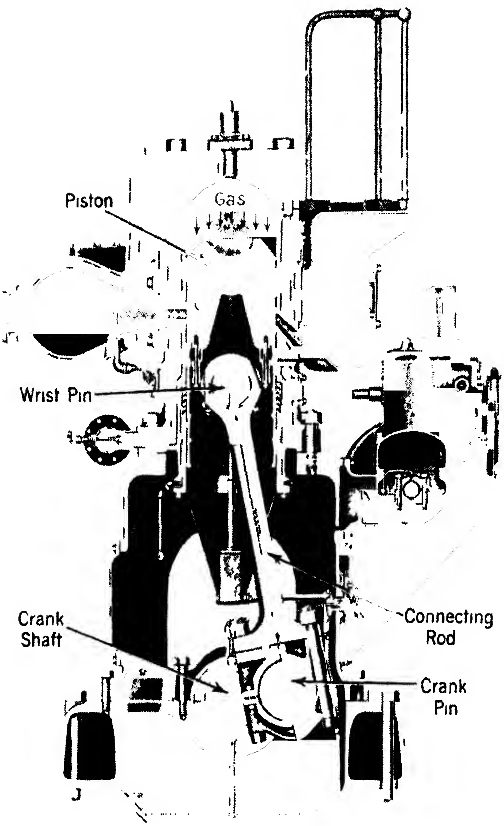
Center of gravity

LOCATING THE CENTER OF GRAVITY

Balance and symmetry

The center of gravity may also be termed the *center of balance*. If you balance a stick on the edge of a knife, the c.g. of the stick will be exactly above the knife edge. Another criterion, and the one which we often rely upon, is that a plane of symmetry of a body will contain the center of gravity.

* Force is thus a vector quantity, whereas you are more accustomed to thinking of so-called scalar quantities Scalars are measured by simple numbers and have no property of direction The area of a room is a scalar quantity.



Burning gas produces the downward force on the piston

Connecting rod crank shaft connection produces the rotation

Courtesy Fairbanks Morse and Company

FIG. 375. SECTION OF A DIESEL ENGINE

While this engine is not an automotive engine, the two possess many points of similarity. Common to both types are the crankshaft, crank pin, connecting rod, wrist pin, piston, and as mentioned in the text, the pressure or force of the gas on the piston during the power stroke. The fundamental difference between the automotive and Diesel engines is that the fuel in the automotive engine is ignited by an electric spark across the gap of the spark plug, while in the Diesel engine (as above), the fuel is ignited because the incoming gases are so highly compressed that they become hot enough to ignite the fuel without the use of a spark. As the crankshaft rotates, the center of the crank pin moves in the circle shown. This movement causes the piston to reciprocate or move up and down. A movement of the piston from top to bottom or *vice versa* is called a *stroke*.

Engine forces

The pressure of the gases on the piston results in a whole series of forces: on the wrist pin, the connecting rod, the crank pin, the crankshaft; and then more forces for members not shown, as, for example, on the shafts and gears of the transmission and differential, on the axles, and finally, frictional forces of the tires on the road, which make it possible for the car to move. It is interesting to observe that without friction it would be impossible to start the car when it is on a level plane.

Symmetry

The connecting rod of Fig. 376 has two planes of symmetry. Since their line of intersection (see page 193) passes lengthwise down the center of the rod, it is only necessary to balance the rod in one other position, either *a* or *b*, as shown in Fig. 376. The c.g. is located at the point where the vertical plane through the knife edge intersects the center line. Now, as stated before, the force of gravity always acts through this point, and it does not matter in what position the connecting rod is placed, as may be seen in Fig. 377.

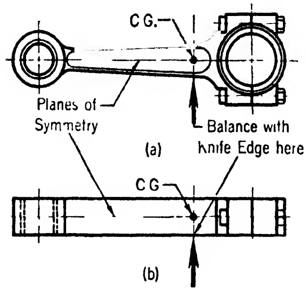


FIG. 376. C.G. OF A CONNECTING ROD

Center of balance

Suppose we drill a hole through the connecting rod at the c.g. in Fig. 376 *a* and support the rod on a smooth shaft in this hole.

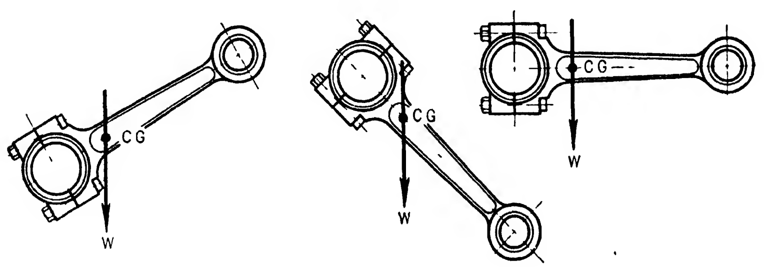


FIG. 377. WEIGHT ACTS THROUGH C.G.

No matter what the position of the connecting rod, the force of gravity acts vertically downward through the center of gravity.

We may then rotate the rod into any position and it will remain in balance as placed (Fig. 378). That is, the c.g. is the center of balance.

Locating the c.g.

Knowing that all planes of symmetry of homogeneous bodies contain the c.g., we conclude that the c.g. of a sphere, cube, cylinder, or of any body symmetrical in relation to three planes, is at the geometric center of the body. We can use this knowledge in locating the force-of-gravity vectors of many structural and machine parts.

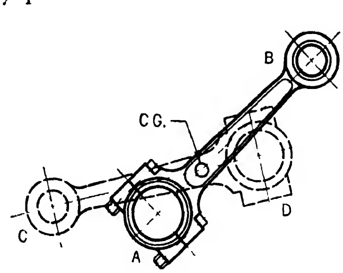


FIG. 378. CENTER OF BALANCE

If it is supported on an axis through the c.g., the connecting rod remains in any position in which it is placed. It will stay in the position *AB*, or in any other position *CD*.

PRINCIPLE OF THE LEVER

Moving the earth

Archimedes (287–212 B.C.), who was one of the earliest and most famous engineers, once said something like this, "Give me a place to stand and a suitable pivot point and I can move the world." He planned on using the principle of the lever, if anyone had called him on his boast. While we still see no practical way of actually moving the earth with a mechanical lever, the principle involved has been of incalculable assistance in technological developments.

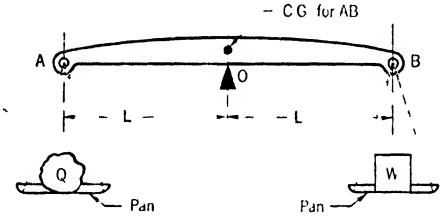


FIG. 379. SIMPLE WEIGHT SCALE

One of the simplest applications of the lever is a pair of balances, Fig. 379. In its elementary form, there will be a pivot at the point O which is directly under the c.g. of the arm AB when the points A and B are on a horizontal line. An object Q in one pan may now be weighed by placing weights, W, in the other pan until the whole system balances with A and B in a horizontal plane. If a weight of $W = 1$ lb. balances Q, then Q weighs 1 lb.

Off-center balance

However, suppose the pivot is moved off center as shown in Fig. 380, and, to simplify the discussion, suppose the arm AB and the pans weigh so little in comparison with the objects in the pans that we may neglect their influences altogether. For a condition of balance under this arrangement, the weight W is not the same as the weight of Q, but, with knowledge of the dimensions a and b, the weight of Q may be computed.

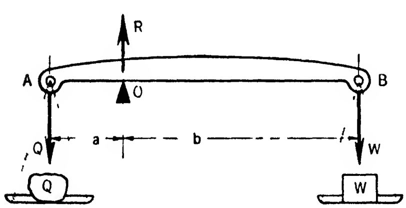


FIG. 380. UNSYMMETRICAL SCALE

The principle that Archimedes discovered was

(1) Weight $W \times$ distance b = weight $Q \times$ distance a.

In technical terms, we say the same thing in this way,

Moment of W about O = moment of Q about O.

Moment of a force

Since W and Q are symbols for the forces of gravity and since a and b are distances from the pivot O to lines along which these forces act, we see that the *moment of a force* about a *particular* point

is the magnitude of the force times the distance from that point to the line of action of the force. This distance, which is called the *moment arm*, is always measured in a direction perpendicular to the vector representing the force.

Definition

(2) $$\text{Moment} = \text{force} \times \text{moment arm.}$$

EXAMPLE.

Let $a = 5$ in., $b = 10$ in., and $W = 2$ lb. How much does Q weigh (Fig. 380)?

Using equation 1, we have

(3) $$Wb = Qa, \text{ or } 2 \times 10 = Q \times 5,$$

from which we obtain

(4) $$Q = \frac{2 \times 10}{5} = 4 \text{ lb.}$$

which is the weight of the body Q.

In equation 3, we have $Wb = Qa$. Rearranging this equation into the form $W/Q = a/b$, we may generalize to obtain the following statement.

Archimedes' principle

The forces on a lever are inversely proportional to their distances from the pivot point, $W:Q = a:b$.

Remember that the distance to a force is measured perpendicularly to the force vector.

EQUILIBRIUM

There is more to Fig. 380 than meets the eye at first. We ought to observe that the system of forces is in *equilibrium*. When a set of forces is in equilibrium, the body on which the forces act is at rest or it has a constant motion.

Loads and reactions

Consider a body, in particular, the balance arm AB. Think of AB as isolated all by itself, free, so to speak, and decide what forces were acting on it when it was a part of the balance. There are the forces Q and W, of course, which act vertically downward. Now if this arm is to be in equilibrium, there must also be found a force or forces acting upward to counteract these downward forces. The equalizing force in this instance is at the pivot O; that is, the pivot O pushes up on the arm with the

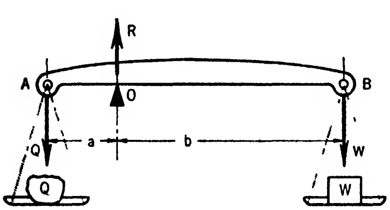


FIG. 380. REPEATED

same force that the arm pushes down on the pivot, and the downward forces are Q and W. To be in equilibrium, the forces on the arm AB must add up to zero. Since all the forces in this case act in a vertical direction, we may write, for equilibrium,

(5) Sum of the vertical forces = 0.

This equation expresses *one* of the conditions of equilibrium. This condition of equilibrium may be used to determine the *reaction* at O. By *reaction*, we mean the force R at O caused by other forces on the arm. To sum forces, we must account for their directions. In this instance, we can let the upward direction be positive; then, the downward direction is negative. Thus, applying equation 5 and summing the forces in Fig. 380 when $W = 2$ lb. and $Q = 4$ lb., we find

Directions and signs

$$R - Q - W = R - 4 - 2 = 0,$$

from which $R = 6$ lb.

Another condition of equilibrium may be stated.

Moment equilibrium

(6) Sum of the moments of all the forces about any point = 0.

Although the rule is good for any point, you will find certain points more convenient than others. In the balance problem, point O is best. Let us take moments about O and equate the sum of all moments to zero in accordance with equation 6. To do this, we must choose positive and negative directions.

A moment has a direction in a clockwise sense or in a counterclockwise sense. Thus, in Fig. 380, the force W tends to turn the arm about the pivot O in a clockwise sense, and the force Q tends to turn it about O in a counterclockwise sense. Suppose we consider clockwise moments to be negative. Then the sum of the moments about O is

Sign of moments

(7) $$Wb - Qa = 0.$$

The force R has no moment about O because its moment arm about this point is zero. Equation 7 is an application of equation 6. But watch carefully as we use a little algebra. Transpose the term Qa to the right-hand side of equation 7. Then you will find

(8) $$Wb = Qa.$$

The result, you notice, is the same as equation 1. You should not be surprised, since all correct methods of solving a problem must give the same answer. The only difference between equa-

Comparison

tions 1 and 6 in this application is simply in the point of view. The point of view in equation 6 is more useful generally. Now you may use the foregoing principles and solve the following problems.

Problems

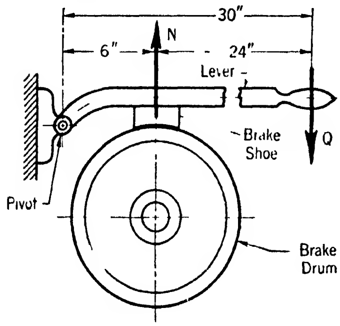


FIG. 381. FORCE ON A BRAKE SHOE, PROBLEM 1

1. A force $Q = 100$ lb. is applied to a 30-in. brake lever as shown in Fig. 381. What is the total force of the brake shoe on the drum? (This force is equal and opposite in direction to the force N of the drum on the shoe, as shown.) *Ans.* 500 lb.

2. A crowbar is used to lift the edge of a crated airplane engine (Fig. 382). It is pivoted on a block of wood at O and the worker exerts a force of $Q = 150$ lb. as shown. What force, F, is applied to the crate? (This force is equal and opposite to the force F of the crate on the crowbar.) Also, what is the reaction R at the pivot?

Ans. $F = 1650$ lb., $R = 1800$ lb.

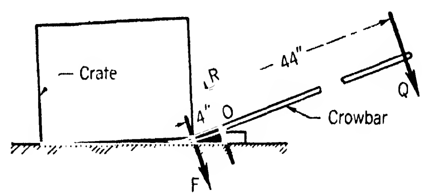


FIG. 382. LEVER ACTION, PROBLEM 2

Radial leverage

3. A simple winch is composed of a cylinder, around which a rope winds, and a crank which the operator uses for turning the cylinder (Fig. 383). Winches are frequently found at water wells. A bucket of water, the total weight of which is $W = 20$ lb., is lifted by a winch (Fig. 383) in which the operator applies a force Q at a distance of 10 in. from the axis O of the winch. Find the value of Q when friction is neglected. *Ans.* 10 lb.

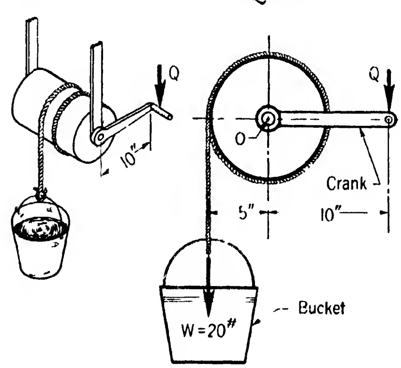


FIG. 383. LEVERAGE OF A WINCH, PROBLEM 3

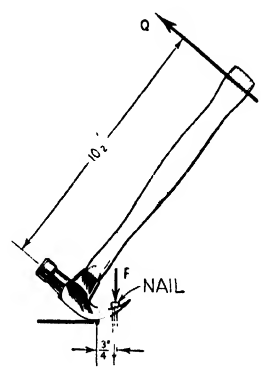


FIG. 384. PULLING FORCE, PROBLEM 4

Nonparallel forces

4. A carpenter had to exert a force of $Q = 50$ lb. on a claw hammer in order to pull a nail when the conditions were as shown in Fig 384. What force F was necessary to move the nail? *Ans.* 700 lb.

COMPONENTS OF FORCES

Direction of force

Suppose you wished to move a loaded industrial truck along a horizontal floor. No doubt you would grab the handle and start pulling. The direction of the force exerted on the truck would be indicated by the inclination of the handle. Assume that the handle makes an angle of b degrees with the horizontal as suggested in Fig. 385. In this event, one effect of the force Q is to tend to lift

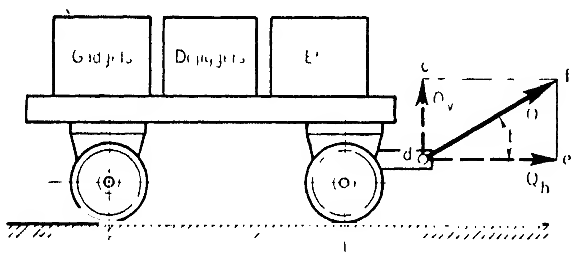


FIG. 385. FORCE COMPONENTS

the front end of the truck off the floor (which is not what you want), and the other effect is the horizontal pull which will cause the truck to move. (Suppose for the moment that there is no friction.) These two effects are measured by what we call *components of the force* Q.

Triangle of forces

The component of Q in a horizontal plane will be called Q_h, and in a vertical plane, Q_v. As seen from Fig. 385, the horizontal component is the leg de of the right triangle def. The vertical component is the leg dc of the right triangle dcf, which is equal in magnitude to the leg ef of triangle def. With these observations and a bit of trigonometry, we find the value of the components to be

$$(9) \qquad Q_h = Q \cos b \quad \text{and} \quad Q_v = Q \sin b.$$

Components of a force are other forces which may replace the given force without changing the action of the body on which they act. That is, we may *remove* Q and use Q_h and Q_v with the same effect on the truck as when Q acted alone. Thus, when you think of components, imagine the original force as removed. There are other components of a force, but we shall specialize on the two described above because they are the most useful.

Use of components

Perhaps you are wondering why we should bother with components when the force itself is known? Well, suppose you tug on the truck and find it so heavily loaded that you need assistance.

(Friction is involved now.) We will assume that a rope is tied to the truck and your assistant pulls on the rope as in Fig. 386 with a force of A lb. (at a degrees with the horizontal) while you pull with B lb. (at b degrees). What total pull results, that is, technically speaking, what is the resultant pull? Let us take some real

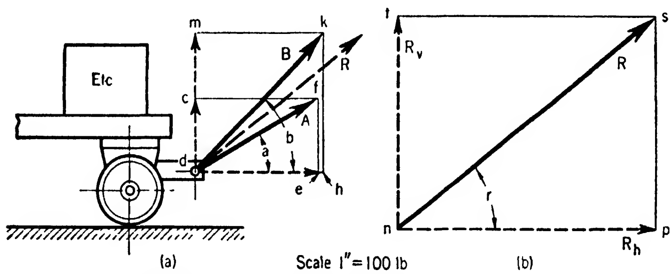


FIG. 386. COMBINING FORCES INTO A RESULTANT

numbers. Your assistant pulls with a force $A = 80$ lb., but you, being more energetic, exert a pull $B = 100$ lb.; let $a = 30°$ and $b = 45°$. What is the resultant pull? First, find the horizontal components of A and B.

(10) $A_h = de = A \cos a = 80 \times \cos 30° = 80 \times 0.866 = 69.3$ lb.

(11) $B_h = dh = B \cos b = 100 \times \cos 45° = 100 \times 0.707 = 70.7$ lb.

Adding collinear forces

Since we know that simple addition works for forces acting in the same direction, we can add A_h and B_h and find

(12) $R_h = A_h + B_h = 69.3 + 70.7 = 140$ lb.,

which is the net *horizontal* force you both exert when pulling together. Now we can find and add the vertical components.

(13) $A_v = dc = A \sin a = 80 \times \sin 30° = 80 \times 0.5 = 40$ lb.

(14) $B_v = dm = B \sin b = 100 \times \sin 45° = 100 \times 0.707 = 70.7$ lb.

(15) $R_v = A_v + B_v = 40 + 70.7 = 110.7$ lb.

Next consider R_h and R_v. These two forces must bear the same relation to some resultant force R as, say, A_h and A_v bear to the force A. This analogy suggests the arrangement of forces in Fig. 386 *b*. The line np is proportional in length to $R_h = 140$ lb., and nt is proportional to $R_v = 110.7$ lb. In thus laying out the forces, we are drawing the vectors to scale.

VECTORS AND TRIGONOMETRY

Resultant vector to scale

The engineer chooses a scale for the forces. In this case, let 1 in. represent 100 lb. of force, so that np is $\frac{140}{100} = 1.40$ in. long,

and nt is $\frac{101.7}{100} = 1.11$ in. long. (Use a ruler and check these figures.) By completing the rectangle $ntsp$ and drawing the diagonal ns, you may determine the magnitude and direction of R. Lay a ruler along ns and you will find that it measures about 1.78 in. If 1 in. represents 100 lb., then 1.78 in. represents $1.78 \times 100 =$ 178 lb., which is the magnitude of R. A protractor is used to measure the angle r, which is found to be about 38°. (*Note:* The vector for R is drawn too heavy for accurate graphical work. In graphical solutions, use a hard pencil with a fine point. Graphical accuracy is usually satisfactory for the solution of engineering problems, the range of error for careful work being one or two per cent.)

Resultant obtained algebraically

To find the resultant R by an algebraic solution, we note that it is the hypotenuse of a right triangle, nps. Its magnitude is therefore the square root of the sum of the squares of the legs (see page 267); thus

$$R = \sqrt{\overline{np}^2 + \overline{ps}^2} = \sqrt{R_h^2 + R_v^2}. \tag{16}$$

Using the numbers of this example, we may write

$$R = \sqrt{140^2 + 110.7^2} = 178.2 \text{ lb.}$$

To find the angle r, we notice that

$$\tan r = \frac{ps}{np} = \frac{R_v}{R_h}. \tag{17}$$

Therefore, since $\tan r = \frac{110.7}{140} = 0.79$, the angle r obtained from the table of tangents on page 566 is 38.3°. This angle defines the direction of R. The actual location of the vector for R, or of any resultant force vector, is such that the resultant's line of action passes through the intersection of its components. Because the original components A and B intersect at d, the *location* of the vector R is as shown by the broken line in Fig. 386 *a*.

Problems

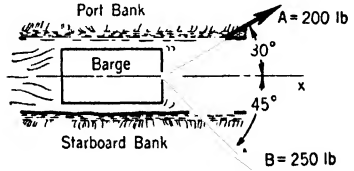


FIG. 387. RESULTANT PULL, PROBLEM 1

1. Two horses are pulling a barge along a canal, Fig. 387. At a certain instant, horse A on the port bank is exerting a pull of $A = 200$ lb., and horse B on the starboard bank is pulling with a force $B = 250$ lb. Determine the magnitude, direction, and location of the resultant force. *Hint.* Watch out for positive and negative components. Observe too that if the horses continue to pull as stated, the barge will soon strike the bank of the canal. Which bank would it strike? *Ans.* $R = 358$ lb., $r = 12.4°$ to the starboard of the x axis.

2. In a tugging contest, four men pull in the directions and with the magnitudes shown in Fig. 388. Determine the resultant force. (Watch out for negative components.) Which side (A and B vs. C and D) wins this contest?

Ans. R = 48.5 lb. pointing downward and leftward at an angle of 36° below the x axis.

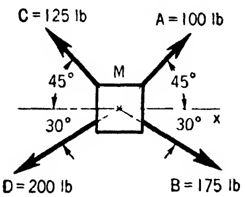


FIG. 388. RESULTANT OF FOUR FORCES, PROBLEM 2

BRIDGE TRUSS

Laws (statics

Now let us consider a problem of equilibrium. Since all parts of a bridge truss are at rest, the forces to be studied in this structure are in equilibrium. Whenever a body is in equilibrium, the resultant force on the body is zero. If the resultant force is zero, its components must also be zero. Hence, the sum of all the horizontal components and the sum of all the vertical components of the forces acting on a body in equilibrium must each add up to zero. We may therefore write these important equations:

(18) Sum of horizontal components = 0.

(In mathematical symbols, $\Sigma H = 0$.)

(19) Sum of vertical components = 0.

(In mathematical symbols, $\Sigma V = 0$.)

Suppose we have a bridge truss loaded as shown in Fig. 389. In order to design the members of the truss, we must determine the load on each member, AB, AC, BC, etc. To do this, we consider

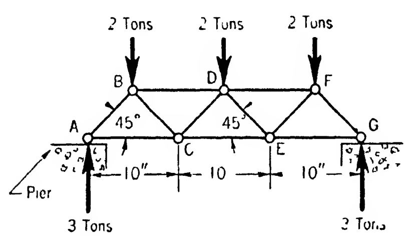


FIG. 389. TRUSS REACTIONS

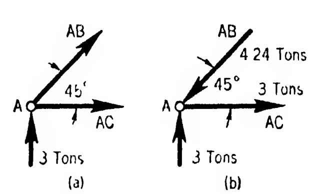


FIG. 390. FORCES ON PIN A

first the pin at A (all members are assumed to be joined by pins). It is acted upon by the upward force or pier reaction of 3 tons (caused by the three 2-ton loads), and by unknown forces in the members AC and AB. In Fig. 390 a, the pin A is pictured with the forces AC and AB acting away from the pin, but we may find that we are wrong about these directions. Anyway, we shall proceed with the summing of components, vertically first. Let the upward direction be positive.

Summing components

Vertical sum of force components $= 3 + AB \sin 45° = 0$.

Solving for AB from this equation, we find

$$AB = -\frac{3}{0.707} = -4.24 \text{ tons.}$$

Here the minus sign tells us that we have chosen the wrong direction for AB and that it should act toward the pin. We then redraw the figure with the corrected direction for AB as shown in Fig. 390 *b*. Evidently the diagonal end post AB is a compression member that pushes on the pin A. Now using the revised drawing, Fig. 390 (*b*), we sum the horizontal force components and find

$$\text{Horizontal sum} = AC - AB \cos 45° = AC - 4.24 \times 0.707 = 0,$$

from which

$$AC = 4.24 \times 0.707 = 3 \text{ tons.}$$

The positive answer indicates that AC is shown acting in the proper direction in Fig. 390. Next we consider the pin B, Fig. 391 *a*, and again the unknown forces, BD and BC at this point are shown

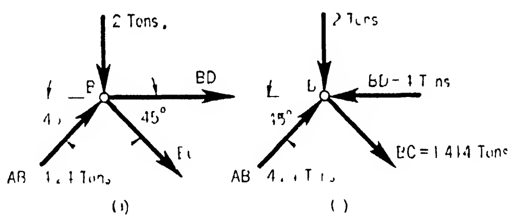


FIG. 391. FORCES ON PIN B

Compression member

acting *away* from the pin. Of course, AB acts toward pin B because the member AB has been found to be in compression and because a compression member pushes outward at both ends (as a compressed spring acts). Summing vertical force components first, we find

$$\text{Vertical sum} = 4.24 \times \cos 45° - 2 - BC \cos 45° = 0,$$

$$BC = \frac{4.24 \times 0.707 - 2}{0.707} = 1.414 \text{ tons.}$$

The positive sign shows that the vector BC is pointed correctly in Fig. 391 *a*. We therefore use the same figure to sum the horizontal force components.

$$\text{Horizontal sum} = BD + BC \cos 45° + AB \cos 45° = 0,$$

$$BD = -1.414 \times 0.707 - 4.24 \times 0.707 = -4 \text{ tons.}$$

Tension member

The negative sign shows that the direction of BD should be reversed, as redrawn in Fig. 391 *b*. When the force in a member points away from a pin, the member is being stretched (in tension) since it is pulling on the pin. If BC, for example, pulls on the pin B, it also pulls on the pin C. To find the forces in all the members of the truss, the foregoing procedure is repeated for each pin. Figures 392 and 393 show typical trusses.

From "Analytic Mechanics," Chambers and Faires

FIG. 392. STEEL TRUSS HIGHWAY BRIDGE

Observe that the members of this structure are so arranged that they always form triangles. A moment's reflection will convince you that the triangular form is the most rigid. Compare pin-connected links in triangular form with pin-connected links in, say, a rectangular form.

Forces in kips

The forces discussed here have been given in tons. American bridge engineers always state all forces in *kips*, a *kip* being 1000 lb. You can therefore multiply all forces in tons given above by 2.0 and they will then be given in *kips*.

Problems

1. Using forces as found above and the pin C in Fig. 389, find the forces CD and CE. *Ans.* $CD = 1.414$ tons, compression; $CE = 5$ tons, tension.

2. Using forces as found above, the results of problem 1, and the pin D in Fig. 389, find the forces DF and DE.
Ans. $DF = 4$ tons, compression; $DE = 1.414$ tons, compression.

3. Using the pin G in Fig. 389, find the forces in FG and EG.
Ans. $FG = 4.24$ tons, compression; $EG = 3$ tons, tension.

4. Using the pin F in Fig 389 and the results of problem 3, find the forces in EF and DF. *Ans.* $EF = 2.83$ kips, tension; $DF = 8$ kips, compression.

FRICTION

Using friction

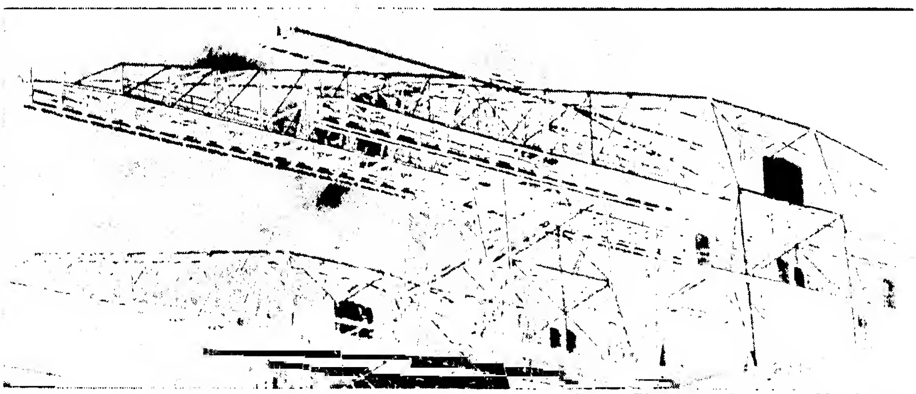
Friction is a contradiction. On the one hand, it serves a very useful purpose in many instances. As a matter of fact, while you might be able to stand up, you could not walk if it were not for friction. Friction holds one foot in place while the weight of the body is shifted to the other foot. On the other hand, in bearings and between pistons and cylinders, for example, we make every effort to reduce friction to a minimum, at least, every effort within reason considering the value of the machine.

Coefficient of friction

If there is rubbing between two surfaces, the frictional force F is computed from two quantities — from what we call the *normal force*, N, and the *coefficient of friction*, f. The relation between these quantities may be expressed mathematically as

$$F = fN. \tag{20}$$

Here we see that the coefficient of friction f is that number which when multiplied by the normal force N gives the frictional force F. The normal force is that part of the reaction between two surfaces in contact which is normal or perpendicular to the surfaces. The frictional force always acts in a direction to resist motion.



Courtesy Westinghouse Electric and Mfg Company

FIG. 393. TRAVELING HAMMERHEAD CRANES

This structure with four such crane trusses was erected for use in building a large dam. How would you like the job of finding the force on each member of this trusswork?

An application of these ideas will help to clarify them. Suppose we wish to tip a heavy object (a machine or a crate, for example), so that grapplers may be inserted under the object with the intent of picking it up by a crane and moving it. We may drive a wedge under one side of the object as suggested by Fig. 394 *a*. To simplify the picture, we shall *assume* that the greased top surface of the wedge is *frictionless* and then make a free body of it as shown in Fig. 394 *b*.

Friction of a wedge

One-half of the weight $W = 2000$ lb. is taken as acting vertically downward on the wedge ($W/2 = 1000$ lb.). Let the coefficient of friction be $f = 0.3$ and find the value of the force Q to move the wedge. The normal force acts upward on the wedge and perpendicular to the surfaces (floor and wedge) in contact. The wedge is to move toward the left; so the frictional force on it acts toward the right. Our free body diagram of the wedge with a frictionless top surface is now complete (Fig. 394 *b*). We sum vertical forces to obtain

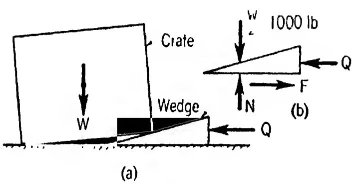


Fig. 394. Wedge used to tilt crate

$$\text{Vertical sum} = N - W/2 = N - 1000 = 0,$$

from which $N = 1000$ lb. Since $f = 0.3$, we next find

$$F = fN = (0.3)(1000) = 300 \text{ lb.}$$

Then,

$$\text{Horizontal sum} = F - Q = 300 - Q = 0,$$

from which $Q = 300$ lb. (While the assumption of a frictionless top surface makes this an incomplete picture of the action of the wedge, it does show how to compute the frictional force.)

Friction coefficients

Typical values of the coefficient of friction, which are seen to depend on the kind of materials in contact, are as follows (dry surfaces):

Wood on wood	0.3	Leather on iron	0.3
Wood on metal	0.4	Leather on wood	0.3
Iron on iron	0.2	Rubber on concrete	0.6

Problems

1. A boy on a tricycle is coasting downhill and decides to stop. Using his foot on the sidewalk as a brake, he presses down with a force of 30 lb. If the coefficient of friction between the leather sole and sidewalk is 0.4, what is the braking (frictional) force? *Ans.* 12 lb.

2. Another youngster, more mindful of the cost of leather than the one in problem 1, uses a stick for a brake and, by virtue of the leverage he obtains, he produces a normal force of 50 lb. with a coefficient of friction of 0.3. What is the braking force? *Ans.* 15 lb.

VELOCITY AND ACCELERATION

Velocity

You are familiar with the concept of *velocity* or *speed*. You can imagine a car going 60 m.p.h., or a plane going 300 m.p.h., and you know that speed is the distance traversed in some unit of time. Instead of using the familiar unit of miles per hour, we find it more convenient in this study to use the number of feet traversed in 1 sec. (that is, ft. per sec.). Thus we may write,

$$(21) \qquad \text{Average speed (or velocity)} = \frac{\text{distance in feet}}{\text{time in seconds}}.$$

Computing speed

Suppose that the distance between two towns is 490,000 ft. and that you drive this distance in 7000 sec. Then your average speed is $\frac{490{,}000}{7000} = 70$ ft. per sec. Of course you do not drive constantly at this rate. You find yourself speeding up and slowing down for one reason or another. When the speed of your car is changing, the car is said to be accelerating. Specifically, the acceleration is the rate at which the speed (or velocity) changes. In equation form, we may write

$$(22) \qquad \text{Average acceleration} = a = \frac{\text{change in speed}}{\text{time}} = \frac{v_2 - v_1}{t},$$

where v_2 represents the final speed, v_1 represents the initial speed, and t represents the time elapsed during the change.

Computing acceleration

For example, a car which changes speed from 30 ft. per sec. to 50 ft. per sec. in 4 sec. has an average acceleration of

$$a = \frac{(50 - 30)\text{ ft. per sec.}}{4\text{ sec.}} = 5\text{ ft. per sec.}^2.$$

Since acceleration is a change in velocity per unit of time, and since the velocity here is in feet per second, the acceleration is in (ft. per sec.) per sec., or *ft. per sec.*^{2}, as given. If the initial speed is zero, that is, if a body starts from rest, equation 22 becomes

$$(23) \qquad a = \frac{v}{t}, \quad \text{or} \quad v = at.$$

It is common knowledge that a falling body gathers speed. Until the time of Galileo (1564–1642), most people thought that heavy bodies would fall faster than light ones.

Galileo's falling bodies

By dropping various sizes of stones from the famous Leaning Tower of Pisa, Galileo showed that a small light stone gains speed as rapidly as a big heavy one. Some people at the time thought he was being tricky and were in favor of executing him as a practitioner of black magic. However, he managed to squeak through before the Inquisition and made many more valuable contributions to science. Anyway, it is a fact that a rock and a feather fall at the same rate when both are in a *vacuum*, that is, when there is no air resistance. Moreover, they fall with a *constant acceleration*, the velocity steadily increasing. This particular constant acceleration is called the *acceleration of gravity* and is represented by the letter g. The value of g is different for different points on the earth's surface, but its average value is about $g = 32.2$ ft. per sec.^{2}.

Problems

1. A train travels 1000 miles in 30 hours. What is the average speed of the trip? *Ans.* $33\frac{1}{3}$ m.p.h.

2. An airplane travels 1000 miles in 3 hours. What is its average speed? *Ans.* $333\frac{1}{3}$ m.p.h.

3. An automobile goes 170 miles in 4 hours. What is its average speed in feet per second? There are 5280 ft. in 1 mile. *Ans.* 62.3 ft. per sec.

4. An automobile starts from rest and attains a speed of 28 m.p.h. in 7 sec. What is the average acceleration? *Ans.* 4 m.p.h. per sec.

5. An automobile going 60 ft. per sec. is brought to a stop in $\frac{1}{2}$ min. What is the average acceleration? *Ans.* 2 ft. per sec.^{2}.

6. An automobile starting from rest has an acceleration of 1.5 ft. per sec.^{2} for 20 sec. What is its final speed? *Ans.* 30 ft. per sec.

7. A stone is dropped from a cliff and falls for 5 sec. If air resistance is neglected, with what speed does it strike the ground? *Ans.* 161 ft. per sec.

WHAT CAUSES ACCELERATION?

Newton's laws

Sir Isaac Newton (1642–1727) first formulated the laws of motion, but not, as the story goes, because he watched an apple fall from a tree. We have already seen the application of one of his laws, to wit, *action and reaction are equal and opposite*, when we recognized, for example, that the force of the lever on the brake drum (Fig. 381) was equal to but directed opposite to the force of the drum on the lever.

Force and acceleration

Another of his laws is: *the acceleration of a body is proportional to the force producing the acceleration.* Thus, we see that a *force* causes acceleration. This simple law leads into all sorts of ramifications

which we cannot explore in this short study. However, we can learn the basic meaning of the statement.

Proportion Since a freely falling body has a known acceleration of g ft. per sec.^{2} and since we know that the force of gravity (weight) is the force which produces this acceleration, we may set up a simple proportion involving these known values of W and g and any other force R producing an acceleration a. Thus we may write,

$$\frac{R}{W} = \frac{a}{g}, \quad \text{or} \quad R = \frac{W}{g}a. \tag{24}$$

This equation may be used to find the resultant force R required to produce a certain acceleration a of a rigid body which weighs W lb. The force R and the acceleration a in equation 24 are *always* in the same direction. Suppose you bring a 3220-lb. car to a stop with an acceleration of 5 ft. per sec.^{2}. The resultant force on the car must be

Force of acceleration

$$R = \frac{W}{g}a = \frac{3220}{32.2} \times 5 = 500 \text{ lb.}$$

This force acts between the tires and the road. Thanks to the principle of the lever (and a hydraulic principle not explained here), you do not have to exert a 500-lb. force on the brake pedal. A foot pressure of about 50 lb. will do the job.

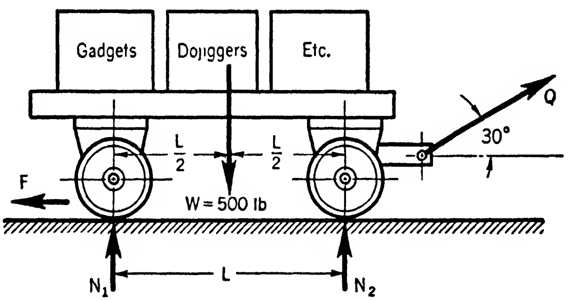


Fig. 395. Acceleration of a truck

Resultant and acceleration We have previously learned something about finding the resultant, and now we see that the resultant of all forces acting on a body is a key to the computation of the acceleration of the body. Reconsider the industrial truck of Fig. 385, redrawn in Fig. 395 where *all* forces acting are represented. Let the pull be $Q = 100$ lb. at 30° with the horizontal. The force $F = 50$ lb. represents the total frictional resistance, a force parallel to and opposite to the

direction of motion. Obviously, this truck, being on a level floor, will move neither up nor down; therefore, the acceleration in the vertical direction is zero and the sum of the vertical forces is zero (by equation 24, where $a = 0$). Consequently, the resultant force is necessarily horizontal, so that the resultant is equal to the horizontal sum. Thus, summing horizontal components, we find

$$\text{Horizontal sum} = Q \cos 30 - F = 100 \cos 30^\circ - 50$$
$$H = R = 86.6 - 50 = 36.6 \text{ lb.}$$

Now using equation 24, we get

$$R = \frac{W}{g} a \quad \text{or} \quad 36.6 = \frac{500}{32.2} a,$$

from which $a = 2.36$ ft. per sec.^{2}, the acceleration of the truck.

Equilibrium again

Naturally, after you get the truck up to your walking speed, you will not need to continue to pull with a force of $Q = 100$ lb., because the force you must then exert is just enough to maintain a steady speed and does not include the effort that would be necessary to cause acceleration. While it is moving at constant speed, it is in equilibrium. By the way, other forces remaining the same, what would be the pull Q to maintain constant speed? In this case, the horizontal sum of the forces is equal to zero. Hence, $Q \cos 30^\circ - F = Q \cos 30^\circ - 50 = 0$. From which, the value of Q is 57.7 lb.

Problems

1. In Fig. 395, suppose $Q = 80$ lb. and $F = 50$ lb. Determine the acceleration and the speed attained after 3 sec.

Ans. $a = 1.24$ ft. per sec.^{2}, $v = 3.72$ ft. per sec.

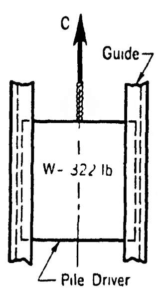


FIG. 396. LIFTING FORCE, PROBLEM 4

2. The driver of a 3220-lb. automobile allows it to coast to a stop from a speed of 50 ft. per sec. If it takes 150 sec., what is the average resisting force? *Ans.* $33\frac{1}{3}$ lb.

3. A train with a total weight of 3220 tons is brought from rest to a speed of 48 ft. per sec. in 60 sec. The total resistance to motion is 30,000 lb. What is the drawbar pull in pounds? (The drawbar pull is the pull of the locomotive on the train.) *Ans.* 190,000 lb.

4. A pile driver (Fig. 396) weighing 322 lb. is raised with an acceleration of 9.6 ft. per sec.^{2}. The total frictional resisting force in the guides is 200 lb. What is the force in the cable C? (Do not overlook the force of gravity in the vertical sum.) *Ans.* 618 lb.

5. A 3000-lb. automobile starts from rest down a 4% grade. (A 4% grade is one that rises 4 ft. in 100 ft. on the horizontal, Fig. 397.) The total constant resistance

to motion (brakes and engine) is $F = 100$ lb. What is the acceleration? Is the car speeding up or slowing down? What is its speed after 1 min.?

Hint. The resultant force acts parallel to the incline; therefore, sum forces in this direction. *Ans.* $a = 0.215$ ft. per sec.^{2}, speeding up, $v = 12.9$ ft. per sec.

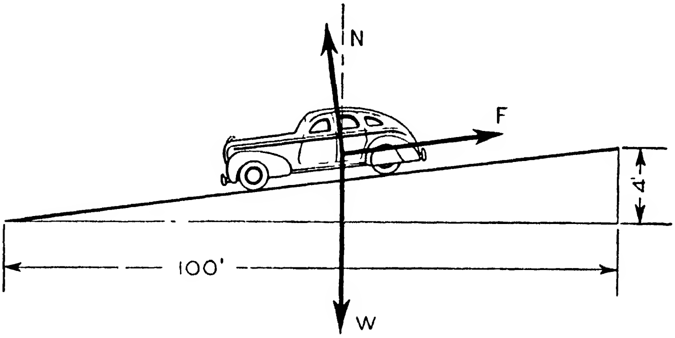


FIG. 397. ACCELERATION OF CAR, PROBLEMS 5 AND 6

6. A 3000-lb. automobile starts moving down a 4% grade (see problem 5) at a speed of 40 ft. per sec., when the total constant resistance to motion is $F = 150$ lb. What is the acceleration? Is the car speeding up or slowing down? What is its speed after 1 min.? How long before it stops?

Ans. $a = 0.322$ ft. per sec.^{2}, slowing down, $v = 20.7$ ft. per sec., $t = 2.07$ min.

MECHANICAL WORK

There are different kinds of work. Some people call it work to write or even to study a textbook. Be that as it may, the kind of work we shall now consider is called *mechanical work* and is done by a force. Remember the industrial truck? Well, here it is again *Force times distance*

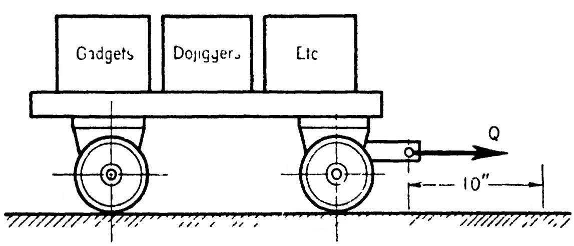


FIG. 398. WORK OF A HORIZONTAL PULL

in Fig. 398. Suppose that you are moving it with a constant horizontal force Q. The work that you do is the product of the force Q times the distance the truck moves. If you pull the truck 10 ft. with a constant force of 50 lb., the work is $10 \times 50 = 500$ ft-lb. *Example*
Thus, in its simplest form, we may state

(25) Work = Force times distance moved in the direction of the force.

For example, the work of an automobile engine develops as the pressure of the gases acting on the top of the piston forces the piston to move down. The force exerted by the gas then acts through the distance measured by the stroke of the piston. Because this gas force is variable, the analysis of the work of this engine is a little too complicated for our present discussion.

Work of a force

If you do not pull on the truck in a *horizontal* direction, then the work is the *horizontal* component of your pull times the distance the truck moves, but only when the truck moves in a *horizontal* direction. In other words, *only that force component which acts along the line of motion does work.* Thus, in Fig. 399, the horizontal

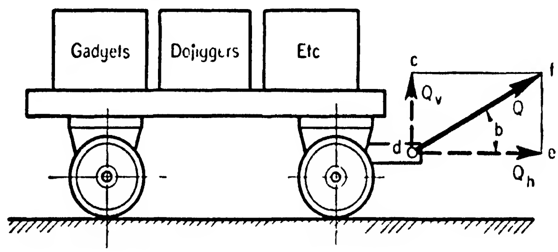


FIG. 399. WORK OF AN INCLINED FORCE

component of Q is Q_h, so that the work done in moving the truck 10 ft. is $Q_h \times 10$ ft-lb. The vertical component Q_v of Q does no work because its point of application does not move *in the direction of* Q_v. Neither would there be any work done by a sideways component of the pull, unless the truck veered off in that direction. We may now make our definition of the work of a force more complete.

Definition

$$(26) \quad \left\{\begin{matrix}\text{Work}\\ \text{of any}\\ \text{force}\end{matrix}\right\} = \left\{\begin{matrix}\text{component of the}\\ \text{force in the}\\ \text{direction of motion}\end{matrix}\right\} \times \left\{\begin{matrix}\text{distance through which}\\ \text{the point of application}\\ \text{of the force moves.}\end{matrix}\right\}$$

Thus, if you pull the truck 10 ft. with a force of $Q = 50$ lb. at an angle $b = 30°$ with the horizontal, as shown in Fig. 399, the work is computed as

$$\text{Work} = Q \cos b \times 10 = 50 \cos 30° \times 10 = 433 \text{ ft-lb.}$$

You pull just as hard as before, but you do less work, even though you do not feel less tired.* Carrying the idea to its limit, you may

* This does not prove that the minimum effort to move the truck from one place to another is done when the pull is horizontal. The effect of the frictional force will cause the optimum pull to incline a bit above the horizontal. However, the computations above are correct for the conditions stated.

huff and puff until you collapse exhausted in trying to move a stalled automobile, but if *it* does *not move*, you have done no work in the sense of this discussion.

WORK AND KINETIC ENERGY

Kinetic energy

There are many forms of energy; electrical, heat, work as just defined, potential, chemical, and others. Kinetic energy is that energy of a body which exists because of the body's motion. Suppose an airplane of weight W is moving with a speed of v ft. per sec.; then, its kinetic energy is

$$KE = \text{kinetic energy} = \frac{Wv^2}{2g}. \tag{27}$$

Of course g is the acceleration of gravity. Thus, if a plane weighs $W = 10{,}000$ lb. and is moving at 300 ft. per sec. (these units *must* always be used in equation 27 when g is used in ft. per sec.^{2}), its kinetic energy is

$$KE = \frac{Wv^2}{2g} = \frac{10{,}000 \times 300^2}{2 \times 32.2} = 13{,}980{,}000 \text{ ft-lb.}$$

Conservation of energy

There are many laws which Congress and the state legislatures have not passed. One of them is the *Law of the Conservation of Energy*. This law says that energy is indestructible and that it cannot be created. It may disappear in one form, but, if so, it reappears in some other form. We cannot prove this law. You would be surprised at how many scientific principles cannot be *proved*. You rightly ask, "Then how do we know that the law is true?" and I answer, "Because we know of no instance where the law has been violated and because we have abundant scientific verification." There may be circumstances beyond our experience (out in celestial space, for example) where this law does not hold. Nevertheless, on this insignificant planet, it does yeoman service for the engineer and scientist and has yet to be found wrong.

Net work, kinetic energy

If we apply this law to a system where there are just two kinds of energy (work and kinetic energy), the only conclusion possible is that

$$\text{Net work done on a body} = \text{change of kinetic energy.}$$

This statement means that energy which "disappears" as work reappears as kinetic energy. Let $Wv_1^2/2g$ be the original kinetic energy and $Wv_2^2/2g$ be the final kinetic energy of some body

after a net amount of work equal to $R \times s$ has been done. Then we may write

$$R \times s = \frac{Wv_2^2}{2g} - \frac{Wv_1^2}{2g}. \tag{28}$$

In this equation R is the resultant force on the body and s is the distance moved by the point of application of the force in the direction of the force. You have probably noted from previous paragraphs that *a body moving in a straight path always moves in the direction of the resultant force.*

Resultant motion

We can now make use of that truck again (Fig. 400). On page 480, we found that for $Q = 100$ lb., $F = 50$ lb., $W = 500$ lb., and $b = 30°$, the resultant $R = 36.6$ lb. Since the resultant force

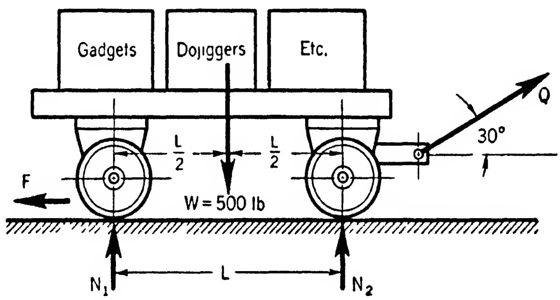


FIG. 400. NET WORK

(which acts horizontally in the direction of motion) times the distance traveled gives the *net work*, that is, the work considering *all forces* which are acting, we can compute the net work for a movement of the truck of, say, 20 ft. Assume that the truck starts from rest, so that $v_1 = 0$ and $KE_1 = 0$. Then we may write

Work and acceleration

$$R \times s = \frac{Wv_2^2}{2g} = 36.6 \times 20 = \frac{500\, v_2^2}{2 \times 32.2}. \tag{29}$$

From this relationship we find $v_2 = 9.71$ ft. per sec., the speed which the truck will attain in a distance of 20 ft. if you pull with $Q = 100$ lb. as specified. This is pretty fast moving, even if you are in a tremendous hurry, since 9.71 ft. per sec. is better than 6.5 m.p.h. (Recall that 4 m.p.h. is brisk walking.) What would probably happen is that before the truck had moved 20 ft., you would let up on your pull, not work so hard, and be satisfied with a speed of 2.5 to 3 m.p.h. Why not compute the distance s to get up a speed of 3 m.p.h. or 4.4 ft. per sec.? You ought to find it to be 4.12 ft.

Problems

1. The truck of Fig. 400 is moved 20 ft. while the frictional resisting force $F = 50$ lb. What is the work done to overcome friction? *Ans.* 1000 ft-lb.

2. The conditions described in problem 1 are assumed and a pull $Q = 100$ lb. at $b = 30°$ with the horizontal is exerted. What work W_Q is done by the force Q? What work W_W is done by the weight W? Using the result from problem 1, determine the net work W'_n and compare with the net work obtained from equation 28, p. 484; that is, compare with $R \times s = 732$.

Ans. $W'_Q = 1732$ ft-lb., $W_W = 0$, $W'_n = 732$ ft-lb.

3. The train described in problem 3, page 480 (weight = 3220 tons, total resistance to motion = 30,000 lb.) is brought from rest to a speed of 48 ft. per sec. by a drawbar pull of 190,000 lb. How far does the train travel in the meanwhile? (Be sure that R = sum of forces in the direction of motion.) *Ans.* 1440 ft.

4. In Fig. 400, $Q = 80$ lb. and $F = 50$ lb. The truck is brought to a speed of 3.72 ft. per sec. from rest. How far does the truck move during this period. (See problem 1, page 480.) *Ans.* 5.58 ft.

5. The driver of a 3220-lb. automobile allows it to coast to a stop from a speed of 50 ft. per sec. on a level road. The constant resisting force (= resultant force) = $33\frac{1}{3}$ lb. How far does the car go? (See problem 2, page 480.) *Ans.* 3750 ft.

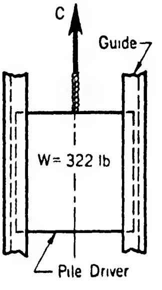


FIG. 401. FORCE ON PILE DRIVER, PROBLEM 6

6. The pile driver of Fig. 401 is released and falls through a distance of 10 ft. to strike a pile. The constant resistance is 100 lb. and the force $C = 0$. With what velocity does it strike the pile? *Ans.* 21 ft. per sec.

7. A 3000-lb. automobile is moving down a 4% grade at a speed of 40 ft. per sec. when the total constant *resistance* to motion is $F = 150$ lb. What is its speed after it has moved 2000 ft.? (See problem 6, page 481.) *Ans.* 17.7 ft. per sec.

8. A 20-lb. projectile from an antiaircraft gun is to be hurled vertically to an altitude of 25,000 ft. What must be its muzzle velocity, if air resistance is neglected? (Note that the resultant force on the projectile after it leaves the gun is the force of gravity.) *Ans.* 1270 ft. per sec.

9. Suppose that Galileo had dropped a 10-lb. stone from the top of the Leaning Tower of Pisa, 179 ft. above the ground. With what velocity would it have struck the ground? (Air resistance neglected, the resultant force is the weight of the stone.) *Ans.* 107.1 ft. per sec.

MOMENTUM

Conservation of momentum

Another law, one that solves many problems which could not be solved by any other principle, is the *Law of the Conservation of Momentum.* If your automobile collides with another car moving in the same straight path, the only thing which is unchanging for that instant before you lose consciousness is momentum; that is,

the momentum of your car plus the momentum of the other car before collision is equal to the sum of the momentums of the two wrecked cars the instant after impact. In other words, *any system of bodies has a constant total momentum as long as no external force acts on the system.*

Loss of momentum

In the instance of the automobile crash, immediately after impact occurs the cars begin to slide about on the road. Since the frictional forces during this sliding constitute an *external* force, momentum is not conserved during this period immediately after the crash, but it is reduced finally to zero when the cars come to rest. On the other hand, the force *between* the cars at the instant of impact is not an external force. It is internal to the system of bodies (the two cars), the force on one car being equal and opposite to the force on the other car.

R
m

Definition of momentum

Assuming that you are now interested in knowing what momentum is, it will not be difficult to inform you.

(30) $$\text{Momentum} = \text{mass} \times \text{velocity}.$$

Simple enough. Mass, as far as the units we are using are concerned, is the weight W divided by the acceleration of gravity g; that is,

(31) $$\text{Mass} = \frac{W}{g}.$$

Therefore, by substitution we may write

(32) $$\text{Momentum} = \frac{W}{g}v.$$

Suppose a body A has an initial velocity of v_{A1} ft. per sec., a final velocity of v_{A2} ft. per sec., and a weight of W_A lb. Then the change of momentum of the body is the second momentum minus the first, or

(33) $$\frac{W_A}{g}v_{A2} - \frac{W_A}{g}v_{A1} = \frac{W_A}{g}(v_{A2} - v_{A1}).$$

Similarly for a body B, the change of momentum is

(34) $$\frac{W_B}{g}v_{B2} - \frac{W_B}{g}v_{B1} = \frac{W_B}{g}(v_{B2} - v_{B1}).$$

Impact and momentum

If these two bodies collide, the decrease (or increase) of momentum of one body will be equal to the increase (or decrease) of momentum of the other body. In short, the algebraic sum of the

above changes is equal to zero, according to the Law of the Conservation of Momentum.

$$\frac{W_A}{g}(v_{A2} - v_{A1}) + \frac{W_B}{g}(v_{B2} - v_{B1}) = 0$$

or

$$\frac{W_A}{g} v_{A1} + \frac{W_B}{g} v_{B1} = \frac{W_A}{g} v_{A2} + \frac{W_B}{g} v_{B2}. \qquad (35)$$

This equation is true for bodies moving in the same straight line, either in the same or in opposite directions. Equation 35 is easy to remember because it says that *the momentum of the bodies before impact is equal to the momentum of the bodies after impact.*

Recoil of a gun

Let a 6440-lb. artillery gun A fire a 64.4-lb. shell B. The muzzle velocity of the shell is 2500 ft. per sec. What is the initial velocity of recoil of the gun?

Analyzing this problem, we note that the gun A and the shell B have an initial velocity of zero, $v_{A1} = 0$, $v_{B1} = 0$. Applying equation 35, we get

$$0 = \frac{6440}{g} v_{A2} + \frac{64.4}{g} \times 2500$$

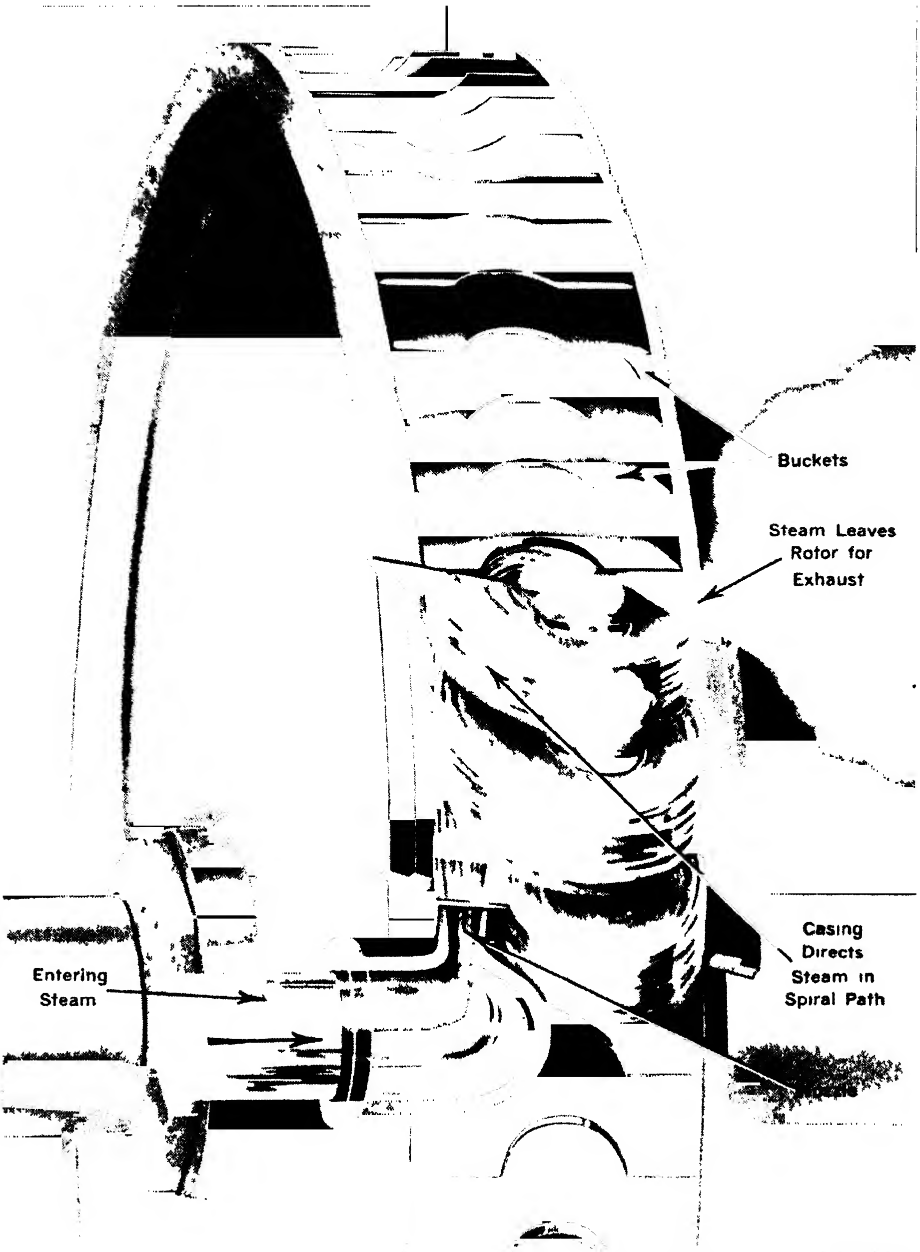
or

$$v_{A2} = -25 \text{ ft. per sec.}$$

The negative sign in this answer simply means that the direction of the velocity of the gun is opposite to the direction of the velocity of the shell, as we already know. The velocity of 25 ft. per sec. is the velocity of the gun the instant after firing and before the recoil mechanism begins to act.

Rocket propulsion

This same principle is used for rocket propulsion. You are familiar with the "sky rockets" which afford fun on July 4th. The burning of the powder charge causes gases to rush out of the back end of the rocket, the gases thus changing their momentum from nothing to something. The change of momentum of the gases results in a change of momentum of the rocket, and away it goes. Observe that, as in the case of the gun and shell, both the rocket and gases are initially at rest and their momentum is therefore zero. Since momentum, like force, is a vector quantity, when the gases gain momentum in, say, a downward direction, the rocket must gain momentum in an upward direction, provided some external force does not hold it in place. These mutual gains of momentum in opposite directions are essential if the sum of the momentums is to remain constant (and zero in this case). This



Courtesy Terry Steam Turbine Company

FIG. 402. CHANGE OF MOMENTUM PRODUCES IMPULSE OF TURBINE ROTOR

This illustration shows the rotor of a re-entry type of turbine. The steam leaves the nozzle and passes into a bucket. The bucket changes the direction of travel of the steam and directs it into a similar channel in a casing. The casing, which is not shown in this picture, is the cover of the rotor. A channel in the casing

principle is also responsible for the "bazooka" gun which has proved of great use as an antitank weapon. It also explains the basic idea that accounts for the operation of a turbine. (See Fig. 402.)

Jet propulsion

A recent contribution of this principle to engineering development is the *jet propulsion* of airplanes. In the operation of this type of plane, the air is first compressed in a rotary compressor. Then fuel is mixed with the air and burned. The heat released by the fuel adds to the energy of the air and increases its volume and temperature. Part of the hot air passes through a turbine which drives the compressor, the remainder rushes through a nozzle, leaving the nozzle as a high-velocity jet, pointed toward the rear. This high velocity corresponds to a large momentum. Since the momentum of the air entering the compressor is less than the momentum of the jet leaving it, the increase in momentum results in a force which moves the plane forward. The processes described are continuous.

Problems

1. A gun weighing 160,000 lb. fires a 900-lb. projectile whose muzzle velocity is 1400 ft. per sec. What is the initial velocity of recoil? *Ans.* 7.85 ft. per sec.

2. A 10-lb. box of sand A, at rest and hanging from a long rope, is struck by a 0.2-lb. bullet B which has a velocity of 2000 ft. per sec. in a horizontal direction. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after impact? (Note that the momentum of the bullet before impact is equal to the momentum of the bullet and box after impact.)
Ans. 39.2 ft. per sec. in the same direction as the bullet.

3. A 10-lb. box of sand A, moving toward the left on a horizontal plane with a velocity of 5 ft. per sec., is struck by a 0.2-lb. bullet B which is moving toward the right with a velocity of 2000 ft. per sec. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after impact and before friction of the box on the plane begins to reduce the momentum?

Hint. Assume rightward direction as positive. Then $v_{A1} = -5$ ft. per sec. and $v_{B1} = +2000$ ft. per sec. in equation 35.
Ans. $+34.3$ ft. per sec. toward the right.

Caption to Fig. 402

redirects the steam back into a bucket on the rotor, where a path similar to the first pass of the steam is repeated. In its journey through the turbine, the steam travels along a spiral-like path. Because the direction of motion and the speed of the steam are changed, there is a change in its momentum. A change of momentum is equal to an impulse, where impulse is defined as force *times* time. Thus, the change of momentum results in a force acting on the rotor, a force which causes the rotor to turn. If the shaft of this rotor is connected to another machine which the turbine drives, useful work will be done by this force in driving the other machine. This type of turbine is manufactured in small sizes only. (See Fig. 407 on p. 500 which shows the rotor of a 40,000-hp. turbine.)

4. A 10-lb. box of sand A, moving toward the left on a horizontal plane with a velocity of 50 ft. per sec., is struck by a 0.2-lb. bullet B which is moving toward the right with a velocity of 2000 ft. per sec. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after the impact?

Ans. $-$ 9.8 ft. per sec. toward the left.

HAVE YOU LEARNED ALL?

Further study

No! Not even if you thoroughly understand every application in this treatment of mechanics. There is much more left to say than is said here. Good books on the subject of analytic mechanics generally contain some 300 to 400 pages and more, yet even these good books do not cover the subject in all of its aspects. If you wish to become an engineer, further study of this subject will yield most useful dividends for you. As a result of the knowledge gained here, you should not find further progress too difficult. At least you have made a good start.

Thermodynamics

THE CONTROL OF HEAT, COLD, AND POWER

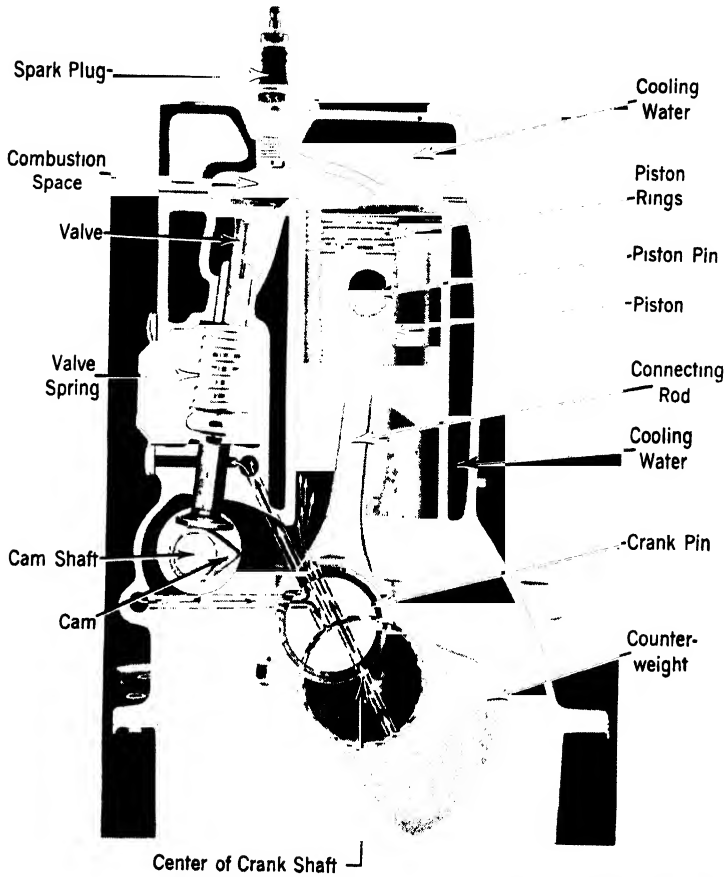
This discussion will deal with a phase of science which goes by the impressive name of *thermodynamics*. The word thermodynamics is derived from two Greek words, *therme* meaning heat, and *dynamis* meaning power. Thus, although the name of this science has a most forbidding appearance, it is simply a study of heat and power. One of the engineer's principal interests in heat is how to get work and power out of it. Our sources of heat are the sun and such fuels as coal, wood, and gasoline. At this writing, the heat from the sun is not used directly for producing work and power in commercial quantities, but it is used indirectly, as for example, in the hydraulic power plant. Without the action of the sun in evaporating water into the atmosphere, from which it is precipitated as rain, the water reservoirs above our power plants would never be filled. *Sources of heat*

HUMAN BODY AS AN ENGINE

Fuel for body The most important engine, to you and me, at least, is the human body, which is quite similar to but much more complicated than mechanical machines for transforming fuel energy into work and power. The food eaten is the fuel for the human body. Instead of burning (oxidizing, see pp. 65 and 80) the fuel at a high temperature, our bodies are luckily designed to "digest" (oxidize) the fuel at 98.6°F., diverting its energy to the body, whence by muscular movements we complete the conversion of heat in the fuel into work. If we just sit still, we must still supply some fuel to keep the internal bodily processes going. This fuel supply corresponds to that which must be supplied to the automotive engine while it is idling. A difference is that we can stop the automotive engine, and save the gasoline fuel, with the expectation of starting it again as desired. This procedure does not yield satisfactory results when applied to the human engine.

Need for oxygen

To make the heat in the fuel available, oxygen is necessary. The automotive engine has a suction stroke during which air is drawn into the engine. The oxygen in the air mixes with the gasoline, the fuel is consumed, and some of its heat is converted into work. The burned gases, containing carbon dioxide (see page 84), then pass through the exhaust. Similarly, the human engine sucks the air into the lungs, where some of the oxygen is used immediately,



Courtesy Studebaker Corporation

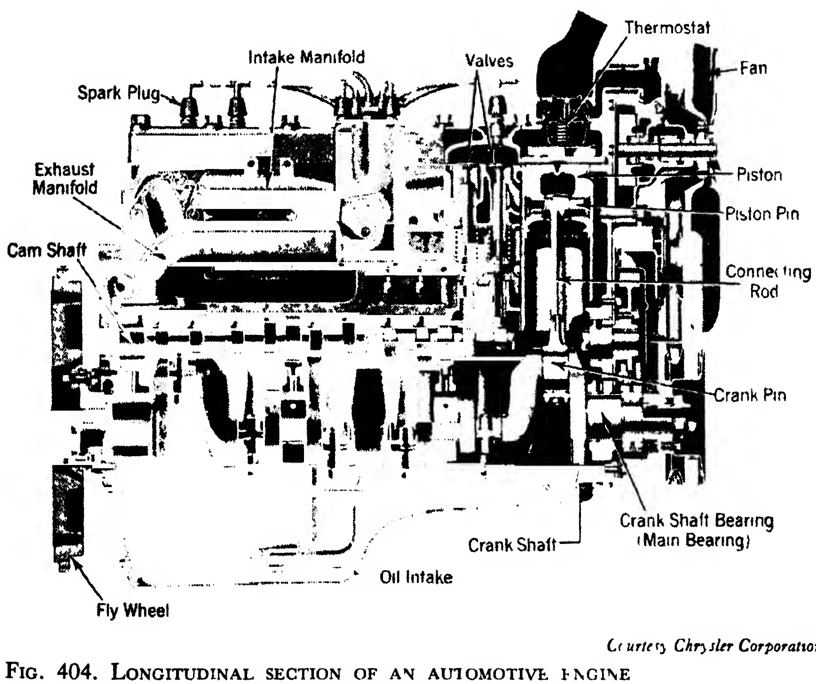
FIG. 403. TRANSVERSE SECTION OF AN AUTOMOTIVE ENGINE

This illustration shows a typical automotive engine. The combustion space is shaped to minimize detonation. The second valve is directly behind the one shown, as may be seen in Fig. 404. As the cam shaft rotates, the various cams on it open each valve at the right moment, and the valve springs close the valves as the cams permit. The arrow chains that are seen in this picture indicate the flow of oil. Study this illustration until you are thoroughly familiar with the names of the parts.

and the "burned" gases, containing carbon dioxide, are expelled (exhausted) from the lungs. This makes the analogy clear although it is an oversimplified explanation of the body processes.

THE GASOLINE ENGINE

Since a gasoline engine, one of a class of engines which are called *internal combustion engines*, is a complete power plant in itself, let us look into the details of its operations. (See Fig. 403 and Fig. 404.) A young man named Otto (1832–1891) first suggested this plan of using fuel for producing mechanical work. ***Automotive engines***



Courtesy Chrysler Corporation

FIG. 404. LONGITUDINAL SECTION OF AN AUTOMOTIVE ENGINE

This view gives a better idea than Fig 403 of the relation of certain parts Observe the two valves, side by side, in the No. 2 cylinder. The construction of the cam shaft and of the crankshaft is clearer than in Fig 403 The intake manifold and the exhaust manifold are seen here. The intake manifold is a series of passageways that lead the fuel-and-air mixture into the various cylinders. Similarly, the exhaust manifold leads the burned gases from the cylinder into the muffler (not shown), whence the gases are discharged into the atmosphere through the exhaust pipe. The thermostat causes a variation in the size of the opening to the water jackets about the cylinders and about the combustion spaces, so that more or less water will flow into these spaces depending upon whether the temperature is rising too high or is too low.

Getting fuel into engine

In Fig. 405 *a*, the piston is at the top of the stroke, the intake valve is open. As the engine turns over, the piston moves down and draws in air and fuel through the intake valve. The intake valve is connected by a passage called the *intake manifold* to the carburetor where the gasoline and air are mixed. In Fig. 405 *b*,

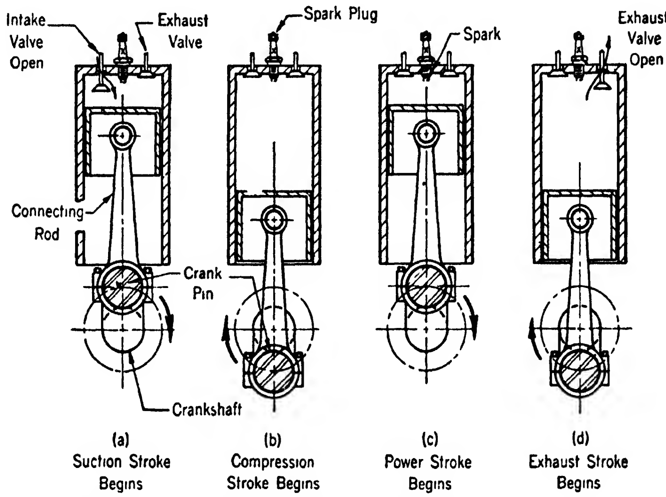


FIG. 405. OPERATION OF AN AUTOMOTIVE TYPE OF ENGINE

the suction stroke is complete, the cylinder is full of fuel and air, and the piston is about to start the compression stroke. Note that *both valves are now closed.*

Consuming the fuel

As the piston moves up in Fig. 405 *b*, the fuel and air are compressed to a pressure of about 140 lb. per sq. in. When the piston has reached the top position, as in Fig. 405 *c*, a spark is caused to jump the gap of the spark plug. This spark ignites the fuel which burns very rapidly. The burned gases become quite hot and therefore tend to expand. However, if little or no expansion is permitted, the pressure rises rapidly instead. Thus, with a large pressure acting on the top of the piston in Fig. 405 *c*, the piston is forced down on the power stroke. When it reaches the bottom of the stroke, Fig. 405 *d*, the exhaust valve opens. As the piston moves up in Fig. 405 *d*, the burned gases are pushed out through the exhaust valve. Finally, when the piston reaches the top position, the exhaust valve closes, the intake valve opens, and the operation starts over from the condition pictured in Fig. 405 *a*. Present-day airplane engines operate on this same cycle of events.

Valve action

WORK AND POWER OF THE GASOLINE ENGINE

Thermal efficiency

In order to get work out of a gasoline engine, fuel must be burned. Generalizing, we may say that *heat must be supplied* in order to complete a thermodynamic cycle which is designed to produce work. As an example, the events which compose the gasoline-engine cycle described above constitute a thermodynamic cycle. The work output of a cycle is often expressed as a percentage number called the *thermal efficiency* of the cycle or engine. Thus, if the efficiency of a cycle or engine is 20 per cent, the work done is 0.20 times the heat supplied. Let Q_s = the heat supplied, e = the thermal efficiency, and W = the work; then

$$\text{Work} = \text{efficiency} \times \text{heat supplied},$$

or, we may write

$$W = e(Q_s). \tag{1}$$

For a gasoline engine, the *heat supplied*, Q_s, is the heating value of the fuel. The heating value of a fuel is approximately the heat that we can get out of it when it is burned.

How to measure heat

Now you must know that the unit which we use for measuring heat is a British thermal unit (abbreviated B.t.u. and commonly called a *B t u*). *A British thermal unit is the amount of heat necessary to raise the temperature of 1 lb. of water through 1° temperature on the Fahrenheit scale* (see page 66 and compare in value with the calorie). The Fahrenheit scale is the one which the weatherman uses to tell us how cold or hot it is.

Heat is a form of energy and a B.t.u. is a small unit of energy. However, the B.t.u. is not so small a unit of energy as the foot-pound with which you became acquainted in the study of forces (page 481). It takes *778 ft-lb. to make a British thermal unit;* that is,

$$\text{No. of B.t.u.} \times 778 = \text{no. of ft-lb.}; \quad \text{or} \quad \frac{\text{no. of ft-lb.}}{778} = \text{no. of B.t.u.} \tag{2}$$

Heating value

Let us get back to heating value. In the case of liquid and solid fuels, heating values are generally expressed as so many B.t.u. per pound. For gaseous fuels, they are usually expressed in B.t.u. per cubic foot. For example, ordinary gasoline will have a heating value of about 20,000 B.t.u. per lb. The heating value of natural gas is about 1000 B.t.u. per cu. ft.

If, therefore, your automobile engine has a thermal efficiency of 20 per cent (not a bad value for this type of engine in ordinary

operation) and if you feed it 1 lb. of gasoline, the work, as obtained from equation 1, is

$$W = e(Q_s) = 0.20 \times 20{,}000 = 4000 \text{ B.t.u. per lb.}$$

Computing work

Since gasoline weighs about 6 lb. per gallon, the work obtained from a gallon is $6 \times 4000 = 24{,}000$ B.t.u., approximately the work required to move your car 15 miles when it is driven at the proper speed to average 15 miles per gallon. (How many ft-lb. is 24,000 B.t.u.? *Ans.* 18,672,000.) Let us now determine the average tractive force necessary to move the car 15 miles on level road for the conditions just stated. This tractive force is that average force which acts on the car through 15 miles. From the section on work (page 481), we recall that

$$\text{Work} = \text{force} \times \text{distance}.$$

Since the work done in moving the car is 24,000 B.t.u. = 18,672,000 ft-lb., we may write

$$18{,}672{,}000 = \text{force} \times 15 \times 5280,$$

where the constant 5280 ft. per mile is inserted to convert 15 miles to feet so that the units on each side of the equation are the same. Solving for the force, we find

$$\text{Force} = \frac{18{,}672{,}000}{15 \times 5280} = 236 \text{ lb.}$$

Hence, a steady pull of 236 lb. would be sufficient to keep the car moving at a uniform speed under the assumed conditions.

What is power

Many people are confused by the difference between work and power; nevertheless, the real distinction is very simple. *Power is a rate of doing work.* Thus, if that pound of gasoline is used to do 4000 B.t.u. of work in 1 *min.*, the power is different from the power corresponding to 4000 B.t.u. of work done in 1 *hour.* Put it another way: suppose you drive up a mile-long hill in 1 min.; the power required to accomplish this is very much greater than if you take 10 min. to drive this mile, even neglecting the effect of the greater air resistance at the higher speed.

Horsepower

A common unit of power is a horsepower. It is defined as work at the rate of 33,000 ft-lb. in 1 min., or work at the rate of 2545 B.t.u. in 1 hour. This knowledge puts us in a position to calculate the horsepower of a car. Suppose you drive so that you use 3 gal. of gasoline an hour and get 24,000 B.t.u. of work from each gallon; then

$$\text{Work per hr.} = 3 \times 24{,}000 = 72{,}000 \text{ B.t.u. per hr.}$$

$$\text{Horsepower} = \frac{72{,}000}{2545} = 28.3,$$

which is the horsepower (hp.) being developed and delivered to the rear wheels of the automobile when it is driven at the speed which consumes 3 gal. of gasoline per hour. For an average of 15 mi. per gal., the consumption of 3 gal. of fuel per hour means $15\,\frac{\text{mi.}}{\text{gal.}} \times 3\,\frac{\text{gal.}}{\text{hr.}} = 45\,\frac{\text{mi.}}{\text{hr.}}$ or 45 m.p.h. While the engine in your car may be *capable* of developing 100 hp., it is actually developing 28.3 hp. for the given conditions of operation.

Problems

1. An engine with a thermal efficiency of 15% is supplied with 40 B.t.u. How much work is obtained? *Ans.* 6 B.t.u.

2. An engine with a thermal efficiency of 25% is supplied with 100,000 ft-lb. of heat. How much work is obtained? *Ans.* 32.2 B.t.u.

3. An engine uses 100 lb. per hr. of a fuel which has a heating value of 2000 B.t.u. per lb. If its thermal efficiency is 18%, what horsepower is being developed? *Ans.* 14.15 hp.

4. An engine developing 100 hp. is using 1 lb. per min. of fuel which has a heating value of 19,000 B.t.u. per lb. What is its thermal efficiency? *Ans.* 22.35%.

5. The fuel used by a natural gas (not gasoline) engine has a heating value of 1000 B.t.u. per cu. ft. If it develops 500 hp. with a thermal efficiency of 25.45%, how many cubic feet of gas are needed per hour? *Ans.* 5000 cu. ft. per hr.

6. Suppose you drive a car so that the engine is delivering 80 hp. with a thermal efficiency of 20%. The gasoline has a heating value of 20,000 B.t.u. per lb. How many gallons of gasoline are used per hour? Gasoline weighs about 6 lb. per gal. (*Note:* This would be roughly equivalent to driving a 3000-lb. car uphill at 60 m.p.h. with a wide-open throttle.) *Ans.* 8.48 gal. per hr.

7. A bomber whose engines develop 8000 hp. with an efficiency of 25% makes a 6-hour trip. The gasoline has a heating value of 20,200 B.t.u. per lb. and weighs about 6.1 lb. per gal. How many gallons are needed for the trip? *Ans.* 3970 gal.

CAN EFFICIENCY OF ENGINES BE INCREASED?

Ideal efficiency

Yes, but not to the extent that some people think. The efficiency of an *ideal* internal combustion engine, that is, an engine with no losses in friction or heat, is expressed by the equation

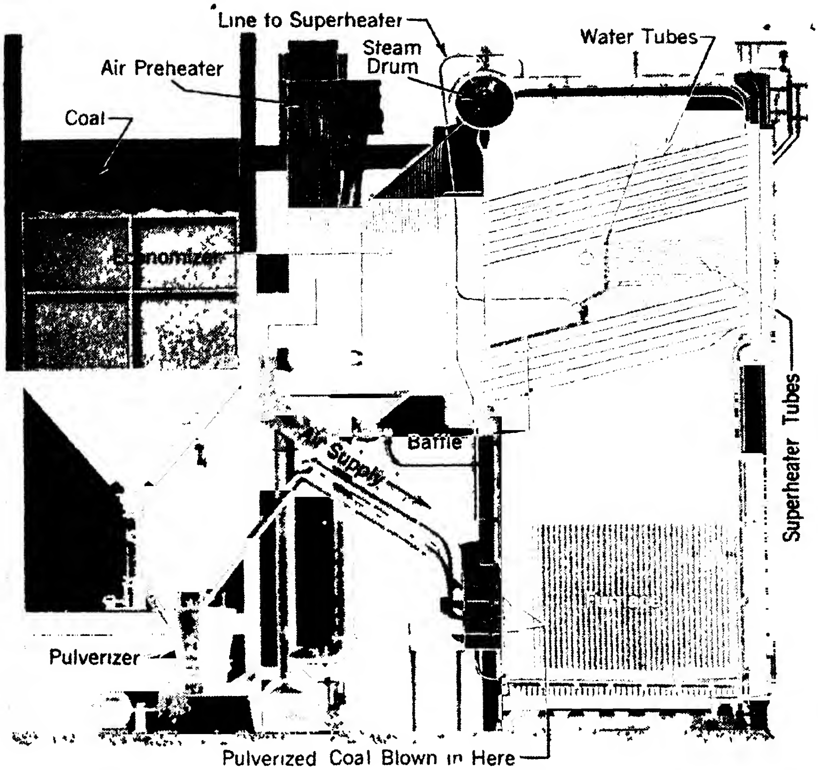
$$e = 1 - \frac{1}{r^{k-1}}, \tag{3}$$

where r stands for the compression ratio and k is about equal to 1.3. The *compression ratio* is equal to the volume of gases in the cylinder

when the piston is at the bottom of the stroke (position, Fig. 405 *b*) divided by the volume of gases in the cylinder when the piston is at the top of the stroke (position, Fig. 405 *a*). If we substitute different values of r into the equation and compute the efficiency, we find that the efficiency of this ideal engine increases as the compression ratio r increases. Since experience shows that changes which increase an ideal efficiency also increase the actual efficiency of an engine, we correctly expect the efficiency of an actual engine to increase when the compression ratio is increased.

Engine knock "Then," the reader may well say, "why don't you go ahead and run the compression ratio up so that I can get at least twice as many miles to the gallon of gasoline, or maybe three times as many?" The reason we cannot do this is because gasoline "knocks" (detonates) when the compression ratio is too high for it. This knock occurs because the fuel burns up too quickly. Since some fuels ignite easier than others and burn quicker in an engine cylinder, some fuels knock more than others. The knocking properties of a gasoline have been expressed by a number called an *octane number*. (See page 115.) For automotive engines in current use (1944), the compression ratio averages about 6.5. For this compression ratio, gasoline with an octane number of about 70 to 75 is ordinarily adequate to eliminate objectionable knocking. Gasoline of higher octane than necessary may be used in any particular engine, but there will be little if any gain in the efficiency of the engine, *unless its compression ratio is also raised*. If the gasoline has too low an octane number for an engine, the engine will knock excessively when it is hot and when the throttle is opened too wide too quickly. Excessive knocking will accelerate wear and it often results in broken pistons or other engine parts.

100 Octane When the octane numbers were established, the highest number attainable in the laboratory was 100; but fuel with an octane number this high was so expensive that none was commercially available. During World War II, when the side with the most efficient engines was able to build the most effective aircraft, the search for high-octane fuels for use in airplane engines was so accelerated that fuels with octane numbers greater than 100 may soon become a commonplace. If this happens, our automotive engines will be designed with a much higher compression ratio and we can expect as a practical result a considerable increase in miles per gallon, that is, in the efficiency of the engine.



Courtesy Babcock and Wilcox Company

FIG. 406. STEAM-GENERATING UNIT

This illustration shows a boiler that is complete with accessories. When there are accessories, such as economizer, air preheater, and pulverizer, the device is usually called a steam-generating unit, rather than a boiler. After a brief survey of this illustration, finish the study of steam power plants and then come back to this illustration and study it in detail.

A *superheater* (see picture) heats the steam to temperatures above its boiling point, the steam coming to the superheater from the steam drum. The *economizer* heats the water entering the boiler with heat from burned gases going to the smokestack. This is a heat-saving process because otherwise the heat would be wasted out of the stack. The *air preheater* is also a heat-saving device, as it uses some of the heat still in the burned gases, after they have left the economizer, for the purpose of heating the air which supports combustion in the furnace. Since this air is necessarily heated in the furnace during the combustion process, all heating done outside of the furnace with heat that would otherwise be lost represents a saving in fuel costs.

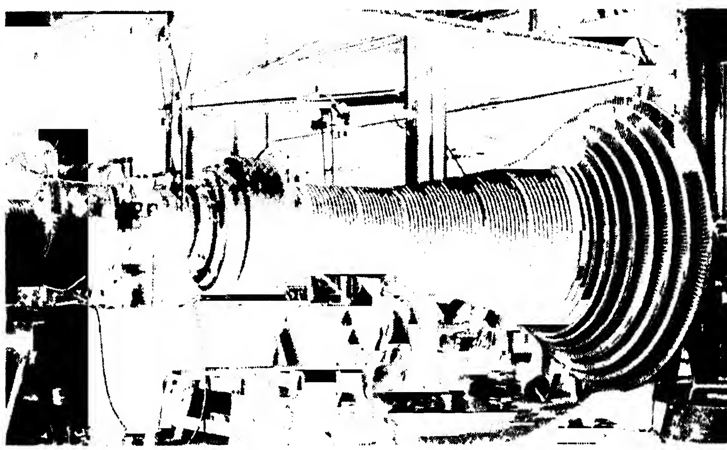
Not all steam-generating units consume pulverized coal. That is a feature of this particular unit. Since pulverized coal burns much more rapidly than lump coal, the capacity of a given size of furnace for generating steam is greater with pulverized coal.

More miles per gallon

For the sake of your pocketbook, observe that the use of high-octane gasoline is of no advantage unless the engine is designed with a compression ratio to fit the octane number. If your engine does not knock on gasoline of a certain octane number, do not buy a higher octane, and you might even try a lower octane gasoline.

HOW TO GET WORK OUT OF STEAM

To get work out of steam, the first problem is to produce some steam. Steam is obtained by evaporating water in a *boiler* or *steam generator* (Fig. 406). Then we need a steam *turbine* (Fig. 407 and



Courtesy Allis-Chalmers Mfg Company

FIG. 407. ROTOR OF A TURBINE

The rotor of a turbine is the rotating element The concentric rings which you see are rows of turbine blades. The steam is directed into these blades and therein it undergoes a change of momentum (see p. 489). Recalling that a change of momentum per unit of time represents force, we see that a force on the turbine blades is produced, thereby turning the rotor and doing work. This rotor is for a 40,000-hp. unit.

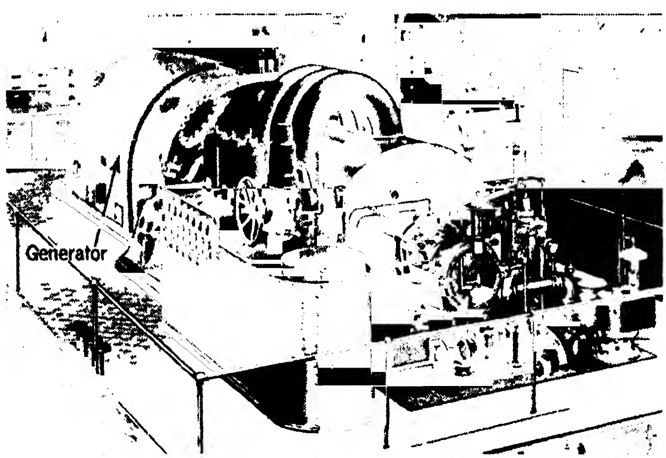
Fig. 408) or a *steam engine* (Fig. 410). These machines are designed to take high-pressure, high-temperature steam, convert some of the energy of the steam into work, and discharge low-pressure, low-temperature steam.

Condensing exhaust steam

The low-pressure steam from the turbine or engine may be discharged into the atmosphere, but in the best regulated power plants, it is discharged into a *condenser* (Fig. 409). A condenser is a device that will condense the steam by using cold water, say, from some convenient lake or river. After the steam is condensed into water, it is returned to the boiler through *pumps*, and the cycle is repeated.

Steam power plant

Combining all of these devices we get a steam power plant as represented diagrammatically in Fig. 411. In this layout, we are assuming that all steam and water lines as well as the turbine itself are perfectly insulated so that there is no loss of heat. Observe that the steam is expanded to a pressure which is below atmospheric pressure (under vacuum) when a condenser is used; hence, the water (condensate) leaving the condenser is first pumped, by a pump called the *condensate pump* (Fig. 411), to atmospheric pressure and into a reservoir. Another pump, called the *feedwater pump*, delivers the water into the boiler. (See Fig. 412 for an idea of the physical arrangement of the elements of a power plant.)



Courtesy Allis-Chalmers Mfg Company

FIG. 408. MEDIUM-SIZE TURBINE

This is a view of an installed turbine of 10,000-hp. capacity. The linkage system at this end of the turbine is part of the governor mechanism which maintains the speed of the turbine constant at 3600 r.p.m. The curved pipes entering the top of the front part are steam lines. The condenser is below this floor level and is not in sight.

First law

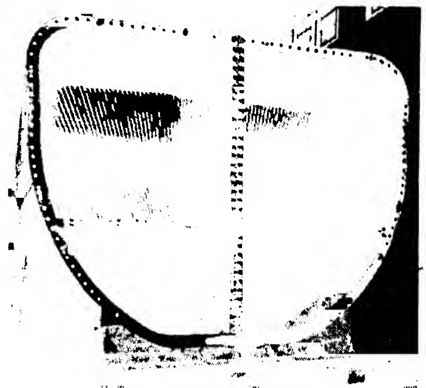
Remember the Law of Conservation of Energy (pages 55 and 483)? Applied to heat engines and heat cycles, this law goes by the formidable name of the *First Law of Thermodynamics*. Let us review some statements of this law; thus

1. Energy is indestructible.
2. Energy may be converted from one form (such as heat) to another form (such as work).
3. The energy entering a device or substance is equal to the energy leaving it (if none is stored).

Actually, the second and third statements are conclusions that follow from the first statement, but they help to simplify our mental processes.

Applying First Law

In applying the *First Law* to a cycle, the only energy quantities with which we are concerned are heat and work. Then if all energy is to be accounted for, if none is stored or taken from storage, the



Courtesy Foster-Wheeler Corporation

FIG. 409. SURFACE CONDENSER

This view shows the end of a condenser with the cover removed. Just inside of the end is a sheet of metal, called the *tube sheet*, to which water tubes are attached. The lines seen on the tube sheet sloping slightly toward the center position are made by the tubes, which do not show up distinctly on this small reproduction. Water flowing through these tubes picks up heat from the exhaust steam surrounding the tubes on the other side of the tube sheet, with the result that the steam condenses into water. Observe in Figs. 411 and 412 how the units described in Figs. 406, 407, 408, 409, and 410 are combined to make up a power-plant cycle.

energy entering the steam and water is equal to the energy leaving the steam and water. Figure 411 depicts these energy quantities by heavy arrows. The heat supplied Q_s is the energy added to the water and steam from the furnace fires in the boiler.

Output and input

The *output* work W_{out} is the work delivered by the turbine; it is energy leaving the steam. The heat Q_r is heat rejected from the steam to the circulating cooling (condenser) water, and it is carried away by this water. The *input* work quantities W_{in1} and

Pump work

W_{in2} represent the energy required to drive the pumps which force the water back into the boiler. Using the First Law of Thermodynamics (Law of the Conservation of Energy), we have

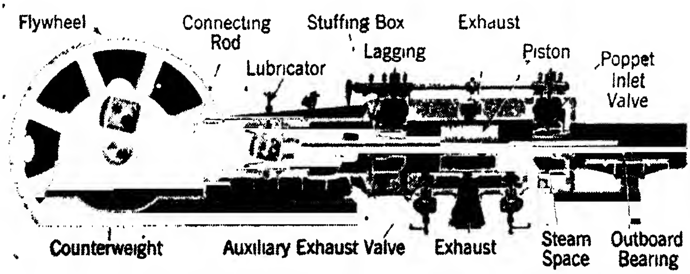
$$\text{Energy in} = \text{Energy out},$$

$$Q_s + W_{in} = Q_r + W_{out} \tag{4}$$

Transposing, we find

(5) $$W_{out} - W_{in} = W_{net} = Q_s - Q_r,$$ *Net work*

where $W_{in} = W_{in1} + W_{in2}$ = total pump work. In many practical applications, the pump work, W_{in}, is so small as to be negligible; so do not be alarmed if we fail to include it in our computations. (Be sure to study the caption to Fig. 411.)



Courtesy Skinner Engine Company

FIG. 410. RECIPROCATING STEAM ENGINE

The reciprocating steam engine is often economical in small power plants The type of engine shown, the most modern, is built in sizes up to about 1100 hp. For large amounts of power (see Fig. 407 and Fig. 408), the only competitor for the steam turbine is the water turbine. In this engine, called a *uniflow engine*, the steam enters through the inlet valve (one at each end of the cylinder) and passes out of the cylinder through exhaust ports cut around the periphery of the cylinder at its central section. The movement of the cylinder uncovers (opens) and then covers (closes) the exhaust ports. High-pressure steam acts first on one side of the piston and then the other.

Equation 5, which says that the *net work* done by a cycle is equal to the heat supplied minus the heat rejected, is a very useful form of the Law of the Conservation of Energy. It applies to *any* kind of a thermodynamic cycle. For example, we may first apply it to the gasoline engine already discussed. We have estimated the work as $W = 4000$ B.t.u. per lb. of gasoline. The heat supplied is $Q_s = 20{,}000$ B.t.u. per lb. Therefore, for each pound of gasoline burned, the amount of rejected heat is

$$Q_r = Q_s - W = 20{,}000 - 4000 = 16{,}000 \text{ B.t.u. per lb.}$$

Some of this heat passes through the exhaust system and some of it goes into the cooling water, but *all* of it is eventually discharged into the atmosphere. Remember that equations 1 and 5 are just as useful for a steam power plant as they are for the gasoline engine. *Rejected heat*

If 1100 B.t.u. per lb. of steam are supplied in a boiler and if 800 B.t.u. per lb. are rejected, determine the work and the thermal efficiency. Assume that 100,000 lb. of steam circulate each hour *Example with steam*

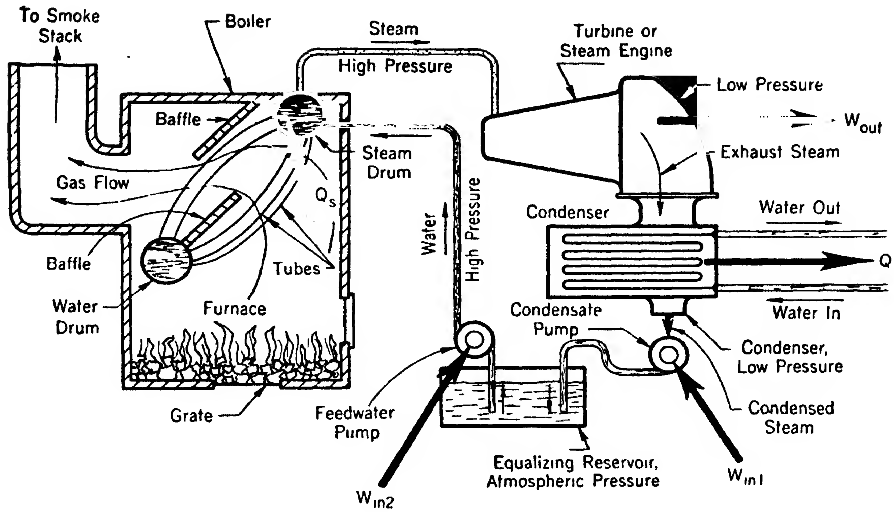


FIG. 411. DIAGRAMMATIC LAYOUT OF A STEAM POWER PLANT

The heavy arrows designate the flow of energy. From the fire in the furnace, heat Q_s flows into the water (in the water drum and tubes) causing the water to boil. Steam rises and collects at the top of the steam drum. This steam, at high pressure and temperature, flows to a turbine or reciprocating steam engine. In the turbine (Fig. 408), the steam expands to a low pressure, and at the same time, some of the energy of the steam is transformed into work output, W_{out}. Usually this work is delivered by the turbine shaft to an electric generator, where it is transformed into electricity. As you know, electricity has many uses, one of which is to do work driving other machines, for example, the pumps in this illustration.

After the steam has expanded to a low pressure, it passes into a condenser (Fig. 409), through which cold "condenser" water is circulated. The condenser water, obtained from any natural source, carries away the heat Q_r taken from the steam as it is condensed. The colder the condenser water, the lower the temperature, and therefore the pressure, of the condensed steam. For this reason, a modern steam power plant is more efficient in the winter than in the summer. The condensed steam (water) from the condenser flows to a pump, called the condensate pump, and is pumped into a reservoir at atmospheric pressure. The reservoir acts as an equalizing chamber which is needed because it is a practical impossibility to keep the rate of flow of water from the condenser exactly equal to the rate of flow into the boiler.

Another accessory, the feedwater pump, forces the water back into the boiler, after which the cycle is repeated. It takes work to run these pumps, this work being an energy *input*. The pumps may be driven by electric motors which utilize a part of the electrical *output* of the generator connected to the turbine. Thus, the net work W_{net} which may be used for duties outside of the power plant is the total output, W_{out}, minus the total input, W_{in}. The pump work $W_{in1} + W_{in2}$ is small as compared to the output of the turbine and is therefore often neglected.

In the furnace, the flow of the hot gases of combustion is directed by *baffles* so as to get the maximum amount of heat transferred to the water and steam. A baffle is simply a partition. Boilers are built in a large variety of forms and arrangements of drums and tubes (see Fig. 406 again).

and compute the horsepower output of the turbine. Neglect the pump work. We find immediately that

$$W = Q_s - Q_r = 1100 - 800 = 300 \text{ B.t.u. per lb.}$$

$$e = \frac{W}{Q_s} = \frac{300}{1100} = 27.3\%$$

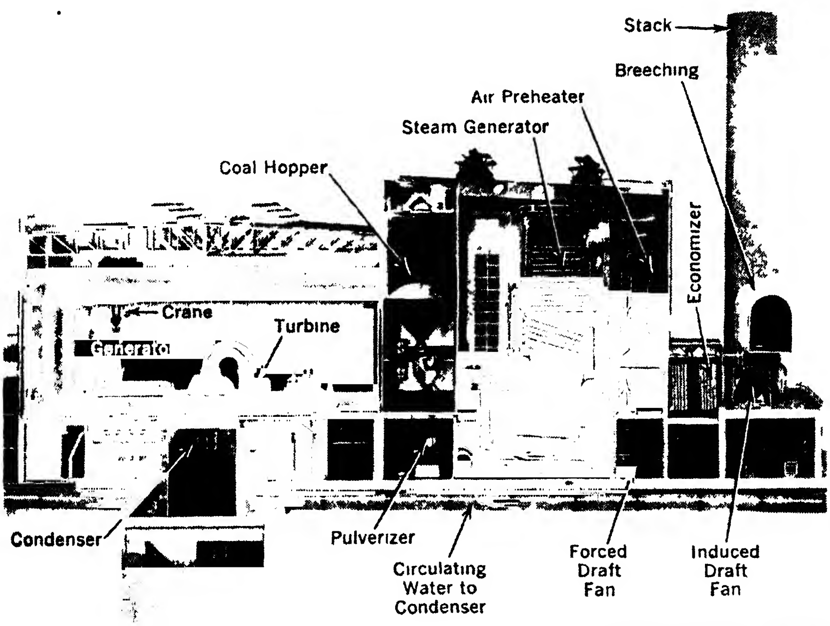
The amount of work per hour is evidently the work per pound of steam times the pounds of steam per hour, or

$$W = 300 \times 100{,}000 = 30{,}000{,}000 \text{ B.t.u. per hr}$$

Then, since a horsepower is equivalent to 2545 B.t.u. per hr., the horsepower of the turbine is *Computing horsepower*

$$\frac{30{,}000{,}000}{2545} = 11{,}780 \text{ hp.}$$

This turbine would be classified as a medium-size turbine.



Courtesy Apex Photo Company

FIG. 412. A POWER PLANT

This illustration, which is a photograph of a model of a large steam power plant, gives you an idea of the actual arrangement of the various units. Study it carefully with frequent comparison with Fig. 411. The *induced draft fan* (right) acts to ***pull*** the air through the furnace, air preheater, and economizer; and the *forced draft fan* acts to push the air through these units. The use of one or more fans for the air supply makes it possible to increase the amount of air entering the furnace, and therefore to burn more fuel. The more fuel burned, the more heat released, and if the design is correct, the more steam generated.

You may have observed that the thermal efficiency in this example was not computed by using the heating value of the fuel, as it was for the internal combustion engine. Although the method given above is commonly used for computing the efficiency of a steam power cycle, the so-called over-all thermal efficiency of a power plant would be based on the heating value of the fuel consumed in the furnace.

Over-all efficiency

Thus, suppose a power plant uses 2 lb. of coal, whose heating value is 13,000 B.t.u. per lb., for 3000 B.t.u. of work. The over-all thermal efficiency is then

$$e = \frac{3000}{2 \times 13{,}000} = 11.52\%.$$

This efficiency represents a typical value for a good medium-size plant. (If you are interested in a little more detail concerning a modern steam power plant, study Fig. 412 carefully.)

Problems

1. An inventor claims that he will be able to get 7,780,000 ft-lb. of work from his engine in a specified time. An engineer investigating the claim finds that, during the same time, the fuel to be supplied to the engine releases 8900 B.t.u. of heat. Do you agree with the engineer's report that this is a crackpot invention? What law is violated?

2. For 1 lb. of steam, the heat supplied in a steam power engine is 1450 B.t.u. and the heat rejected is 900 B.t.u. If 1,000,000 lb. of steam are used per hour, determine the thermal efficiency and the horsepower developed.
Ans. 37.9%, 216,000 hp.

3. In a steam cycle, 1,000,000 B.t.u. per min. are supplied when the thermal efficiency is 20%. Determine the heat rejected and the horsepower developed.
Ans. 800,000 B.t.u. per min., 4720 hp.

4. The best performance obtained so far in steam power plants is about 1 hp-hr. (= 2545 B.t.u.) of work for 1 lb. of coal burned. If the coal has a heating value of 14,000 B.t.u. per lb., what is the over-all thermal efficiency? *Ans*. 18.2%.

ANOTHER LAW

Waste heat

Perhaps you have speculated on the heat being rejected in the foregoing examples and wondered why something is not done to get more than, say, 20 to 25 per cent of the heat supplied converted into useful work. Some people have been heard to remark to this effect, "Of 20,000 B.t.u. in 1 lb. of gasoline, 16,000 B.t.u. are wasted. When the engineers learn how to make an engine 100 per cent efficient, we will get 75 miles to a gallon of gasoline instead of 15." However, it is not so simple as that. We may some

day drive cars 75 miles on a gallon of gasoline (motorized bicycles go that far on a gallon now), but if we do, it will not be because the motor is 100 per cent efficient. Why? Because with our present knowledge, it can be demonstrated that no conceivable heat engine can have a 100 per cent thermal efficiency — even on paper.

Carnot efficiency

A young Frenchman named Carnot (1796–1832) has shown beyond doubt that the maximum conceivable thermal efficiency is

$$e_{\max} = \frac{t_1 - t_2}{t_1 + 460}, \tag{6}$$

where t_1 is the highest temperature in the cycle and t_2 is the lowest naturally available temperature. *By naturally available temperature*, we mean a temperature provided by nature, such as the temperature of the air, or of a near-by river or lake. The law which we are leading up to is called *the Second Law of Thermodynamics* and is an outcome of Carnot's study. This law says in effect that no engine can convert all the heat supplied to it into work, that some of the heat must be rejected. As seen from equation 6, t_2 must equal $-460°$ F. if $e = 100$ per cent; but $-460°$ F. is absolute zero temperature, far below any naturally available temperature. To illustrate with numbers, assume that the highest temperature of the steam in a steam turbine is 1000° F. This is about the highest that we can use at the present time because we have no suitable material which can withstand a higher temperature and retain its strength, although some are on the way. Assume also that the naturally available temperature at the time is 80° F. The corresponding maximum possible efficiency is

$$e_{\max} = \frac{1000 - 80}{1000 + 460} = 63\%.$$

Room for improvement

To be sure, this efficiency is much higher than any actually obtained, which suggests that there is room for improvement in the processes of obtaining power. There is room, and improvements are being made. However, there is not so much opportunity for improvement as the high ideal efficiency implies, because there are some inevitable losses due to friction and loss of heat, and because of the practical impossibility of adjusting the actual processes of a cycle into an ideal combination. It is probable that the best *actual thermal efficiency* will not soon exceed about one-half of this *theoretical maximum efficiency* in turbine-equipped power plants.

Problems

1. An inventor claims that his engine will produce 1000 B.t.u. of work from a certain amount of fuel. Upon investigation an engineer finds that 1500 B.t.u. are supplied to the engine and that the highest temperature in the cycle is 600° F. while the lowest natural temperature is 70° F. Could the inventor be justified in his claim? Explain.

2. The inventor mentioned in problem 1 later changes his claim to 750 B.t.u. of work from 1500 B.t.u. supplied. Is this possible? What would you guess as the work that might be obtained from this engine, assuming that it is really a good one?

Ans. Actual work, say, 375 B.t.u.

THE ELECTRIC REFRIGERATOR

Reversed cycle

For the electric refrigerator in your kitchen, you should be thankful to the man who first conceived of the thermodynamic cycle for refrigeration. This cycle is, in a sense, the reverse of cycles devised for the production of power. Whereas, in the power cycle, heat is supplied from which work is obtained, in the refrigerating cycle, work is supplied in order to obtain refrigeration. In other words, the *output* of a power cycle is work and the *input* is heat; the *output* of a refrigerating cycle is refrigeration (a removal of heat) and the *input* is work.

It is well at this stage to point out that a particular substance, water, for example, will boil at a temperature which has a definite relation to the pressure on the substance. (See page 71.) At atmospheric pressure of 14.7 lb. per sq. in., water boils at 212° F. At any other pressure, the boiling temperature of water is not 212° F.

Boiling temperatures

At a pressure of 100 lb. per sq. in. absolute,* water boils at 327.8° F. As an illustration of another substance, sulfur dioxide at 100 lb. per sq. in. boils at 110° F. Any substance that will exist in a liquid phase, say, iron or ammonia, has its own characteristic boiling temperature which is related to the pressure on the liquid.

Refrigerating with steam

Steam can be and is used for refrigeration. Steam could be, but is not, used to freeze water. When the pressure on water is reduced to 0.0885 lb. per sq. in. absolute, it will boil at 32° F. If the pressure is reduced below 0.0885 lb. per sq. in., steam will exist at a temperature below 32° F. However, except under certain favorable conditions, steam (and water) does not make a satisfactory

* A pressure gage (an instrument for reading pressures) measures the difference between the *absolute pressure* and the *atmospheric pressure*. At sea level, the atmosphere exerts a pressure of 14.7 lb. per sq. in. Thus, if a gage reads 85.3 lb. per sq. in., the absolute pressure (or the pressure measured from a state of no pressure) is 14.7 + 85.3 = 100 lb. per sq. in.

refrigerant (a substance used in a refrigerating cycle), because, for one reason, these very low pressures (vacuums) are difficult to maintain.

Sulfur dioxide

A common refrigerant for household refrigerators is sulfur dioxide. (See page 100.) At 15.21 lb. per sq. in. absolute, sulfur dioxide boils at 15° F., a temperature sufficiently low for freezing water, and, as mentioned before, it boils at 110° F. when the absolute pressure is 100 lb. per sq. in. Note these pressures and temperatures carefully, as we shall use them in the following discussion.

Compressing the refrigerant

How does a refrigerating cycle operate? Let us start at *A*, Fig. 413, where vaporous sulfur dioxide (SO_2), say, is entering the *compressor* at a low pressure of 15.21 lb. per sq. in. and a temperature of 15° F. The compressor pumps the SO_2 to a high pressure, for example, 100 lb. per sq. in. which raises its temperature to 110° F. Now that the refrigerant is hotter than the surrounding air (or other naturally available temperature), it passes into a *condenser*, where the refrigerant is condensed by cooling because heat flows from the hot SO_2 at 110° F. to the cooler outside air at 80° F. In larger installations, the condenser uses water, instead of air, for carrying away heat.

Expansion valve

As the SO_2 condenses into a liquid at about 90° F., it flows into the *receiver*, which is simply a convenient equalizing reservoir. (See Fig. 411 again.) The liquid outlet of the receiver is a tube which extends nearly to the bottom of the receiver, to be sure that nothing but liquid enters it. The liquid at high pressure and 90° temperature passes through an *expansion valve*, which is a device that introduces a restriction to flow in the line — much as the ordinary water faucet is a restriction to the flow of water when it is not wide open. The expansion valve separates the high-pressure and low-pressure regions, as observed in Fig. 413, just as there is high-pressure water on the pipe side of the water faucet and low pressure on the outlet side. With a drop in pressure, there is a drop in temperature.

Boiling the refrigerant

The liquid refrigerant then enters an *expansion coil*, in this case, the space around the freezing compartment (Fig. 413). Notice that the SO_2 is now in the low-pressure region where it will boil at a temperature, in our illustration, of 15° F. But to make it boil, heat must be supplied to it. Since the home refrigerator has an inside temperature of some 40° to 50° F., the heat flows from the contents of the box into the SO_2 in the coil, and the SO_2 therefore

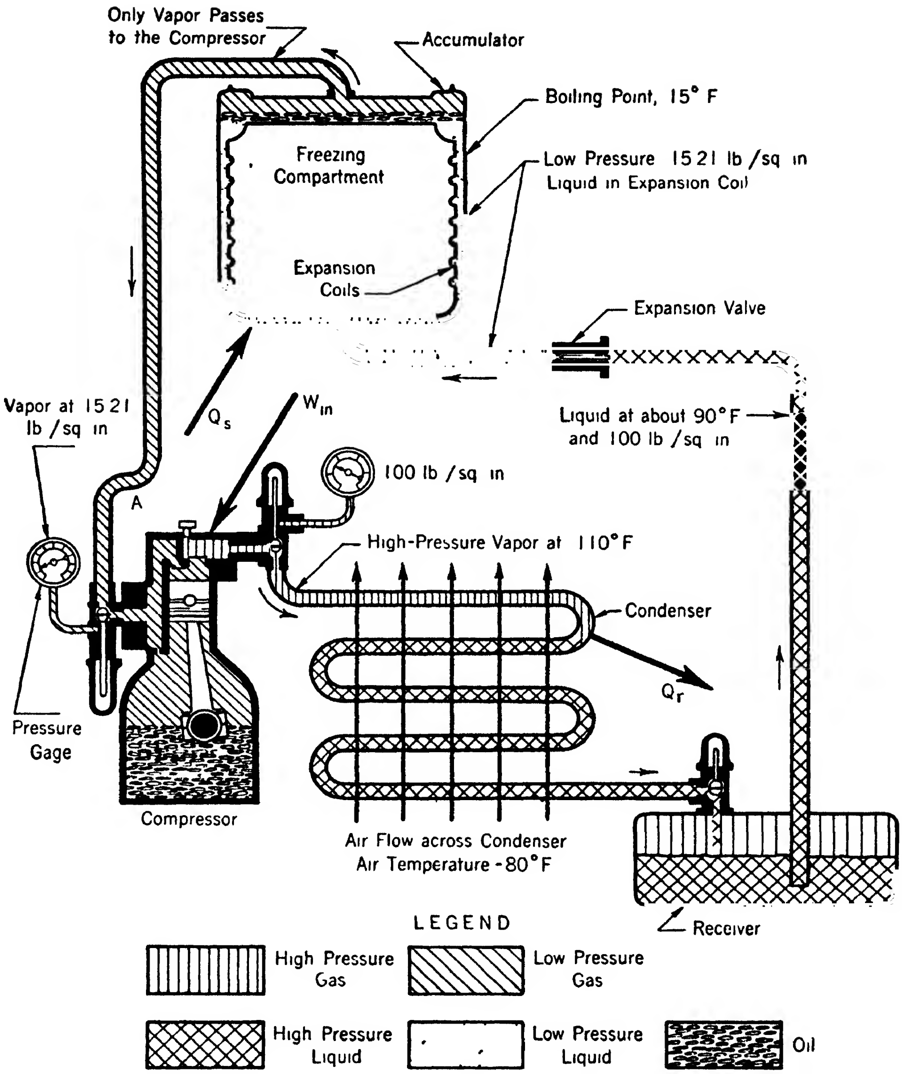


FIG. 413. DIAGRAMMATIC LAYOUT OF A REFRIGERATING UNIT

With certain refinements and controls omitted, this illustration shows how the refrigerating cycle operates. (See text.) At the condenser, heat flows from the 110° F. vapor to the 80° F. air. The condenser on a home refrigerator usually is located on the back of the box. Inside the box, heat flows from the contents of the box into the 15° F. liquid, causing it to boil (evaporate). Low-temperature vapor passes to the compressor where it is compressed to a high pressure and temperature, whence it restarts its cycle. The heavy arrows indicate the flow of energy. From the law of the conservation of energy, we see that $Q_s + W_{in} = Q_r$, or heat to be removed $\doteq$ heat absorbed from box plus work of compressor.

boils (evaporates). The evaporated SO_2 then flows on to the compressor, entering at A, and the cycle starts over.

Controlling temperature

You may wonder why the box does not cool on down, nearly to a temperature of 15° F. The principal reason is that automatic controls stop the operation of the compressor when the temperature reaches the point for which the controls are set. Then the pressure, and the corresponding boiling temperature, in the expansion coil is allowed to rise until the temperature in the box reaches its higher limit, at which point the compressor starts again and pumps SO_2 out of the expansion coil until the lower limit of temperature is reached again. The operation is so governed that the refrigerating cycle pumps out of the box all the heat that comes into the box from the outside and the heat which must be discharged to cool the contents.

Pumping heat

Observe that heat flows into the SO_2 while it is in the box (or other space to be cooled) and that the compressor serves to pump the SO_2 to a higher temperature than the naturally available temperature outside, so that the heat picked up in the box is discharged outside. The refrigerating cycle is therefore a *heat pump*, that is, a cycle which pumps heat from a low-temperature region to a high-temperature region. A moment's reflection reminds us that *heat naturally flows from the hotter body to the colder body.* You know by experience that you cannot raise the temperature of a hot stove with your finger. Of course, there are other means of obtaining a refrigerating cycle than the one described (which is the most common), but in all cases we must go to considerable trouble and expense to force the heat from a cold body to a warmer body. The arrangement of an actual domestic refrigerating unit is shown in Fig. 414.

Other refrigerants

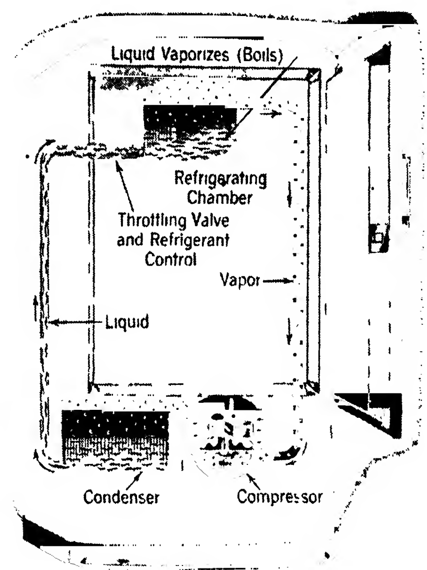
There are several substances, other than sulfur dioxide, which are used in refrigerating cycles. Ammonia (page 107) is used for large commercial ice plants since it has good thermodynamic properties for refrigerating purposes and is relatively cheap. Dichlorodifluoromethane or *freon* has found many applications, especially in air-conditioning systems, because it is not poisonous, it is not inflammable, and it too has good thermodynamic properties for the purpose.

The reason for the existence of a power cycle is to produce work which can be devoted to many useful purposes. The modern power plant generates electricity (one form of energy) which is readily

converted into work. Some of this electricity runs the electric motor of Fig. 414 which supplies the work to keep the refrigerating cycle operating.

Coefficient of performance

One nice thing about the refrigerating cycle is that by efficient design and construction we can get more B.t.u. of refrigeration



Courtesy Frigidaire Division, General Motors

FIG. 414. DOMESTIC REFRIGERATING UNIT

Freon refrigerant

This unit operates with freon as the refrigerant. After the freon evaporates in the refrigerating chamber (which is equivalent to the cooling coils in commercial machines), it enters the compressor as a vapor. This compressor, a rotary type, pumps the vapor to a high pressure and delivers it to the condenser. The natural flow of air across the condenser carries away sufficient heat to cause the freon to condense into a liquid. From the condenser, the liquid passes through the control device and is throttled to the lower pressure which exists in the refrigerating chamber. At this lower pressure, it boils, thus picking up heat from the inside of the cabinet. The vapor then starts the cycle anew, going to the compressor.

than the B.t.u. of work required. Thus, if the efficiency is thought of as being *output* over *input*, the output is the refrigeration and the input is the work. Since this ratio is likely to be greater than unity above 100 per cent, we do not call it efficiency. It goes by the name of the *coefficient of performance* and is abbreviated *C.O.P.* Thus, the coefficient of performance is

$$C.O.P. = \frac{\text{refrigeration}}{\text{work}}. \tag{7}$$

Suppose the refrigeration is 200 B.t.u. per min. and the work supplied is 80 B.t.u. per min. Then

$$C.O.P. = \frac{200}{80} = 2.5,$$

which means that for every B.t.u. of work done by the motor, we shall get 2.5 B.t.u. of refrigeration, that is, 2.5. B.t.u. are removed from the space being refrigerated. This is not a case of getting something for nothing since the refrigerator does not produce heat energy — it merely transports it.

AIR CONDITIONING

What is air conditioning

How would you like to have your home air conditioned? It probably is conditioned some of the time, because anything that we do to atmospheric air, such as heating it in the wintertime, is a process of conditioning air. However, no doubt when you think of air conditioning you think of cooling it for greater comfort in the summertime. As the engineer thinks of it, air conditioning means much more than either heating or cooling. A complete job of air conditioning involves the following processes:

1. Cleaning the air of dust, smoke, pollen, and odors.
2. Heating the air in the winter, or cooling it in the summer.
3. Regulating the humidity of the air — removing moisture in summer, adding moisture in winter.
4. Circulating or delivering air to the proper spaces in the proper quantities.

Humidity

Let us first consider what is meant by *humidity.* All atmospheric air contains steam. Even if you are reading these pages when the outside temperature is $-20°$ F. (20° below zero), that outside air contains steam. Of course, the pressure of the steam must be quite low, below 0.0885 lb. per sq. in. (page 508). While atmospheric pressure is about 14.7 lb. per sq. in., this *total pressure* of 14.7 lb. per sq. in. is exerted by several separate and distinct substances. Air is mostly oxygen and nitrogen, but it also contains argon, steam, and traces of several other gases. Each one of these gases exerts some part of the pressure which we call *atmospheric pressure,* so that the total pressure of the atmosphere is the sum of the pressures, called *partial pressures,* exerted by the individual gases. Hence, the steam or water vapor may and does exist at extremely low pressures under certain conditions, and it would rarely, if ever, have a partial pressure as high as 1 lb. per sq. in. in atmospheric air.

Adding steam to air

Let us consider the air in a closed room. It has some steam in it, but not so much as it might have. If we wished to add more steam, we could put a pan of water in the room, and water would evaporate just as puddles of water evaporate from the road after a rain. If we put several pans of water in the room, some of the water will evaporate, but after a certain point is reached, evaporation will be balanced by an equal condensation. When this limit is reached, the air is said to be *saturated*, meaning that it contains all the steam or moisture that it can "hold." The amount of steam that a particular amount of air can hold depends upon the temperature of the air — the higher the temperature, the greater the amount of steam possible. However, it is important to note that for each air temperature, there is a fixed maximum quantity of steam that may be retained in the air.

The importance of this limitation upon evaporation is demonstrated by an amusing experience. Someone once designed a drying room in a gymnasium for drying the wet clothes of the athletes. A number of steam pipes were installed, so that the room was extremely hot, but it was unventilated. Consequently, the clothes never got dry, because the air became saturated and then refused to take more moisture from the clothes. When the room was ventilated so that unsaturated air entered, it made an excellent drying room.

Dew point

Starting over again with our roomful of air, we shall not put any pans of water in it, but instead we shall cool the air. Since the capacity of the air for holding steam decreases as the temperature goes down, a point is eventually reached where the moisture originally in the air is enough to saturate the air. This temperature at which the cooled air becomes saturated is called the *dew point. If air is cooled below the dew point, some of the steam will condense out as water.* Surely you have noticed moisture on cold windowpanes in the winter or on some other cold surface in a warm room. This water is deposited on the cold surface because the air adjacent to the surface has been cooled below its dew point. If one has too much trouble with such condensation, the solution of the problem is to decrease the humidity of the air in the room or to increase the temperature of the cold surface.

Relative humidity

The amount of steam that air contains is usually expressed by its *relative humidity*, which is the ratio of the weight of steam in a certain amount of air divided by the weight of steam that the

same air can hold at that temperature when it is saturated. Suppose that the relative humidity is 70 per cent (= 0.70). This means that the air contains 70 per cent as much steam as it can hold. If the relative humidity is 100 per cent, the air is saturated air; at 0 per cent humidity, the air contains no steam and is said to be dry air.

Although the actual amount of steam in atmospheric air is small, its presence plays an important role in our feeling of comfort. No doubt you have heard the expression, "It isn't the heat; it's the humidity." Humidity can certainly affect your comfort or discomfort; here's how it works.

Evaporation of perspiration

Your body is perspiring even when it feels dry. If you are not exercising strenuously and if the relative humidity is not high, the perspiration evaporates as fast as it forms. This evaporation goes far toward making you feel comfortable on a warm day, because as the perspiration evaporates there is a cooling effect on the body. An alcohol rub on a feverish body lowers the temperature of the body for the same reason — because the evaporation of the alcohol, which takes place at a more rapid rate than the evaporation of water, carries away heat. Thus, on hot days of high humidity, since the air is nearly saturated, the *rate of evaporation* of perspiration is very low, moisture stays on the skin, and in the absence of the cooling effect of evaporation you feel uncomfortably warm.

Amount of perspiration

The amount of moisture that the body normally gives up through perspiration is surprisingly high. An average person sitting at rest for 24 hours loses about half a gallon of water through perspiration. A man sawing wood perspires at the rate of about 4 gallons in 24 hours. Thus, one of the principal functions of air conditioning in the summer is to *remove* enough *moisture from the air* so that it is "dry" enough to readily evaporate the perspiration from the skin.

Humidity of heated air

In the wintertime, the opposite condition often exists in heated spaces. Suppose that cold atmospheric air at 20° F., which may have an average relative humidity of 60 per cent, is heated to a room temperature of 70° F. without the addition of moisture. Since its capacity for holding moisture increases with the temperature and since the original amount of moisture remains unchanged, the air at 70° F. will have a much lower relative humidity, in fact about 9 per cent. Such air is quite dry and is not considered to be healthful. Moreover, this dry air results in a high rate of evaporation of moisture from the skin, so that if you are not warmly clad, you will feel uncomfortable. If moisture is added

to this 70° F. air until the relative humidity is brought back to 60 per cent, the rate of moisture evaporation from the skin decreases, less heat is carried away from the body, and comfort is obtained with much lighter clothing. In a typical five-room house, for twenty air changes daily, this increase of humidity would require the evaporation of about 8 gallons of water each day. However, it is not at all necessary for health reasons to bring the humidity up to 60 per cent. Moisture from cooking and perspiration ordinarily keeps the humidity in our homes at a reasonable level.

Winter conditioning

With these phenomena in mind we see that for best results in the winter, the air needs to be not only heated but also humidified. Also, if your city is smoky and dusty, you will find further comfort and convenience when the air introduced into your living spaces is cleaned.

Clean air

Air is cleaned in various ways: (1) by passing it through filters, which operate on the same principle as the oil filter of your automobile; (2) by passing it through a water spray where the minute (invisible) particles of dirt are picked up by a fine spray of water (the water spray may also be used to add moisture for winter air conditioning); (3) by an *electric precipitator*, which shows great promise but has not yet been widely used.

PROBLEMS OF SUMMER CONDITIONING

Cooling air by evaporation

First, consider the situation in dry climates where the relative humidity is low. In this case, the cooling of air is often fairly simple. It takes heat to evaporate water. If air is blown through a spray of water (even at the same temperature as the air) or through a mesh of some sort which is saturated with water, some of the water will evaporate into the air. Part of the heat necessary to vaporize it comes from the air, so that the air will be cooled by this abstraction of heat. For saturated air, none of the water will be evaporated and no cooling will occur (unless the water is colder than the air). If the air is nearly saturated, some cooling may occur, but the resulting air will have such a high relative humidity that it may not be comfortable.

Excessive humidity

Possibly you have entered small theaters advertised as air conditioned, observed the cooler air all right, but when you got up to leave, found that the back and seat of the chair were covered with moisture. If so, no doubt the water was there because the air was too nearly saturated to absorb your perspiration as fast as it

occurred. However, if the air has a *low humidity*, as stated at first, some moisture may be added to it without the loss of comfort and without ending up with an excessive humidity. Yet a decided cooling of the air may be obtained at the same time by evaporating the water.

Saturated air

Most people in this country live in areas where the foregoing system of cooling will not operate satisfactorily, because the atmospheric air already contains too much moisture. Remember, when air with a fixed amount of moisture is cooled, its *relative* humidity goes up. Suppose the outside air is at 95° F. with a relative humidity of 80 per cent, and suppose it is cooled by refrigeration to a temperature of 75° F. At this second condition, the air will be saturated. As a matter of fact, it becomes saturated at about 88° F. and stays saturated through any further cooling. The 88° F. temperature is the dew point, so that further cooling results in some of the steam condensing out. However, what we started to point out was that even though a 75° F. temperature sounded cool and comfortable (compared to 95° F. outside), saturated air at 75° F. would not be at all comfortable because *the perspiration would not leave the skin.*

Dehumidification

Our problem then is to remove moisture from the cooled air, and it is for this reason that summer air conditioning is relatively expensive. The most common method of removing the unwanted moisture is by refrigerating the air, as described in the discussion of refrigeration, carrying the temperature down to values below the desired room temperature. Then, *sometimes, the cooled air must be artificially heated again* (yes, in the summertime) in order to obtain comfortable room conditions. We see now that the processes for complete summer conditioning would be: (1) cleaning, (2) refrigerating and drying, (3) reheating, (4) distributing, or getting the conditioned air into the right spaces. Study Fig. 415 and its caption carefully.

QUANTITY OF MOISTURE IN AIR

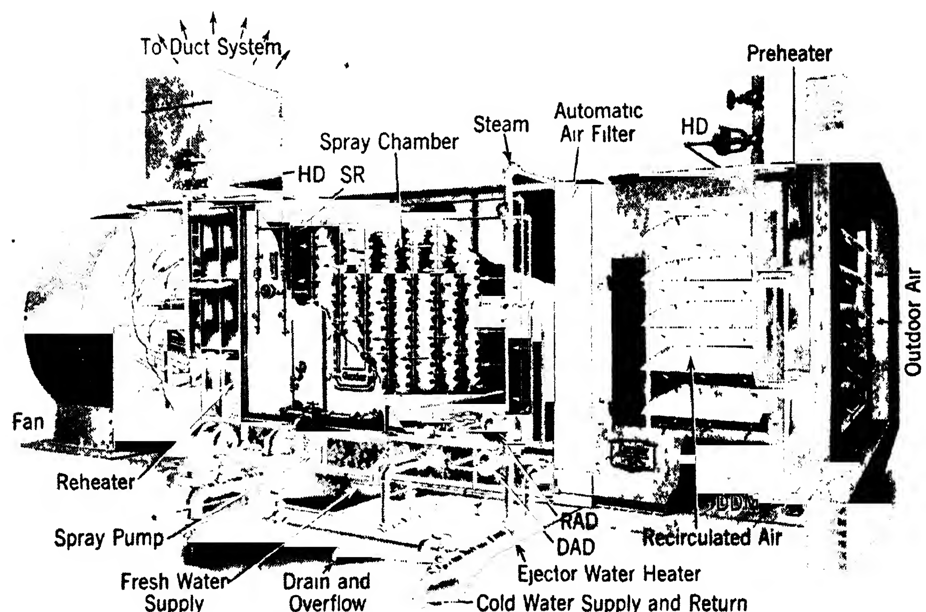
Dry air

In the accompanying table, page 519, we have given the weight of steam or moisture in atmospheric air in pounds of steam per pound of dry air. Now what is *dry air?* Well, dry air is air that has no moisture in it. Since atmospheric air always contains moisture, we would have to go to some trouble to produce really dry air. Nevertheless, in engineering, it is customary to use a pound of dry air as the basis of our calculations.

Weight of moisture in air

From the table; we find beside the 95° temperature the number 0.0365 lb. Therefore, if we add moisture to a pound of *dry air* at 95° F. until it becomes saturated, that is, until it has picked up as much moisture as it can hold, we will have added 0.0365 lb. If, however, we had added moisture only until the relative humidity reached 80 per cent, the air would have received only 80 per cent of the amount of moisture that it could hold. Hence, for each pound of dry air,

Moisture in air at 95° F. and 80% humidity $= 0.80 \times 0.0365 = 0.0292$ lb.



Courtesy Carrier Engineering Corporation

FIG. 415. AIR-CONDITIONING UNIT

This picture shows a large central station unit. The air conditioning is completed here and the conditioned air is carried to various rooms through a duct system (upper left). Ordinarily, the circulated air is a mixture of fresh outside air and recirculated air (at right). This mixture passes through a filter, which removes some of the dust particles, thence across some steam coils which are used in winter conditioning as a means of regulating the humidity of the air. In summer conditioning, the spray chamber is provided with water which has been cooled by a refrigerating process. In passing through the spray chamber, the air is cooled below its dew point, so that the moisture content of the air is reduced to a predetermined level. That is, the air loses moisture while passing through the water spray in summer conditioning. (In winter conditioning, it would gain moisture.) The water spray also further cleans the air. If the air leaving the spray chamber is too cold to be introduced into living spaces, another heater is provided (reheater, on the left). The fan furnishes the motive power for distributing the conditioned air.

Suppose you wish to compute the weight of moisture in a particular room. First, compute the weight of dry air in the room. This weight may be found approximately by using the following formula. Weight of dry air is

Moisture in a room

(8) $$w = 0.073 \times V,$$

where V is the volume of the room in cubic feet. Suppose the room is 12 × 12 × 10 ft. Its volume is 1440 cu. ft., and the weight of dry air is nearly

$$w = 0.073 \times 1440 = 105 \text{ lb}$$

WEIGHT OF STEAM IN SATURATED AIR

Temperature ° F.	*Weight of Steam per Pound of Dry Air*	*Temperature ° F.*	*Weight of Steam per Pound of Dry Air*
20	0.00214	60	0.0110
25	0.00272	65	0.0132
30	0.00344	70	0.0158
35	0.00427	75	0.0187
40	0.00519	80	0.0223
45	0.00631	85	0.0263
50	0.00722	90	0.0311
55	0.00921	95	0.0365

Now look up in the table the temperature of the room and find the corresponding weight of moisture in saturated air. Let the temperature of the room be 75° F., and find (in table) the factor 0.0187 lb. of moisture per pound of dry air. But since we have 105 lb. of dry air, the total amount of moisture that the air can hold is

$$\text{Moisture in saturated air} = 105 \times 0.0187 = 1.96 \text{ lb.}$$

Ordinarily, however, the air is not saturated. Suppose the relative humidity is 60 per cent. (The humidity can be determined by any one of several instruments.) Then the weight of moisture in the room is

$$60\% \times 1.96 = 0.60 \times 1.96 = 1.176 \text{ lb.}$$

Let us try another example. If the outside air is at 95° F. and 80 per cent relative humidity, what would be the relative humidity if this air were cooled to 55° F. and then reheated to 80° F. (as may happen in summer air conditioning)?

Summer conditioning

The weight of moisture that enters the system with 1 lb. of outside air is

Relative humidity$\times$value from table at 95° F. $= 0.80 \times 0.0365 = 0.0292$ lb.

Humidity change

If this air is cooled to 55° F., it is then *below its dew point*, so some moisture condenses out and the air is just saturated. We see that this is true because the moisture in saturated air at 55° F. (0.00921 lb. from the table) is less than the value just computed (0.0292 lb.). Now when the saturated air at 55° F. is reheated to 80° F., the amount of moisture remains constant, inasmuch as the condensed moisture has been removed. So at 80° F. the amount of moisture is 0.00921 lb., but the amount that the air could hold is 0.0223 lb., as found from the table corresponding to 80° F. Thus, the relative humidity is

$$\frac{\text{Amount of moisture in the air (80° F.)}}{\text{Amount of moisture that the air could hold}} = \frac{0.00921}{0.0223} = 41.3\%.$$

It is only fair to warn the reader that the simple method used above is not exactly the correct way to make this computation (and others involving relative humidity), but it gives fair results if the temperature range is not very large.

ELEMENTS OF COMFORT

By way of summing up the foregoing discussion of air conditioning, let us consider the factors which enter into our feeling of comfort.

Temperature of air

The lower the air temperature (below 98.6° F., the temperature of the human body), the greater the difference in temperature between your body and the air, and the faster heat flows from your body to the air — unless, of course, you insulate your body with clothing.

Moisture in air

If the temperature of the air is above 98.6° F., your body will not lose heat to the air by radiation and conduction. It will lose heat by evaporation of moisture from the skin. At normal atmospheric temperatures, relatively dry air feels cooler than more humid air, particularly if there is any movement of the air, because the drier air absorbs moisture faster. In this respect, dry air is advantageous in the summer, but disadvantageous in the winter. Because of its relative slowness in absorbing moisture, humid air has an advantage in the winter in that you feel more comfortable at a lower temperature in humid air than in dry air. This lower

temperature for comfort results in a lower heat loss from the building and therefore a saving of fuel.

Movement of air

When we use fans in the summer, our purpose is to circulate air around the body faster than it would circulate of its own accord. The more rapid movement of air causes a greater evaporation from the skin in a given time and thereby makes us feel cooler.

The temperature of the surrounding surfaces is an important factor which we have not mentioned and which is frequently overlooked as an element in our comfort. We have said that "heat naturally flows from the hotter body to the colder body." However, this statement should not be interpreted to mean that *no* heat flows from the colder body to the hotter body. What it actually means is that *more* heat flows from the hotter body to the colder, than from the colder to the hotter body.

Wall temperature

To understand this point, imagine yourself in a room. If the walls are hotter than body temperature, your body *receives* more heat from the hot walls than it *sends* to the walls, so that the walls tend to warm you. If the walls are below body temperature, your body *sends* more heat to the cold walls than it *receives* from them, and therefore the walls tend to cool you. Note that this action is entirely separate from the effect of the temperature of the air. Thus, in cold weather, it is desirable to have warm walls for comfort.

Cold walls are frequently the cause of discomfort in winter, even when the temperature of the air is so high that you think you ought to be comfortable. On the other hand, in hot summer weather, it is desirable to have cool walls for comfort. The heat stored in hot walls and furniture, including your bed mattress, is a common cause of discomfort, and this is true even at night when cool outside air is coming into the room. Wall and ceiling insulation improves both summer and winter comfort by keeping the walls and ceiling warmer in winter and cooler in summer, and for this reason, insulation adds to year-round comfort.

Insulation

Problems

1. What is the approximate weight of moisture in 1 lb. of dry air which is at 60° F. and 90% relative humidity? *Ans.* 0.0099 lb.

2. A room has the dimensions 14 × 20 × 9 ft. and the temperature in it is 90° F., while the relative humidity is 85%. What is the weight of moisture in the air of this room? *Ans.* 4.86 lb.

3. The air in the room of the previous problem (90° F., 85% relative humidity) is cooled to 60° F. What weight of moisture remains in the air?

Hint. It falls below the dew point, does it not? How much moisture condenses out in cooling to 60° F.? *Ans.* 2.02 lb., 2.84 lb.

4. Saturated air at 30° F. is heated to 75° F. What is the approximate relative humidity after the heating? *Ans.* 18.4%.

5. If the 75° F. air of the previous problem has moisture added to it until the relative humidity is 70%, how much moisture must be added per pound of dry air? *Ans.* 0.00965 lb.

6. A house with cubic contents of 10,000 cu. ft. is supplied with one change of air per hour. The air undergoes the process described in the previous problem. How many pounds of water must be evaporated to change the humidity as described during a period of ten hours? If the water weighs 8.33 lb. per gallon, how many gallons are necessary? *Ans.* 70.4 lb., 8.46 gal.

WHAT TO STUDY NEXT ABOUT HEAT

It is too bad that the subject of heat and power cannot be taught to everyone in six easy lessons, because it is a fascinating study. Unfortunately, however, there are a host of other ideas, not hinted at here, which added together define the science of thermodynamics. If you care to pursue further knowledge in this field, you should first acquire a good book on physics and prepare for yourself a foundation in fundamental science. You will then be prepared for a more comprehensive study of applied thermodynamics, which tells in greater detail of engineering applications of this science of heat, cold, and power. Good Luck!

Logarithmic and Trigonometric Tables

These tables are reprinted from the *Macmillan Logarithmic and Trigonometric Tables*

INSTRUCTIONS FOR USE OF THE MACMILLAN TABLES FOR ENGINEERING PREVIEW

TABLE I
Pages 525–543

COMMON LOGARITHMS OF NUMBERS

The characteristic is given only on the first page of these tables. The mantissa is given to five places. Locate the first three digits of the number in column N and the next digit in row N. The mantissa desired is found at the intersection of the proper row and column. The Proportional Parts table is for interpolation between adjacent entries in any row. It provides the correction which is from 1/10 to 9/10 of the difference between entries. The first two digits of the mantissa, when omitted, are given above, or when starred are found below.

TABLE Ia
Page 544

CONDENSED LOGARITHMS TO FIFTEEN DECIMAL PLACES

To illustrate the use of this table we shall compute the logarithm of g (32.16, the average acceleration of gravity) to five places. First divide 32.16 by 3, the first digit, obtaining 10.72. Then divide 10.72 by 10 obtaining 1.072. Then divide 1.072 by 1.07 obtaining 1.0019. Accordingly,

$$g = 32.16 = 3(10)(1.07)(1.0019)$$

The logarithms of each factor can be obtained separately and added to give

$$\log g = 0.47712 + 1.00000 + 0.02938 + 0.00082 = 1.50732$$

This value may be checked in Table I. By following this process, and by using Table Ia, logarithms can be obtained to fifteen decimal places.

TABLE Ib
Page 545

IMPORTANT CONSTANTS

The values given here are needed to more than slide rule accuracy in certain calculations. The number of significant figures given varies with the probable need.

Page 546

CHART OF TRIGONOMETRIC FUNCTIONS

These curves illustrate the variations of the trigonometric functions and show their positive and negative ranges for angles from $-\pi/2$ to π.

TABLE II
Pages 547–569

TRIGONOMETRIC FUNCTIONS

Angles below 45° are given at the top of the table. Select column headed by correct trigonometric function and move downward along left-hand minute column to locate row. Missing digits of function are found immediately above. For angles above 45° use function headings at bottom of table and minute column at right which reads upward.

TABLE III
Page 570

DEGREES, MINUTES, AND SECONDS TO RADIANS

An angle in degrees, minutes, and seconds can be changed into radians by the addition of equivalents. From the first three double columns, change degrees to radians. Use the fourth double column to change minutes to radians and the last double column for seconds. Then add these three values to obtain the whole angle in radians.

TABLE IV
Pages 571–572

RADIAN MEASURE TRIGONOMETRIC FUNCTIONS

Functions of angles in hundredths of a radian up to 1.60 radians are given in the table. The right-hand column is the equivalent angle in degrees and minutes to tenths of a minute.

TABLE V
Page 573

COMPOUND INTEREST

One dollar invested at the interest rate chosen from row n after a period of years chosen from column n will increase, when compounded annually, to the number of dollars found at the intersection of the proper row and column.

TABLE VI
Page 574

AMOUNT OF AN ANNUITY

One dollar per year invested at the interest rate chosen from row n accumulating for the number of years chosen from column n will amount to the number of dollars found at the intersection of the proper row and column. The interest is compounded annually.

LOGARITHMIC AND TRIGONOMETRIC TABLES

TABLE I

COMMON LOGARITHMS OF NUMBERS

FROM

1 TO 10 000

TO

FIVE DECIMAL PLACES

1 — 100

N	Log	N	Log	N	Log	N	Log	N	Log
0	———	**20**	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309
1	0.00 000	21	1.32 222	41	1.61 278	61	1.78 533	81	1.90 849
2	0.30 103	22	1.34 242	42	1.62 325	62	1.79 239	82	1.91 381
3	0.47 712	23	1.36 173	43	1.63 347	63	1.79 934	83	1.91 908
4	0.60 206	24	1.38 021	44	1.64 345	64	1.80 618	84	1.92 428
5	0.69 897	25	1.39 794	45	1.65 321	65	1.81 291	85	1.92 942
6	0.77 815	26	1.41 497	46	1.66 276	66	1.81 954	86	1.93 450
7	0.84 510	27	1.43 136	47	1.67 210	67	1.82 607	87	1.93 952
8	0.90 309	28	1.44 716	48	1.68 124	68	1.83 251	88	1.94 448
9	0.95 424	29	1.46 240	49	1.69 020	69	1.83 885	89	1.94 939
10	1.00 000	**30**	1.47 712	**50**	1.69 897	**70**	1.84 510	**90**	1.95 424
11	1.04 139	31	1.49 136	51	1.70 757	71	1.85 126	91	1.95 904
12	1.07 918	32	1.50 515	52	1.71 600	72	1.85 733	92	1.96 379
13	1.11 394	33	1.51 851	53	1.72 428	73	1.86 332	93	1.96 848
14	1.14 613	34	1.53 148	54	1.73 239	74	1.86 923	94	1.97 313
15	1.17 609	35	1.54 407	55	1.74 036	75	1.87 506	95	1.97 772
16	1.20 412	36	1.55 630	56	1.74 819	76	1.88 081	96	1.98 227
17	1.23 045	37	1.56 820	57	1.75 587	77	1.88 649	97	1.98 677
18	1.25 527	38	1.57 978	58	1.76 343	78	1.89 209	98	1.99 123
19	1.27 875	39	1.59 106	59	1.77 085	79	1.89 763	99	1.99 564
N	**Log**	**N**	**Log**	**N**	**Log**	**N**	**Log**	**N**	**Log**

N.	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	29	28	27	26	25	24	23	22	21
1	2 9	2 8	2 7	2 6	2 5	2 4	2 3	2 2	2 1
2	5 8	5 6	5 4	5 2	5 0	4 8	4 6	4 4	4 2
3	8 7	8 4	8 1	7 8	7 5	7 2	6 9	6 6	6 3
4	11 6	11 2	10 8	10 4	10 0	9 6	9 2	8 8	8 4
5	14 5	14 0	13 5	13 0	12 5	12 0	11 5	11 0	10 5
6	17 4	16 8	16 2	15 6	15 0	14 4	13 8	13 2	12 6
7	20 3	19 6	18 9	18 2	17 5	16 8	16 1	15 4	14 7
8	23 2	22 4	21 6	20 8	20 0	19 2	18 4	17 6	16 8
9	26 1	25 2	24 3	23 4	22 5	21 6	20 7	19 8	18 9

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 2
=.30102 99957

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20	19
1	2.0	1.9
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

M
$= \log_{10} e$
$= \log_{10} 2.718\cdots$
$= .43429\ 44819$

	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 3 =.47712 12547

log π =.49714 98727

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	317	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	216	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	401	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

log M
$= \log [\log e]$
$= 9.63778\,431$
$- 10$

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 5 =.69897 00043

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 7
=.84509 80400

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	515	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	715
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	810	815	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	941	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	019	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

Prop. Pts.

CONDENSED LOGARITHMS TO FIFTEEN DECIMAL PLACES

[The first digits of n are given in the first row at the top, the last digit of n in the left-hand column. The first column of logarithms are those of 1, 2, 3, ···, 9. The remaining columns give log $(1 + x)$, where $x = (0.1)^k$ times 1, 2, ···, 9.]

Last Digit ↓	First Digit of n →	1	1.0	1.00
	Log n	First Digits of log n →	0	.00
1	00000 00000 00000	04139 26851 58225	0432 13737 82643	043 40774 79319
2	30102 99956 63981	07918 12460 47625	0860 01717 61918	086 77215 31227
3	47712 12547 19662	11394 33523 06837	1283 72247 05172	130 09330 20418
4	60205 99913 27962	14612 80356 78238	1703 33392 98780	173 37128 09001
5	69897 00043 36019	17609 12590 55681	2118 92990 69938	216 60617 56508
6	77815 12503 83614	20411 99826 55925	2530 58652 64770	259 79807 19909
7	84509 80400 14257	23044 89213 78274	2938 37776 85210	302 94705 53618
8	90308 99869 91944	25527 25051 03306	3342 37554 86950	346 05321 09506
9	95424 25094 39325	27875 36009 52829	3742 64979 40624	389 11662 36911

(continuation)

	1.000	1.0000	1.00000	1.000000	1.0000000	1.00000000
	000	0000	00000	000000	0000000	00000000
1	04 34272 76863	0 43429 23104	04342 94265	0434 29446	043 42945	04 34294
2	08 68502 11619	0 86858 02780	08685 88095	0868 58888	086 85890	08 68589
3	13 02688 05227	1 30286 39028	13028 81491	1302 88325	130 28834	13 02883
4	17 36830 58463	1 73714 31850	17371 74453	1737 17758	173 71779	17 37178
5	21 70929 72230	2 17141 81245	21714 66981	2171 47187	217 14724	21 71472
6	26 04985 47390	2 60568 87215	26057 59071	2605 76611	260 57668	26 05767
7	30 38997 84812	3 03995 49761	30400 50733	3040 06031	304 00613	30 40061
8	34 72966 85361	3 47421 68881	34743 41958	3474 35447	347 43557	34 74356
9	39 06892 49910	3 90847 44354	39086 32748	3908 64858	390 86502	39 08650

[For $x < 0.0000001$, log $(1 + x) = x\,M$ to within 3 in the 17th place, where $M = 0.43429448$. Hence the last column gives multiples of M except for the decimal place. All the columns that would follow have the same significant digits displaced each time one place.]

CONDENSED ANTILOGARITHMS TO TEN DECIMAL PLACES

[The first digits of n are given in the first row at the top, $n = (0.1)^k x$, $x = 1, 2, 3, \cdots, 9$ are given in the left-hand column. The first digits in 10^n are given in the second row at the top.]

x	$n = 0.1x$	$0.01x$	$0.001x$	$0.0001x$	$(0.1)^5x$	$(0.1)^6x$	$(0.1)x$
	10^n	1	1.0	1.00	1.000	1.0000	1.00000
1	1 25892 54118	02329 29922	0230 52381	023 02850	02 30261	0 23026	02303
2	1 58489 31925	04712 85481	0461 57903	046 06231	04 60528	0 46052	04605
3	1 99526 23150	07151 93052	0693 16689	069 10142	06 90799	0 69078	06908
4	2 51188 64315	09647 81961	0925 28861	092 14583	09 21076	0 92104	09210
5	3 16227 76602	12201 84543	1157 94543	115 19535	11 51359	1 15130	11513
6	3 98107 17055	14815 36215	1391 13857	138 25058	13 81646	1 38156	13816
7	5 01187 23363	17489 75549	1624 86929	161 31092	16 11939	1 61182	16118
8	6 30957 34448	20226 44346	1859 13881	184 37657	18 42238	1 84209	18421
9	7 94328 23472	23026 87708	2093 94837	207 44753	20 72541	2 07235	20723

[For $n < 0.000001$, $10^n = 1 + n\,(1/M)$ to within 3 in the 12th decimal place, where $(1/M) = 2.302585$. Hence the last column gives multiples of $(1/M)$ except for the decimal place. All the columns that would follow contain the same significant digits displaced one place for each new column.]

Table I b — Important Constants

Logarithms of Important Constants

n = number	Value of n	$\mathrm{Log}_{10}\,n$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Naperian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812263
$\pi \div 180$ = radians in 1°	0.01745329	8.24187737
$\pi \div 10800$ = radians in 1′	0.0002908882	6.46372612
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811133	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37 (exact legal value)	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g (average value)	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231 (exact legal value)	2.3636120
ft. lb. per sec. in 1 H. P.	550 (exact legal value)	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1. 10445
watts in 1 H. P.	745.957	2.8727135

Several Numbers Very Accurately

π =	3.14159	26535	89793	23846	26433	83280
e =	2.71828	18284	59045	23536	02874	71353
M =	0.43429	44819	03251	82765	11289	18917
$1 \div M$ =	2.30258	50929	94045	68401	79914	54684
$\log_{10} \pi$ =	0.49714	98726	94133	85435	12682	88291
$\log_{10} M$ =	9.63778	43113	00536	78912		

Certain Convenient Values for $n = 1$ to $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\mathrm{Log}_{10}\,n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

TABLE II

ACTUAL VALUES

OF THE

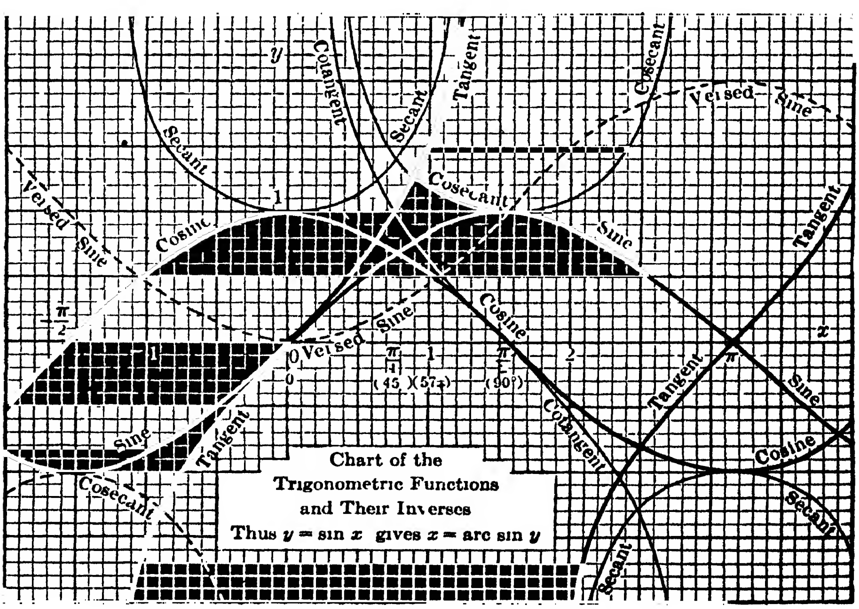
TRIGONOMETRIC FUNCTIONS

FROM

0° TO 90° AT INTERVALS OF ONE MINUTE

TO

FIVE DECIMAL PLACES



′	Sin	Tan	Ctn	Cos	
0	.00000	.00000	——	1.0000	**60**
1	029	029	3437.7	000	59
2	058	058	1718.9	000	58
3	087	087	1145.9	000	57
4	116	116	859.44	000	56
5	.00145	.00145	687.55	1.0000	**55**
6	175	175	572.96	000	54
7	204	204	491.11	000	53
8	233	233	429.72	000	52
9	262	262	381.97	000	51
10	.00291	.00291	343.77	1.0000	**50**
11	320	320	312.52	.99999	49
12	349	349	286.48	999	48
13	378	378	264.44	999	47
14	407	407	245.55	999	46
15	.00436	.00436	229.18	.99999	**45**
16	465	465	214.86	999	44
17	495	495	202.22	999	43
18	524	524	190.98	999	42
19	553	553	180.93	998	41
20	.00582	.00582	171.89	.99998	**40**
21	611	611	163.70	998	39
22	640	640	156.26	998	38
23	669	669	149.47	998	37
24	698	698	143.24	998	36
25	.00727	.00727	137.51	.99997	**35**
26	756	756	132.22	997	34
27	785	785	127.32	997	33
28	814	815	122.77	997	32
29	844	844	118.54	996	31
30	.00873	.00873	114.59	.99996	**30**
31	902	902	110.89	996	29
32	931	931	107.43	996	28
33	960	960	104.17	995	27
34	.00989	.00989	101.11	995	26
35	.01018	.01018	98.218	.99995	**25**
36	047	047	95.489	995	24
37	076	076	92.908	994	23
38	105	105	90.463	994	22
39	134	135	88.144	994	21
40	.01164	.01164	85.940	.99993	**20**
41	193	193	83.844	993	19
42	222	222	81.847	993	18
43	251	251	79.943	992	17
44	280	280	78.126	992	16
45	.01309	.01309	76.390	.99991	**15**
46	338	338	74.729	991	14
47	367	367	73.139	991	13
48	396	396	71.615	990	12
49	425	425	70.153	990	11
50	.01454	.01455	68.750	.99989	**10**
51	483	484	67.402	989	9
52	513	513	66.105	989	8
53	542	542	64.858	988	7
54	571	571	63.657	988	6
55	.01600	.01600	62.499	.99987	**5**
56	629	629	61.383	987	4
57	658	658	60.306	986	3
58	687	687	59.266	986	2
59	716	716	58.261	985	1
60	.01745	.01746	57.290	.99985	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

89°

′	Sin	Tan	Ctn	Cos	
0	.01745	.01746	57.290	.99985	**60**
1	774	775	56.351	984	59
2	803	804	55.442	984	58
3	832	833	54.561	983	57
4	862	862	53.709	983	56
5	.01891	.01891	52.882	.99982	**55**
6	920	920	52.081	982	54
7	949	949	51.303	981	53
8	.01978	.01978	50.549	980	52
9	.02007	.02007	49.816	980	51
10	.02036	.02036	49.104	.99979	**50**
11	065	066	48.412	979	49
12	094	095	47.740	978	48
13	123	124	47.085	977	47
14	152	153	46.449	977	46
15	.02181	.02182	45.829	.99976	**45**
16	211	211	45.226	976	44
17	240	240	44.639	975	43
18	269	269	44.066	974	42
19	298	298	43.508	974	41
20	.02327	.02328	42.964	.99973	**40**
21	356	357	42.433	972	39
22	385	386	41.916	972	38
23	414	415	41.411	971	37
24	443	444	40.917	970	36
25	.02472	.02473	40.436	.99969	**35**
26	501	502	39.965	969	34
27	530	531	39.506	968	33
28	560	560	39.057	967	32
29	589	589	38.618	966	31
30	.02618	.02619	38.188	.99966	**30**
31	647	648	37.769	965	29
32	676	677	37.358	964	28
33	705	706	36.956	963	27
34	734	735	36.563	963	26
35	.02763	.02764	36.178	.99962	**25**
36	792	793	35.801	961	24
37	821	822	35.431	960	23
38	850	851	35.070	959	22
39	879	881	34.715	959	21
40	.02908	.02910	34.368	.99958	**20**
41	938	939	34.027	957	19
42	967	968	33.694	956	18
43	.02996	.02997	33.366	955	17
44	.03025	.03026	33.045	954	16
45	.03054	.03055	32.730	.99953	**15**
46	083	084	32.421	952	14
47	112	114	32.118	952	13
48	141	143	31.821	951	12
49	170	172	31.528	950	11
50	.03199	.03201	31.242	.99949	**10**
51	228	230	30.960	948	9
52	257	259	30.683	947	8
53	286	288	30.412	946	7
54	316	317	30.145	945	6
55	.03345	.03346	29.882	.99944	**5**
56	374	376	29.624	943	4
57	403	405	29.371	942	3
58	432	434	29.122	941	2
59	461	463	28.877	940	1
60	.03490	.03492	28.636	.99939	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

88°

′	Sin	Tan	Ctn	Cos	
0	.03490	.03492	28.636	.99939	**60**
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	.03635	.03638	27.490	.99934	**55**
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	725	26.845	931	52
9	752	754	.637	930	51
10	.03781	.03783	26.432	.99929	**50**
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	.03926	.03929	25.452	.99923	**45**
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	.04071	.04075	24.542	.99917	**40**
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	.04217	.04220	23.695	.99911	**35**
26	246	250	.532	910	34
27	275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	.04362	.04366	22.904	.99905	**30**
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	483	.308	900	26
35	.04507	.04512	22.164	.99898	**25**
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	.04653	.04658	21.470	.99892	**20**
41	682	687	.337	890	19
42	711	716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	.04798	.04803	20.819	.99885	**15**
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	.04943	.04949	20.206	.99878	**10**
51	.04972	.04978	20.087	876	9
52	.05001	.05007	19.970	875	8
53	030	037	.855	873	7
54	059	066	.740	872	6
55	.05088	.05095	19.627	.99870	**5**
56	117	124	.516	869	4
57	146	153	.405	867	3
58	175	182	.296	866	2
59	205	212	.188	864	1
60	.05234	.05241	19.081	.99863	**0**
	Cos	Ctn	Tan	Sin	′

87°

′	Sin	Tan	Ctn	Cos	
0	.05234	.05241	19.081	.99863	**60**
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	.05379	.05387	18.564	.99855	**55**
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	495	503	.171	849	51
10	.05524	.05533	18.075	.99847	**50**
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	.05669	.05678	17.611	.99839	**45**
16	698	708	.521	838	44
17	727	737	.431	836	43
18	756	766	.343	834	42
19	785	795	.256	833	41
20	.05814	.05824	17.169	.99831	**40**
21	844	854	17.084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	.05960	.05970	16.750	.99822	**35**
26	.05989	.05999	.668	821	34
27	.06018	.06029	.587	819	33
28	047	058	.507	817	32
29	076	087	.428	815	31
30	.06105	.06116	16.350	.99813	**30**
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	.06262	15.969	.99804	**25**
36	279	291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	.06395	.06408	15.605	.99795	**20**
41	424	438	.534	793	19
42	453	467	.464	792	18
43	482	496	.394	790	17
44	511	525	.325	788	16
45	.06540	.06554	15.257	.99786	**15**
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	.06685	.06700	14.924	.99776	**10**
51	714	730	.860	774	9
52	743	759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	.06831	.06847	14.606	.99766	**5**
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	**0**
	Cos	Ctn	Tan	Sin	′

86°

′	Sin	Tan	Ctn	Cos	
0	.06976	.06993	14.301	.99756	60
1	.07005	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	.07121	.07139	14.008	.99746	55
6	150	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	256	.782	738	51
10	.07266	.07285	13.727	.99736	50
11	295	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	.510	727	46
15	.07411	.07431	13.457	.99725	45
16	440	461	.404	723	44
17	469	490	.352	721	43
18	498	519	.300	719	42
19	527	548	.248	716	41
20	.07556	.07578	13.197	.99714	40
21	585	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695	12.996	705	36
25	.07701	.07724	12.947	.99703	35
26	730	753	.898	701	34
27	759	782	.850	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	.07846	.07870	12.706	.99692	30
31	875	899	.659	689	29
32	904	929	.612	687	28
33	933	958	.566	685	27
34	962	.07987	.520	683	26
35	.07991	.08017	12.474	.99680	25
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	.339	673	22
39	107	134	.295	671	21
40	.08136	.08163	12.251	.99668	20
41	165	192	.207	666	19
42	194	221	.163	664	18
43	223	251	.120	661	17
44	252	280	.077	659	16
45	.08281	.08309	12.035	.99657	15
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	.08426	.08456	11.826	.99644	10
51	455	485	.785	642	9
52	484	514	.745	639	8
53	513	544	.705	637	7
54	542	573	.664	635	6
55	.08571	.08602	11.625	.99632	5
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	0
	Cos	Ctn	Tan	Sin	′

85°

′	Sin	Tan	Ctn	Cos	
0	.08716	.08749	11.430	.99619	60
1	745	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	.08860	.08895	11.242	.99607	55
6	889	925	.205	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	.09042	11.059	.99594	50
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	.09150	.09189	10.883	.99580	45
16	179	218	.848	578	44
17	208	247	.814	575	43
18	237	277	.780	572	42
19	266	306	.746	570	41
20	.09295	.09335	10.712	.99567	40
21	324	365	.678	564	39
22	353	394	.645	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	.09440	.09482	10.546	.99553	35
26	469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	.09585	.09629	10.385	.99540	30
31	614	658	.354	537	29
32	642	688	.322	534	28
33	671	717	.291	531	27
34	700	746	.260	528	26
35	.09729	.09776	10.229	.99526	25
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	.09874	.09923	10.078	.99511	20
41	903	952	.048	508	19
42	932	.09981	10.019	506	18
43	961	.10011	9.9893	503	17
44	.09990	040	.9601	500	16
45	.10019	.10069	9.9310	.99497	15
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135	187	.8164	485	11
50	.10164	.10216	9.7882	.99482	10
51	192	246	.7601	479	9
52	221	275	.7322	476	8
53	250	305	.7044	473	7
54	279	334	.6768	470	6
55	.10308	.10363	9.6493	.99467	5
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	0
	Cos	Ctn	Tan	Sin	′

84°

′	Sin	Tan	Ctn	Cos	
0	.10453	.10510	9.5144	.99452	**60**
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	.10597	.10657	9.3831	.99437	**55**
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	746	.3060	428	52
9	713	775	.2806	424	51
10	.10742	.10805	9.2553	.99421	**50**
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	.10887	.10952	9.1309	.99406	**45**
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	040	.0579	396	42
19	.11002	070	.0338	393	41
20	.11031	.11099	9.0098	.99390	**40**
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	.11176	.11246	8.8919	.99374	**35**
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	263	335	.8225	364	32
29	291	364	.7996	360	31
30	.11320	.11394	8.7769	.99357	**30**
31	349	423	.7542	354	29
32	378	452	.7317	351	28
33	407	482	.7093	347	27
34	436	511	.6870	344	26
35	.11465	.11541	8.6648	.99341	**25**
36	494	570	.6427	337	24
37	523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	.11609	.11688	8.5555	.99324	**20**
41	638	718	.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	.11836	8.4490	.99307	**15**
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	.11898	.11983	8.3450	.99290	**10**
51	927	.12013	.3245	286	9
52	956	042	.3041	283	8
53	.11985	072	.2838	279	7
54	.12014	101	.2636	276	6
55	.12043	.12131	8.2434	.99272	**5**
56	071	160	.2234	269	4
57	100	190	.2035	265	3
58	129	219	.1837	262	2
59	158	249	.1640	258	1
60	.12187	.12278	8.1443	.99255	**0**
	Cos	Ctn	Tan	Sin	′

83°

′	Sin	Tan	Ctn	Cos	
0	.12187	.12278	8.1443	.99255	**60**
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	.12331	.12426	8.0476	.99237	**55**
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	515	7.9906	226	52
9	447	544	.9718	222	51
10	.12476	.12574	7.9530	.99219	**50**
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	.12620	.12722	7.8606	.99200	**45**
16	649	751	.8424	197	44
17	678	781	.8243	193	43
18	706	810	.8062	189	42
19	735	840	.7882	186	41
20	.12764	.12869	7.7704	.99182	**40**
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	.12908	.13017	7.6821	.99163	**35**
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	.13053	.13165	7.5958	.99144	**30**
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	254	.5449	133	27
34	168	284	.5281	129	26
35	.13197	.13313	7.5113	.99125	**25**
36	226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	.13341	.13461	7.4287	.99106	**20**
41	370	491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	.3800	094	17
44	456	580	.3639	091	16
45	.13485	.13609	7.3479	.99087	**15**
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	728	.2844	071	11
50	.13629	.13758	7.2687	.99067	**10**
51	658	787	.2531	063	9
52	687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	.13773	.13906	7.1912	.99047	**5**
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	**0**
	Cos	Ctn	Tan	Sin	′

82°

′	Sin	Tan	Ctn	Cos	
0	.13917	.14054	7.1154	.99027	60
1	946	084	.1004	023	59
2	.13975	113	.0855	019	58
3	.14004	143	.0706	015	57
4	033	173	.0558	011	56
5	.14061	.14202	7.0410	.99006	55
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	291	6.9972	994	52
9	177	321	.9827	990	51
10	.14205	.14351	6.9682	.98986	50
11	234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	.14349	.14499	6.8969	.98965	45
16	378	529	.8828	961	44
17	407	559	.8687	957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	.14648	6.8269	.98944	40
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	767	.7720	927	36
25	.14637	.14796	6.7584	.98923	35
26	666	826	.7448	919	34
27	695	856	.7313	914	33
28	723	886	.7179	910	32
29	752	915	.7045	906	31
30	.14781	.14945	6.6912	.98902	30
31	810	.14975	.6779	897	29
32	838	.15005	.6646	893	28
33	867	034	.6514	889	27
34	896	064	.6383	884	26
35	.14925	.15094	6.6252	.98880	25
36	954	124	.6122	876	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	867	22
39	040	213	.5734	863	21
40	.15069	.15243	6.5606	.98858	20
41	097	272	.5478	854	19
42	126	302	.5350	849	18
43	155	332	.5223	845	17
44	184	362	.5097	841	16
45	.15212	.15391	6.4971	.98836	15
46	241	421	.4846	832	14
47	270	451	.4721	827	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	.15356	.15540	6.4348	.98814	10
51	385	570	.4225	809	9
52	414	600	.4103	805	8
53	442	630	.3980	800	7
54	471	660	.3859	796	6
55	.15500	.15689	6.3737	.98791	5
56	529	719	.3617	787	4
57	557	749	.3496	782	3
58	586	779	.3376	778	2
59	615	809	.3257	773	1
60	.15643	.15838	6.3138	.98769	0
	Cos	Ctn	Tan	Sin	′

81°

′	Sin	Tan	Ctn	Cos	
0	.15643	.15838	6.3138	.98769	60
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2666	751	56
5	.15787	.15988	6.2549	.98746	55
6	816	.16017	.2432	741	54
7	845	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	.2085	728	51
10	.15931	.16137	6.1970	.98723	50
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	256	.1515	704	46
15	.16074	.16286	6.1402	.98700	45
16	103	316	.1290	695	44
17	132	346	.1178	690	43
18	160	376	.1066	686	42
19	189	405	.0955	681	41
20	.16218	.16435	6.0844	.98676	40
21	246	465	.0734	671	39
22	275	495	.0624	667	38
23	304	525	.0514	662	37
24	333	555	.0405	657	36
25	.16361	.16585	6.0296	.98652	35
26	390	615	.0188	648	34
27	419	645	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865	633	31
30	.16505	.16734	5.9758	.98629	30
31	533	764	.9651	624	29
32	562	794	.9545	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	.16648	.16884	5.9228	.98604	25
36	677	914	.9124	600	24
37	706	944	.9019	595	23
38	734	.16974	.8915	590	22
39	763	.17004	.8811	585	21
40	.16792	.17033	5.8708	.98580	20
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	.8298	561	16
45	.16935	.17183	5.8197	.98556	15
46	964	213	.8095	551	14
47	.16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	.17078	.17333	5.7694	.98531	10
51	107	363	.7594	526	9
52	136	393	.7495	521	8
53	164	423	.7396	516	7
54	193	453	.7297	511	6
55	.17222	.17483	5.7199	.98506	5
56	250	513	.7101	501	4
57	279	543	.7004	496	3
58	308	573	.6906	491	2
59	336	603	.6809	486	1
60	.17365	.17633	5.6713	.98481	0
	Cos	Ctn	Tan	Sin	′

80°

′	Sin	Tan	Ctn	Cos	
0	.17365	.17633	5.6713	.98481	**60**
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425	466	57
4	479	753	.6329	461	56
5	.17508	.17783	5.6234	.98455	**55**
6	537	813	.6140	450	54
7	565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435	51
10	.17651	.17933	5.5764	.98430	**50**
11	680	963	.5671	425	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	.18083	5.5301	.98404	**45**
16	823	113	.5209	399	44
17	852	143	.5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	.17937	.18233	5.4845	.98378	**40**
21	966	263	.4755	373	39
22	.17995	293	.4665	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	.18081	.18384	5.4397	.98352	**35**
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195	504	.4043	331	31
30	.18224	.18534	5.3955	.98325	**30**
31	252	564	.3868	320	29
32	281	594	.3781	315	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	.18367	.18684	5.3521	.98299	**25**
36	395	714	.3435	294	24
37	424	715	.3349	288	23
38	452	775	.3263	283	22
39	481	805	.3178	277	21
40	.18509	.18835	5.3093	.98272	**20**
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	.18652	.18986	5.2672	.98245	**15**
46	681	.19016	.2588	240	14
47	710	046	.2505	234	13
48	738	076	.2422	229	12
49	767	106	.2339	223	11
50	.18795	.19136	5.2257	.98218	**10**
51	824	166	.2174	212	9
52	852	197	.2092	207	8
53	881	227	.2011	201	7
54	910	257	.1929	196	6
55	.18938	.19287	5.1848	.98190	**5**
56	967	317	.1767	185	4
57	.18995	347	.1686	179	3
58	.19024	378	.1606	174	2
59	052	408	.1526	168	1
60	.19081	.19438	5.1446	.98163	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

79°

′	Sin	Tan	Ctn	Cos	
0	.19081	.19438	5.1446	.98163	**60**
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	.19224	.19589	5.1049	.98135	**55**
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	309	680	.0814	118	52
9	338	710	.0736	112	51
10	.19366	.19740	5.0658	.98107	**50**
11	395	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.0350	084	46
15	.19509	.19891	5.0273	.98079	**45**
16	538	921	.0197	073	44
17	566	952	.0121	067	43
18	595	.19982	5.0045	061	42
19	623	.20012	4.9969	056	41
20	.19652	.20042	4.9894	.98050	**40**
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	766	164	.9594	027	36
25	.19794	.20194	4.9520	.98021	**35**
26	823	224	.9446	016	34
27	851	254	.9372	010	33
28	880	285	.9298	.98004	32
29	908	315	.9225	.97998	31
30	.19937	.20345	4.9152	.97992	**30**
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	466	.8860	969	26
35	.20079	.20497	4.8788	.97963	**25**
36	108	527	.8716	958	24
37	136	557	.8644	952	23
38	165	588	.8573	946	22
39	193	618	.8501	940	21
40	.20222	.20648	4.8430	.97934	**20**
41	250	679	.8359	928	19
42	279	709	.8288	922	18
43	307	739	.8218	916	17
44	336	770	.8147	910	16
45	.20364	.20800	4.8077	.97905	**15**
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	450	891	.7867	887	12
49	478	921	.7798	881	11
50	.20507	.20952	4.7729	.97875	**10**
51	535	.20982	.7659	869	9
52	563	.21013	.7591	863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	.20649	.21104	4.7385	.97845	**5**
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	4.7046	.97815	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

78°

′	Sin	Tan	Ctn	Cos	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	655	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	.22495	.23087	4.3315	.97437	0
	Cos	Ctn	Tan	Sin	′

77°

′	Sin	Tan	Ctn	Cos	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	.23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	.23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24316	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	9
52	966	686	.0509	086	8
53	.23995	717	.0459	079	7
54	.24023	747	.0408	072	6
55	.24051	.24778	4.0358	.97065	5
56	079	809	.0308	058	4
57	108	840	.0257	051	3
58	136	871	.0207	044	2
59	164	902	.0158	037	1
60	.24192	.24933	4.0108	.97030	0
	Cos	Ctn	Tan	Sin	′

76°

′	Sin	Tan	Ctn	Cos	
0	.24192	.24933	4.0108	.97030	60
1	220	964	.0058	023	59
2	249	.24995	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	362	118	.9812	987	54
7	390	149	.9763	980	53
8	418	180	.9714	973	52
9	446	211	.9665	966	51
10	.24474	.25242	3.9617	.96959	50
11	503	273	.9568	952	49
12	531	304	.9520	945	48
13	559	335	.9471	937	47
14	587	366	.9423	930	46
15	.24615	.25397	3.9375	.96923	45
16	644	428	.9327	916	44
17	672	459	.9279	909	43
18	700	490	.9232	902	42
19	728	521	.9184	894	41
20	.24756	.25552	3.9136	.96887	40
21	784	583	.9089	880	39
22	813	614	.9042	873	38
23	841	645	.8995	866	37
24	869	676	.8947	858	36
25	.24897	.25707	3.8900	.96851	35
26	925	738	.8854	844	34
27	954	769	.8807	837	33
28	.24982	800	.8760	829	32
29	.25010	831	.8714	822	31
30	.25038	.25862	3.8667	.96815	30
31	066	893	.8621	807	29
32	094	924	.8575	800	28
33	122	955	.8528	793	27
34	151	.25986	.8482	786	26
35	.25179	.26017	3.8436	.96778	25
36	207	048	.8391	771	24
37	235	079	.8345	764	23
38	263	110	.8299	756	22
39	291	141	.8254	749	21
40	.25320	.26172	3.8208	.96742	20
41	348	203	.8163	734	19
42	376	235	.8118	727	18
43	404	266	.8073	719	17
44	432	297	.8028	712	16
45	.25460	.26328	3.7983	.96705	15
46	488	359	.7938	697	14
47	516	390	.7893	690	13
48	545	421	.7848	682	12
49	573	452	.7804	675	11
50	.25601	.26483	3.7760	.96667	10
51	629	515	.7715	660	9
52	657	546	.7671	653	8
53	685	577	.7627	645	7
54	713	608	.7583	638	6
55	.25741	.26639	3.7539	.96630	5
56	769	670	.7495	623	4
57	798	701	.7451	615	3
58	826	733	.7408	608	2
59	854	764	.7364	600	1
60	.25882	.26795	3.7321	.96593	0
	Cos	Ctn	Tan	Sin	′

75°

′	Sin	Tan	Ctn	Cos	
0	.25882	.26795	3.7321	.96593	60
1	910	826	.7277	585	59
2	938	857	.7234	578	58
3	966	888	.7191	570	57
4	.25994	920	.7148	562	56
5	.26022	.26951	3.7105	.96555	55
6	050	.26982	.7062	547	54
7	079	.27013	.7019	540	53
8	107	044	.6976	532	52
9	135	076	.6933	524	51
10	.26163	.27107	3.6891	.96517	50
11	191	138	.6848	509	49
12	219	169	.6806	502	48
13	247	201	.6764	494	47
14	275	232	.6722	486	46
15	.26303	.27263	3.6680	.96479	45
16	331	294	.6638	471	44
17	359	326	.6596	463	43
18	387	357	.6554	456	42
19	415	388	.6512	448	41
20	.26443	.27419	3.6470	.96440	40
21	471	451	.6429	433	39
22	500	482	.6387	425	38
23	528	513	.6346	417	37
24	556	545	.6305	410	36
25	.26584	.27576	3.6264	.96402	35
26	612	607	.6222	394	34
27	640	638	.6181	386	33
28	668	670	.6140	379	32
29	696	701	.6100	371	31
30	.26724	.27732	3.6059	.96363	30
31	752	764	.6018	355	29
32	780	795	.5978	347	28
33	808	826	.5937	340	27
34	836	858	.5897	332	26
35	.26864	.27889	3.5856	.96324	25
36	892	921	.5816	316	24
37	920	952	.5776	308	23
38	948	.27983	.5736	301	22
39	.26976	.28015	.5696	293	21
40	.27004	.28046	3.5656	.96285	20
41	032	077	.5616	277	19
42	060	109	.5576	269	18
43	088	140	.5536	261	17
44	116	172	.5497	253	16
45	.27144	.28203	3.5457	.96246	15
46	172	234	.5418	238	14
47	200	266	.5379	230	13
48	228	297	.5339	222	12
49	256	329	.5300	214	11
50	.27284	.28360	3.5261	.96206	10
51	312	391	.5222	198	9
52	340	423	.5183	190	8
53	368	454	.5144	182	7
54	396	486	.5105	174	6
55	.27424	.28517	3.5067	.96166	5
56	452	549	.5028	158	4
57	480	580	.4989	150	3
58	508	612	.4951	142	2
59	536	643	.4912	134	1
60	.27564	.28675	3.4874	.96126	0
	Cos	Ctn	Tan	Sin	′

74°

16° — Values of Trigonometric Functions — 17°

′	Sin	Tan	Ctn	Cos	
0	.27564	.28675	3.4874	.96126	60
1	592	706	.4836	118	59
2	620	738	.4798	110	58
3	648	769	.4760	102	57
4	676	801	.4722	094	56
5	.27704	.28832	3.4684	.96086	55
6	731	864	.4646	078	54
7	759	895	.4608	070	53
8	787	927	.4570	062	52
9	815	958	.4533	054	51
10	.27843	.28990	3.4495	.96046	50
11	871	.29021	.4458	037	49
12	899	053	.4420	029	48
13	927	084	.4383	021	47
14	955	116	.4346	013	46
15	.27983	.29147	3.4308	.96005	45
16	.28011	179	.4271	.95997	44
17	039	210	.4234	989	43
18	067	242	.4197	981	42
19	095	274	.4160	972	41
20	.28123	.29305	3.4124	.95964	40
21	150	337	.4087	956	39
22	178	368	.4050	948	38
23	206	400	.4014	940	37
24	234	432	.3977	931	36
25	.28262	.29463	3.3941	.95923	35
26	290	495	.3904	915	34
27	318	526	.3868	907	33
28	346	558	.3832	898	32
29	374	590	.3796	890	31
30	.28402	.29621	3.3759	.95882	30
31	429	653	.3723	874	29
32	457	685	.3687	865	28
33	485	716	.3652	857	27
34	513	748	.3616	849	26
35	.28541	.29780	3.3580	.95841	25
36	569	811	.3544	832	24
37	597	843	.3509	824	23
38	625	875	.3473	816	22
39	652	906	.3438	807	21
40	.28680	.29938	3.3402	.95799	20
41	708	.29970	.3367	791	19
42	736	.30001	.3332	782	18
43	764	033	.3297	774	17
44	792	065	.3261	766	16
45	.28820	.30097	3.3226	.95757	15
46	847	128	.3191	749	14
47	875	160	.3156	740	13
48	903	192	.3122	732	12
49	931	224	.3087	724	11
50	.28959	.30255	3.3052	.95715	10
51	.28987	287	.3017	707	9
52	.29015	319	.2983	698	8
53	042	351	.2948	690	7
54	070	382	.2914	681	6
55	.29098	.30414	3.2879	.95673	5
56	126	446	.2845	664	4
57	154	478	.2811	656	3
58	182	509	.2777	647	2
59	209	541	.2743	639	1
60	.29237	.30573	3.2709	.95630	0
	Cos	Ctn	Tan	Sin	′

73°

′	Sin	Tan	Ctn	Cos	
0	.29237	.30573	3.2709	.95630	60
1	265	605	.2675	622	59
2	293	637	.2641	613	58
3	321	669	.2607	605	57
4	348	700	.2573	596	56
5	.29376	.30732	3.2539	.95588	55
6	404	764	.2506	579	54
7	432	796	.2472	571	53
8	460	828	.2438	562	52
9	487	860	.2405	554	51
10	.29515	.30891	3.2371	.95545	50
11	543	923	.2338	536	49
12	571	955	.2305	528	48
13	599	.30987	.2272	519	47
14	626	.31019	.2238	511	46
15	.29654	.31051	3.2205	.95502	45
16	682	083	.2172	493	44
17	710	115	.2139	485	43
18	737	147	.2106	476	42
19	765	178	.2073	467	41
20	.29793	.31210	3.2041	.95459	40
21	821	242	.2008	450	39
22	849	274	.1975	441	38
23	876	306	.1943	433	37
24	904	338	.1910	424	36
25	.29932	.31370	3.1878	.95415	35
26	960	402	.1845	407	34
27	.29987	434	.1813	398	33
28	.30015	466	.1780	389	32
29	043	498	.1748	380	31
30	.30071	.31530	3.1716	.95372	30
31	098	562	.1684	363	29
32	126	594	.1652	354	28
33	154	626	.1620	345	27
34	182	658	.1588	337	26
35	.30209	.31690	3.1556	.95328	25
36	237	722	.1524	319	24
37	265	754	.1492	310	23
38	292	786	.1460	301	22
39	320	818	.1429	293	21
40	.30348	.31850	3.1397	.95284	20
41	376	882	.1366	275	19
42	403	914	.1334	266	18
43	431	946	.1303	257	17
44	459	.31978	.1271	248	16
45	.30486	.32010	3.1240	.95240	15
46	514	042	.1209	231	14
47	542	074	.1178	222	13
48	570	106	.1146	213	12
49	597	139	.1115	204	11
50	.30625	.32171	3.1084	.95195	10
51	653	203	.1053	186	9
52	680	235	.1022	177	8
53	708	267	.0991	168	7
54	736	299	.0961	159	6
55	.30763	.32331	3.0930	.95150	5
56	791	363	.0899	142	4
57	819	396	.0868	133	3
58	846	428	.0838	124	2
59	874	460	.0807	115	1
60	.30902	.32492	3.0777	.95106	0
	Cos	Ctn	Tan	Sin	′

72°

′	Sin	Tan	Ctn	Cos	
0	.30902	.32492	3.0777	.95106	60
1	929	524	.0746	097	59
2	957	556	.0716	088	58
3	.30985	588	.0686	079	57
4	.31012	621	.0655	070	56
5	.31040	.32653	3.0625	.95061	55
6	068	685	.0595	052	54
7	095	717	.0565	043	53
8	123	749	.0535	033	52
9	151	782	.0505	024	51
10	.31178	.32814	3.0475	.95015	50
11	206	846	.0445	.95006	49
12	233	878	.0415	.94997	48
13	261	911	.0385	988	47
14	289	943	.0356	979	46
15	.31316	.32975	3.0326	.94970	45
16	344	.33007	.0296	961	44
17	372	040	.0267	952	43
18	399	072	.0237	943	42
19	427	104	.0208	933	41
20	.31454	.33136	3.0178	.94924	40
21	482	169	.0149	915	39
22	510	201	.0120	906	38
23	537	233	.0090	897	37
24	565	266	.0061	888	36
25	.31593	.33298	3.0032	.94878	35
26	620	330	3.0003	869	34
27	648	363	2.9974	860	33
28	675	395	.9945	851	32
29	703	427	.9916	842	31
30	.31730	.33460	2.9887	.94832	30
31	758	492	.9858	823	29
32	786	524	.9829	814	28
33	813	557	.9800	805	27
34	841	589	.9772	795	26
35	.31868	.33621	2.9743	.94786	25
36	896	654	.9714	777	24
37	923	686	.9686	768	23
38	951	718	.9657	758	22
39	.31979	751	.9629	749	21
40	.32006	.33783	2.9600	.94740	20
41	034	816	.9572	730	19
42	061	848	.9544	721	18
43	089	881	.9515	712	17
44	116	913	.9487	702	16
45	.32144	.33945	2.9459	.94693	15
46	171	.33978	.9431	684	14
47	199	.34010	.9403	674	13
48	227	043	.9375	665	12
49	254	075	.9347	656	11
50	.32282	.34108	2.9319	.94646	10
51	309	140	.9291	637	9
52	337	173	.9263	627	8
53	364	205	.9235	618	7
54	392	238	.9208	609	6
55	.32419	.34270	2.9180	.94599	5
56	447	303	.9152	590	4
57	474	335	.9125	580	3
58	502	368	.9097	571	2
59	529	400	.9070	561	1
60	.32557	.34433	2.9042	.94552	0
	Cos	Ctn	Tan	Sin	′

71°

′	Sin	Tan	Ctn	Cos	
0	.32557	.34433	2.9042	.94552	60
1	584	465	.9015	542	59
2	612	498	.8987	533	58
3	639	530	.8960	523	57
4	667	563	.8933	514	56
5	.32694	.34596	2.8905	.94504	55
6	722	628	.8878	495	54
7	749	661	.8851	485	53
8	777	693	.8824	476	52
9	804	726	.8797	466	51
10	.32832	.34758	2.8770	.94457	50
11	859	791	.8743	447	49
12	887	824	.8716	438	48
13	914	856	.8689	428	47
14	942	889	.8662	418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	954	.8609	399	44
17	.33024	.34987	.8582	390	43
18	051	.35020	.8556	380	42
19	079	052	.8529	370	41
20	.33106	.35085	2.8502	.94361	40
21	134	118	.8476	351	39
22	161	150	.8449	342	38
23	189	183	.8423	332	37
24	216	216	.8397	322	36
25	.33244	.35248	2.8370	.94313	35
26	271	281	.8344	303	34
27	298	314	.8318	293	33
28	326	346	.8291	284	32
29	353	379	.8265	274	31
30	.33381	.35412	2.8239	.94264	30
31	408	445	.8213	254	29
32	436	477	.8187	245	28
33	463	510	.8161	235	27
34	490	543	.8135	225	26
35	.33518	.35576	2.8109	.94215	25
36	545	608	.8083	206	24
37	573	641	.8057	196	23
38	600	674	.8032	186	22
39	627	707	.8006	176	21
40	.33655	.35740	2.7980	.94167	20
41	682	772	.7955	157	19
42	710	805	.7929	147	18
43	737	838	.7903	137	17
44	764	871	.7878	127	16
45	.33792	.35904	2.7852	.94118	15
46	819	937	.7827	108	14
47	846	.35969	.7801	098	13
48	874	.36002	.7776	088	12
49	901	035	.7751	078	11
50	.33929	.36068	2.7725	.94068	10
51	956	101	.7700	058	9
52	.33983	134	.7675	049	8
53	.34011	167	.7650	039	7
54	038	199	.7625	029	6
55	.34065	.36232	2.7600	.94019	5
56	093	265	.7575	.94009	4
57	120	298	.7550	.93999	3
58	147	331	.7525	989	2
59	175	364	.7500	979	1
60	.34202	.36397	2.7475	.93969	0
	Cos	Ctn	Tan	Sin	′

70°

20° — Values of Trigonometric Functions — 21°

′	Sin	Tan	Ctn	Cos	
0	.34202	.36397	2.7475	.93969	**60**
1	229	430	.7450	959	59
2	257	463	.7425	949	58
3	284	496	.7400	939	57
4	311	529	.7376	929	56
5	.34339	.36562	2.7351	.93919	**55**
6	366	595	.7326	909	54
7	393	628	.7302	899	53
8	421	661	.7277	889	52
9	448	694	.7253	879	51
10	.34475	.36727	2.7228	.93869	**50**
11	503	760	.7204	859	49
12	530	793	.7179	849	48
13	557	826	.7155	839	47
14	584	859	.7130	829	46
15	.34612	.36892	2.7106	.93819	**45**
16	639	925	.7082	809	44
17	666	958	.7058	799	43
18	694	.36991	.7034	789	42
19	721	.37024	.7009	779	41
20	.34748	.37057	2.6985	.93769	**40**
21	775	090	.6961	759	39
22	803	123	.6937	748	38
23	830	157	.6913	738	37
24	857	190	.6889	728	36
25	.34884	.37223	2.6865	.93718	**35**
26	912	256	.6841	708	34
27	939	289	.6818	698	33
28	966	322	.6794	688	32
29	.34993	355	.6770	677	31
30	.35021	.37388	2.6746	.93667	**30**
31	048	422	.6723	657	29
32	075	455	.6699	647	28
33	102	488	.6675	637	27
34	130	521	.6652	626	26
35	.35157	.37554	2.6628	.93616	**25**
36	184	588	.6605	606	24
37	211	621	.6581	596	23
38	239	654	.6558	585	22
39	266	687	.6534	575	21
40	.35293	.37720	2.6511	.93565	**20**
41	320	754	.6488	555	19
42	347	787	.6464	544	18
43	375	820	.6441	534	17
44	402	853	.6418	524	16
45	.35429	.37887	2.6395	.93514	**15**
46	456	920	.6371	503	14
47	484	953	.6348	493	13
48	511	.37986	.6325	483	12
49	538	.38020	.6302	472	11
50	.35565	.38053	2.6279	.93462	**10**
51	592	086	.6256	452	9
52	619	120	.6233	441	8
53	647	153	.6210	431	7
54	674	186	.6187	420	6
55	.35701	.38220	2.6165	.93410	**5**
56	728	253	.6142	400	4
57	755	286	.6119	389	3
58	782	320	.6096	379	2
59	810	353	.6074	368	1
60	.35837	.38386	2.6051	.93358	**0**
	Cos	Ctn	Tan	Sin	′

69°

′	Sin	Tan	Ctn	Cos	
0	.35837	.38386	2.6051	.93358	**60**
1	864	420	.6028	348	59
2	891	453	.6006	337	58
3	918	487	.5983	327	57
4	945	520	.5961	316	56
5	.35973	.38553	2.5938	.93306	**55**
6	.36000	587	.5916	295	54
7	027	620	.5893	285	53
8	054	654	.5871	274	52
9	081	687	.5848	264	51
10	.36108	.38721	2.5826	.93253	**50**
11	135	754	.5804	243	49
12	162	787	.5782	232	48
13	190	821	.5759	222	47
14	217	854	.5737	211	46
15	.36244	.38888	2.5715	.93201	**45**
16	271	921	.5693	190	44
17	298	955	.5671	180	43
18	325	.38988	.5649	169	42
19	352	.39022	.5627	159	41
20	.36379	.39055	2.5605	.93148	**40**
21	406	089	.5583	137	39
22	434	122	.5561	127	38
23	461	156	.5539	116	37
24	488	190	.5517	106	36
25	.36515	.39223	2.5495	.93095	**35**
26	542	257	.5473	084	34
27	569	290	.5452	074	33
28	596	324	.5430	063	32
29	6[illegible]	357	.5408	052	31
30	.36650	.39391	2.5386	.93042	**30**
31	677	425	.5365	031	29
32	704	458	.5343	020	28
33	731	492	.5322	.93010	27
34	758	526	.5300	.92999	26
35	.36785	.39559	2.5279	.92988	**25**
36	812	593	.5257	978	24
37	839	626	.5236	967	23
38	867	660	.5214	956	22
39	894	694	.5193	945	21
40	.36921	.39727	2.5172	.92935	**20**
41	948	761	.5150	924	19
42	.36975	795	.5129	913	18
43	.37002	829	.5108	902	17
44	029	862	.5086	892	16
45	.37056	.39896	2.5065	.92881	**15**
46	083	930	.5044	870	14
47	110	963	.5023	859	13
48	137	.39997	.5002	849	12
49	164	.40031	.4981	838	11
50	.37191	.40065	2.4960	.92827	**10**
51	218	098	.4939	816	9
52	245	132	.4918	805	8
53	272	166	.4897	794	7
54	299	200	.4876	784	6
55	.37326	.40234	2.4855	.92773	**5**
56	353	267	.4834	762	4
57	380	301	.4813	751	3
58	407	335	.4792	740	2
59	434	369	.4772	729	1
60	.37461	.40403	2.4751	.92718	**0**
	Cos	Ctn	Tan	Sin	′

68°

′	Sin	Tan	Ctn	Cos	
0	.37461	.40403	2.4751	.92718	60
1	488	436	.4730	707	59
2	515	470	.4709	697	58
3	542	504	.4689	686	57
4	569	538	.4668	675	56
5	.37595	.40572	2.4648	.92664	55
6	622	606	.4627	653	54
7	649	640	.4606	642	53
8	676	674	.4586	631	52
9	703	707	.4566	620	51
10	.37730	.40741	2.4545	.92609	50
11	757	775	.4525	598	49
12	784	809	.4504	587	48
13	811	843	.4484	576	47
14	838	877	.4464	565	46
15	.37865	.40911	2.4443	.92554	45
16	892	945	.4423	543	44
17	919	.40979	.4403	532	43
18	946	.41013	.4383	521	42
19	973	047	.4362	510	41
20	.37999	.41081	2.4342	.92499	40
21	.38026	115	.4322	488	39
22	053	149	.4302	477	38
23	080	183	.4282	466	37
24	107	217	.4262	455	36
25	.38134	.41251	2.4242	.92444	35
26	161	285	.4222	432	34
27	188	319	.4202	421	33
28	215	353	.4182	410	32
29	241	387	.4162	399	31
30	.38268	.41421	2.4142	.92388	30
31	295	455	.4122	377	29
32	322	490	.4102	366	28
33	349	524	.4083	355	27
34	376	558	.4063	343	26
35	.38403	.41592	2.4043	.92332	25
36	430	626	.4023	321	24
37	456	660	.4004	310	23
38	483	694	.3984	299	22
39	510	728	.3964	287	21
40	.38537	.41763	2.3945	.92276	20
41	564	797	.3925	265	19
42	591	831	.3906	254	18
43	617	865	.3886	243	17
44	644	899	.3867	231	16
45	.38671	.41933	2.3847	.92220	15
46	698	.41968	.3828	209	14
47	725	.42002	.3808	198	13
48	752	036	.3789	186	12
49	778	070	.3770	175	11
50	.38805	.42105	2.3750	.92164	10
51	832	139	.3731	152	9
52	859	173	.3712	141	8
53	886	207	.3693	130	7
54	912	242	.3673	119	6
55	.38939	.42276	2.3654	.92107	5
56	966	310	.3635	096	4
57	.38993	345	.3616	085	3
58	.39020	379	.3597	073	2
59	046	413	.3578	062	1
60	.39073	.42447	2.3559	.92050	0
	Cos	Ctn	Tan	Sin	′

67°

′	Sin	Tan	Ctn	Cos	
0	.39073	.42447	2.3559	.92050	60
1	100	482	.3539	039	59
2	127	516	.3520	028	58
3	153	551	.3501	016	57
4	180	585	.3483	.92005	56
5	.39207	.42619	2.3464	.91994	55
6	234	654	.3445	982	54
7	260	688	.3426	971	53
8	287	722	.3407	959	52
9	314	757	.3388	948	51
10	.39341	.42791	2.3369	.91936	50
11	367	826	.3351	925	49
12	394	860	.3332	914	48
13	421	894	.3313	902	47
14	448	929	.3294	891	46
15	.39474	.42963	2.3276	.91879	45
16	501	.42998	.3257	868	44
17	528	.43032	.3238	856	43
18	555	067	.3220	845	42
19	581	101	.3201	833	41
20	.39608	.43136	2.3183	.91822	40
21	635	170	.3164	810	39
22	661	205	.3146	799	38
23	688	239	.3127	787	37
24	715	274	.3109	775	36
25	.39741	.43308	2.3090	.91764	35
26	768	343	.3072	752	34
27	795	378	.3053	741	33
28	822	412	.3035	729	32
29	848	447	.3017	718	31
30	.39875	.43481	2.2998	.91706	30
31	902	516	.2980	694	29
32	928	550	.2962	683	28
33	955	585	.2944	671	27
34	.39982	620	.2925	660	26
35	.40008	.43654	2.2907	.91648	25
36	035	689	.2889	636	24
37	062	724	.2871	625	23
38	088	758	.2853	613	22
39	115	793	.2835	601	21
40	.40141	.43828	2.2817	.91590	20
41	168	862	.2799	578	19
42	195	897	.2781	566	18
43	221	932	.2763	555	17
44	248	.43966	.2745	543	16
45	.40275	.44001	2.2727	.91531	15
46	301	036	.2709	519	14
47	328	071	.2691	508	13
48	355	105	.2673	496	12
49	381	140	.2655	484	11
50	.40408	.44175	2.2637	.91472	10
51	434	210	.2620	461	9
52	461	244	.2602	449	8
53	488	279	.2584	437	7
54	514	314	.2566	425	6
55	.40541	.44349	2.2549	.91414	5
56	567	384	.2531	402	4
57	594	418	.2513	390	3
58	621	453	.2496	378	2
59	647	488	.2478	366	1
60	.40674	.44523	2.2460	.91355	0
	Cos	Ctn	Tan	Sin	′

66°

′	Sin	Tan	Ctn	Cos	
0	.40674	.44523	2.2460	.91355	**60**
1	700	558	.2443	343	59
2	727	593	.2425	331	58
3	753	627	.2408	319	57
4	780	662	.2390	307	56
5	.40806	.44697	2.2373	.91295	**55**
6	833	732	.2355	283	54
7	860	767	.2338	272	53
8	886	802	.2320	260	52
9	913	837	.2303	248	51
10	.40939	.44872	2.2286	.91236	**50**
11	966	907	.2268	224	49
12	.40992	942	.2251	212	48
13	.41019	.44977	.2234	200	47
14	045	.45012	.2216	188	46
15	.41072	.45047	2.2199	.91176	**45**
16	098	082	.2182	164	44
17	125	117	.2165	152	43
18	151	152	.2148	140	42
19	178	187	.2130	128	41
20	.41204	.45222	2.2113	.91116	**40**
21	231	257	.2096	104	39
22	257	292	.2079	092	38
23	284	327	.2062	080	37
24	310	362	.2045	068	36
25	.41337	.45397	2.2028	.91056	**35**
26	363	432	.2011	044	34
27	390	467	.1994	032	33
28	416	502	.1977	020	32
29	443	538	.1960	.91008	31
30	.41469	.45573	2.1943	.90996	**30**
31	496	608	.1926	984	29
32	522	643	.1909	972	28
33	549	678	.1892	960	27
34	575	713	.1876	948	26
35	.41602	.45748	2.1859	.90936	**25**
36	628	784	.1842	924	24
37	655	819	.1825	911	23
38	681	854	.1808	899	22
39	707	889	.1792	887	21
40	.41734	.45924	2.1775	.90875	**20**
41	760	960	.1758	863	19
42	787	.45995	.1742	851	18
43	813	.46030	.1725	839	17
44	840	065	.1708	826	16
45	.41866	.46101	2.1692	.90814	**15**
46	892	136	.1675	802	14
47	919	171	.1659	790	13
48	945	206	.1642	778	12
49	972	242	.1625	766	11
50	.41998	.46277	2.1609	.90753	**10**
51	.42024	312	.1592	741	9
52	051	348	.1576	729	8
53	077	383	.1560	717	7
54	104	418	.1543	704	6
55	.42130	.46454	2.1527	.90692	**5**
56	156	489	.1510	680	4
57	183	525	.1494	668	3
58	209	560	.1478	655	2
59	235	595	.1461	643	1
60	.42262	.46631	2.1445	.90631	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

65°

′	Sin	Tan	Ctn	Cos	
0	.42262	.46631	2.1445	.90631	**60**
1	288	666	.1429	618	59
2	315	702	.1413	606	58
3	341	737	.1396	594	57
4	367	772	.1380	582	56
5	.42394	.46808	2.1364	.90569	**55**
6	420	843	.1348	557	54
7	446	879	.1332	545	53
8	473	914	.1315	532	52
9	499	950	.1299	520	51
10	.42525	.46985	2.1283	.90507	**50**
11	552	.47021	.1267	495	49
12	578	056	.1251	483	48
13	604	092	.1235	470	47
14	631	128	.1219	458	46
15	.42657	.47163	2.1203	.90446	**45**
16	683	199	.1187	433	44
17	709	234	.1171	421	43
18	736	270	.1155	408	42
19	762	305	.1139	396	41
20	.42788	.47341	2.1123	.90383	**40**
21	815	377	.1107	371	39
22	841	412	.1092	358	38
23	867	448	.1076	346	37
24	894	483	.1060	334	36
25	.42920	.47519	2.1044	.90321	**35**
26	946	555	.1028	309	34
27	972	590	.1013	296	33
28	.42999	626	.0997	284	32
29	.43025	662	.0981	271	31
30	.43051	.47698	2.0965	.90259	**30**
31	077	733	.0950	246	29
32	104	769	.0934	233	28
33	130	805	.0918	221	27
34	156	840	.0903	208	26
35	.43182	.47876	2.0887	.90196	**25**
36	209	912	.0872	183	24
37	235	948	.0856	171	23
38	261	.47984	.0840	158	22
39	287	.48019	.0825	146	21
40	.43313	.48055	2.0809	.90133	**20**
41	340	091	.0794	120	19
42	366	127	.0778	108	18
43	392	163	.0763	095	17
44	418	198	.0748	082	16
45	.43445	.48234	2.0732	.90070	**15**
46	471	270	.0717	057	14
47	497	306	.0701	045	13
48	523	342	.0686	032	12
49	549	378	.0671	019	11
50	.43575	.48414	2.0655	.90007	**10**
51	602	450	.0640	.89994	9
52	628	486	.0625	981	8
53	654	521	.0609	968	7
54	680	557	.0594	956	6
55	.43706	.48593	2.0579	.89943	**5**
56	733	629	.0564	930	4
57	759	665	.0549	918	3
58	785	701	.0533	905	2
59	811	737	.0518	892	1
60	.43837	.48773	2.0503	.89879	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

64°

′	Sin	Tan	Ctn	Cos	
0	.43837	.48773	2.0503	.89879	60
1	863	809	.0488	867	59
2	889	845	.0473	854	58
3	916	881	.0458	841	57
4	942	917	.0443	828	56
5	.43968	.48953	2.0428	.89816	55
6	.43994	.48989	.0413	803	54
7	.44020	.49026	.0398	790	53
8	046	062	.0383	777	52
9	072	098	.0368	764	51
10	.44098	.49134	2.0353	.89752	50
11	124	170	.0338	739	49
12	151	206	.0323	726	48
13	177	242	.0308	713	47
14	203	278	.0293	700	46
15	.44229	.49315	2.0278	.89687	45
16	255	351	.0263	674	44
17	281	387	.0248	662	43
18	307	423	.0233	649	42
19	333	459	.0219	636	41
20	.44359	.49495	2.0204	.89623	40
21	385	532	.0189	610	39
22	411	568	.0174	597	38
23	437	604	.0160	584	37
24	464	640	.0145	571	36
25	.44490	.49677	2.0130	.89558	35
26	516	713	.0115	545	34
27	542	749	.0101	532	33
28	568	786	.0086	519	32
29	594	822	.0072	506	31
30	.44620	.49858	2.0057	.89493	30
31	646	894	.0042	480	29
32	672	931	.0028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	.44750	.50040	1.9984	.89428	25
36	776	076	.9970	415	24
37	802	113	.9955	402	23
38	828	149	.9941	389	22
39	854	185	.9926	376	21
40	.44880	.50222	1.9912	.89363	20
41	906	258	.9897	350	19
42	932	295	.9883	337	18
43	958	331	.9868	324	17
44	.44984	368	.9854	311	16
45	.45010	.50404	1.9840	.89298	15
46	036	441	.9825	285	14
47	062	477	.9811	272	13
48	088	514	.9797	259	12
49	114	550	.9782	245	11
50	.45140	.50587	1.9768	.89232	10
51	166	623	.9754	219	9
52	192	660	.9740	206	8
53	218	696	.9725	193	7
54	243	733	.9711	180	6
55	.45269	.50769	1.9697	.89167	5
56	295	806	.9683	153	4
57	321	843	.9669	140	3
58	347	879	.9654	127	2
59	373	916	.9640	114	1
60	.45399	.50953	1.9626	.89101	0
	Cos	Ctn	Tan	Sin	′

63°

′	Sin	Tan	Ctn	Cos	
0	.45399	.50953	1.9626	.89101	60
1	425	.50989	.9612	087	59
2	451	.51026	.9598	074	58
3	477	063	.9584	061	57
4	503	099	.9570	048	56
5	.45529	.51136	1.9556	.89035	55
6	554	173	.9542	021	54
7	580	209	.9528	.89008	53
8	606	246	.9514	.88995	52
9	632	283	.9500	981	51
10	.45658	.51319	1.9486	.88968	50
11	684	356	.9472	955	49
12	710	393	.9458	942	48
13	736	430	.9444	928	47
14	762	467	.9430	915	46
15	.45787	.51503	1.9416	.88902	45
16	813	540	.9402	888	44
17	839	577	.9388	875	43
18	865	614	.9375	862	42
19	891	651	.9361	848	41
20	.45917	.51688	1.9347	.88835	40
21	942	724	.9333	822	39
22	968	761	.9319	808	38
23	.45994	798	.9306	795	37
24	.46020	835	.9292	782	36
25	.46046	.51872	1.9278	.88768	35
26	072	909	.9265	755	34
27	097	946	.9251	741	33
28	123	.51983	.9237	728	32
29	149	.52020	.9223	715	31
30	.46175	.52057	1.9210	.88701	30
31	201	094	.9196	688	29
32	226	131	.9183	674	28
33	252	168	.9169	661	27
34	278	205	.9155	647	26
35	.46304	.52242	1.9142	.88634	25
36	330	279	.9128	620	24
37	355	316	.9115	607	23
38	381	353	.9101	593	22
39	407	390	.9088	580	21
40	.46433	.52427	1.9074	.88566	20
41	458	464	.9061	553	19
42	484	501	.9047	539	18
43	510	538	.9034	526	17
44	536	575	.9020	512	16
45	.46561	.52613	1.9007	.88499	15
46	587	650	.8993	485	14
47	613	687	.8980	472	13
48	639	724	.8967	458	12
49	664	761	.8953	445	11
50	.46690	.52798	1.8940	.88431	10
51	716	836	.8927	417	9
52	742	873	.8913	404	8
53	767	910	.8900	390	7
54	793	947	.8887	377	6
55	.46819	.52985	1.8873	.88363	5
56	844	.53022	.8860	349	4
57	870	059	.8847	336	3
58	896	096	.8834	322	2
59	921	134	.8820	308	1
60	.46947	.53171	1.8807	.88295	0
	Cos	Ctn	Tan	Sin	′

62°

′	Sin	Tan	Ctn	Cos	
0	.46947	.53171	1.8807	.88295	60
1	973	208	.8794	281	59
2	.46999	246	.8781	267	58
3	.47024	283	.8768	254	57
4	050	320	.8755	240	56
5	.47076	.53358	1.8741	.88226	55
6	101	395	.8728	213	54
7	127	432	.8715	199	53
8	153	470	.8702	185	52
9	178	507	.8689	172	51
10	.47204	.53545	1.8676	.88158	50
11	229	582	.8663	144	49
12	255	620	.8650	130	48
13	281	657	.8637	117	47
14	306	694	.8624	103	46
15	.47332	.53732	1.8611	.88089	45
16	358	769	.8598	075	44
17	383	807	.8585	062	43
18	409	844	.8572	048	42
19	434	882	.8559	034	41
20	.47460	.53920	1.8546	.88020	40
21	486	957	.8533	.88006	39
22	511	.53995	.8520	.87993	38
23	537	.54032	.8507	979	37
24	562	070	.8495	965	36
25	.47588	.54107	1.8482	.87951	35
26	614	145	.8469	937	34
27	639	183	.8456	923	33
28	665	220	.8443	909	32
29	690	258	.8430	896	31
30	.47716	.54296	1.8418	.87882	30
31	741	333	.8405	868	29
32	767	371	.8392	854	28
33	793	409	.8379	840	27
34	818	446	.8367	826	26
35	.47844	.54484	1.8354	.87812	25
36	869	522	.8341	798	24
37	895	560	.8329	784	23
38	920	597	.8316	770	22
39	946	635	.8303	756	21
40	.47971	.54673	1.8291	.87743	20
41	.47997	711	.8278	729	19
42	.48022	748	.8265	715	18
43	048	786	.8253	701	17
44	073	824	.8240	687	16
45	.48099	.54862	1.8228	.87673	15
46	124	900	.8215	659	14
47	150	938	.8202	645	13
48	175	.54975	.8190	631	12
49	201	.55013	.8177	617	11
50	.48226	.55051	1.8165	.87603	10
51	252	089	.8152	589	9
52	277	127	.8140	575	8
53	303	165	.8127	561	7
54	328	203	.8115	546	6
55	.48354	.55241	1.8103	.87532	5
56	379	279	.8090	518	4
57	405	317	.8078	504	3
58	430	355	.8065	490	2
59	456	393	.8053	476	1
60	.48481	.55431	1.8040	.87462	0
	Cos	Ctn	Tan	Sin	′

61°

′	Sin	Tan	Ctn	Cos	
0	.48481	.55431	1.8040	.87462	60
1	506	469	.8028	448	59
2	532	507	.8016	434	58
3	557	545	.8003	420	57
4	583	583	.7991	406	56
5	.48608	.55621	1.7979	.87391	55
6	634	659	.7966	377	54
7	659	697	.7954	363	53
8	684	736	.7942	349	52
9	710	774	.7930	335	51
10	.48735	.55812	1.7917	.87321	50
11	761	850	.7905	306	49
12	786	888	.7893	292	48
13	811	926	.7881	278	47
14	837	.55964	.7868	264	46
15	.48862	.56003	1.7856	.87250	45
16	888	041	.7844	235	44
17	913	079	.7832	221	43
18	938	117	.7820	207	42
19	964	156	.7808	193	41
20	.48989	.56194	1.7796	.87178	40
21	.49014	232	.7783	164	39
22	040	270	.7771	150	38
23	065	309	.7759	136	37
24	090	347	.7747	121	36
25	.49116	.56385	1.7735	.87107	35
26	141	424	.7723	093	34
27	166	462	.7711	079	33
28	192	501	.7699	064	32
29	217	539	.7687	050	31
30	.49242	.56577	1.7675	.87036	30
31	268	616	.7663	021	29
32	293	654	.7651	.87007	28
33	318	693	.7639	.86993	27
34	344	731	.7627	978	26
35	.49369	.56769	1.7615	.86964	25
36	394	808	.7603	949	24
37	419	846	.7591	935	23
38	445	885	.7579	921	22
39	470	923	.7567	906	21
40	.49495	.56962	1.7556	.86892	20
41	521	.57000	.7544	878	19
42	546	039	.7532	863	18
43	571	078	.7520	849	17
44	596	116	.7508	834	16
45	.49622	.57155	1.7496	.86820	15
46	647	193	.7485	805	14
47	672	232	.7473	791	13
48	697	271	.7461	777	12
49	723	309	.7449	762	11
50	.49748	.57348	1.7437	.86748	10
51	773	386	.7426	733	9
52	798	425	.7414	719	8
53	824	464	.7402	704	7
54	849	503	.7391	690	6
55	.49874	.57541	1.7379	.86675	5
56	899	580	.7367	661	4
57	924	619	.7355	646	3
58	950	657	.7344	632	2
59	.49975	696	.7332	617	1
60	.50000	.57735	1.7321	.86603	0
	Cos	Ctn	Tan	Sin	′

60°

′	Sin	Tan	Ctn	Cos	
0	.50000	.57735	1.7321	.86603	**60**
1	025	774	.7309	588	59
2	050	813	.7297	573	58
3	076	851	.7286	559	57
4	101	890	.7274	544	56
5	.50126	.57929	1.7262	.86530	**55**
6	151	.57968	.7251	515	54
7	176	.58007	.7239	501	53
8	201	046	.7228	486	52
9	227	085	.7216	471	51
10	50252	.58124	1.7205	.86457	**50**
11	277	162	.7193	442	49
12	302	201	.7182	427	48
13	327	240	.7170	413	47
14	352	279	.7159	398	46
15	.50377	.58318	1.7147	.86384	**45**
16	403	357	7136	369	44
17	428	396	.7124	354	43
18	453	435	.7113	340	42
19	478	474	.7102	325	41
20	.50503	.58513	1.7090	.86310	**40**
21	528	552	.7079	295	39
22	553	591	.7067	281	38
23	578	631	.7056	266	37
24	603	670	.7045	251	36
25	.50628	.58709	1.7033	.86237	**35**
26	654	748	.7022	222	34
27	679	787	.7011	207	33
28	704	826	.6999	192	32
29	729	865	.6988	178	31
30	.50754	.58905	1.6977	.86163	**30**
31	779	944	.6965	148	29
32	804	.58983	.6954	133	28
33	829	.59022	.6943	119	27
34	854	061	.6932	104	26
35	50879	.59101	1.6920	.86089	**25**
36	904	140	.6909	074	24
37	929	179	.6898	059	23
38	954	218	.6887	045	22
39	50979	258	.6875	030	21
40	.51004	.59297	1 6864	.86015	**20**
41	029	336	.6853	.86000	19
42	054	376	.6842	.85985	18
43	079	415	.6831	970	17
44	104	454	.6820	956	16
45	.51129	.59494	1.6808	.85941	**15**
46	154	533	.6797	926	14
47	179	573	.6786	911	13
48	204	612	.6775	896	12
49	229	651	.6764	881	11
50	.51254	.59691	1.6753	.85866	**10**
51	279	730	.6742	851	9
52	304	770	.6731	836	8
53	329	809	.6720	821	7
54	354	849	.6709	806	6
55	.51379	.59888	1 6698	.85792	**5**
56	404	928	.6687	777	4
57	429	.59967	.6676	762	3
58	454	.60007	.6665	747	2
59	479	046	.6654	732	1
60	.51504	.60086	1 6643	.85717	**0**
	Cos	Ctn	Tan	Sin	′

59°

′	Sin	Tan	Ctn	Cos	
0	.51504	.60086	1.6643	.85717	**60**
1	529	126	.6632	702	59
2	554	165	.6621	687	58
3	579	205	.6610	672	57
4	604	245	.6599	657	56
5	.51628	.60284	1.6588	.85642	**55**
6	653	324	.6577	627	54
7	678	364	.6566	612	53
8	703	403	.6555	597	52
9	728	443	.6545	582	51
10	.51753	.60483	1 6534	.85567	**50**
11	778	522	.6523	551	49
12	803	562	.6512	536	48
13	828	602	.6501	521	47
14	852	642	.6490	506	46
15	.51877	.60681	1.6479	.85491	**45**
16	902	721	.6469	476	44
17	927	761	.6458	461	43
18	952	801	.6447	446	42
19	.51977	841	.6436	431	41
20	.52002	.60881	1.6426	.85416	**40**
21	026	921	.6415	401	39
22	051	.60960	.6404	385	38
23	076	.61000	.6393	370	37
24	101	040	.6383	355	36
25	.52126	.61080	1.6372	.85340	**35**
26	151	120	.6361	325	34
27	175	160	.6351	310	33
28	200	200	.6340	294	32
29	225	240	.6329	279	31
30	.52250	.61280	1.6319	.85264	**30**
31	275	320	.6308	249	29
32	299	360	.6297	234	28
33	324	400	.6287	218	27
34	349	440	.6276	203	26
35	.52374	.61480	1.6265	.85188	**25**
36	399	520	.6255	173	24
37	423	561	.6244	157	23
38	448	601	.6234	142	22
39	473	641	.6223	127	21
40	.52498	.61681	1.6212	.85112	**20**
41	522	721	.6202	096	19
42	547	761	.6191	081	18
43	572	801	.6181	066	17
44	597	842	.6170	051	16
45	.52621	.61882	1.6160	.85035	**15**
46	646	922	.6149	020	14
47	671	.61962	.6139	.85005	13
48	696	.62003	.6128	.84989	12
49	720	043	.6118	974	11
50	.52745	.62083	1.6107	.84959	**10**
51	770	124	.6097	943	9
52	794	164	.6087	928	8
53	819	204	.6076	913	7
54	844	245	.6066	897	6
55	.52869	.62285	1.6055	.84882	**5**
56	893	325	.6045	866	4
57	918	366	.6034	851	3
58	943	406	.6024	836	2
59	967	446	.6014	820	1
60	.52992	.62487	1.6003	.84805	**0**
	Cos	Ctn	Tan	Sin	′

58°

′	Sin	Tan	Ctn	Cos	
0	.52992	.62487	1.6003	.84805	60
1	.53017	527	.5993	789	59
2	041	568	.5983	774	58
3	066	608	.5972	759	57
4	091	649	.5962	743	56
5	.53115	.62689	1.5952	.84728	55
6	140	730	.5941	712	54
7	164	770	.5931	697	53
8	189	811	.5921	681	52
9	214	852	.5911	666	51
10	.53238	.62892	1.5900	.84650	50
11	263	933	.5890	635	49
12	288	.62973	.5880	619	48
13	312	.63014	.5869	604	47
14	337	055	.5859	588	46
15	.53361	.63095	1.5849	.84573	45
16	386	136	.5839	557	44
17	411	177	.5829	542	43
18	435	217	.5818	526	42
19	460	258	.5808	511	41
20	.53484	.63299	1.5798	.84495	40
21	509	340	.5788	480	39
22	534	380	.5778	464	38
23	558	421	.5768	448	37
24	583	462	.5757	433	36
25	.53607	.63503	1.5747	.84417	35
26	632	544	.5737	402	34
27	656	584	.5727	386	33
28	681	625	.5717	370	32
29	705	666	.5707	355	31
30	.53730	63707	1 5697	.84339	30
31	754	748	.5687	324	29
32	779	789	.5677	308	28
33	804	830	.5667	292	27
34	828	871	.5657	277	26
35	.53853	.63912	1.5647	.84261	25
36	877	953	.5637	245	24
37	902	.63994	.5627	230	23
38	926	.64035	.5617	214	22
39	951	076	.5607	198	21
40	.53975	.64117	1.5597	.84182	20
41	.54000	158	.5587	167	19
42	024	199	.5577	151	18
43	049	240	.5567	135	17
44	073	281	.5557	120	16
45	.54097	.64322	1.5547	.84104	15
46	122	363	.5537	088	14
47	146	404	.5527	072	13
48	171	446	.5517	057	12
49	195	487	.5507	041	11
50	.54220	.64528	1.5497	.84025	10
51	244	569	.5487	.84009	9
52	269	610	.5477	.83994	8
53	293	652	.5468	978	7
54	317	693	.5458	962	6
55	.54342	.64734	1.5448	.83946	5
56	366	775	.5438	930	4
57	391	817	.5428	915	3
58	415	858	.5418	899	2
59	440	899	.5408	883	1
60	.54464	.64941	1 5399	.83867	0
	Cos	Ctn	Tan	Sin	′

57°

′	Sin	Tan	Ctn	Cos	
0	.54464	.64941	1.5399	.83867	60
1	488	.64982	.5389	851	59
2	513	.65024	.5379	835	58
3	537	065	.5369	819	57
4	561	106	.5359	804	56
5	54586	.65148	1.5350	83788	55
6	610	189	.5340	772	54
7	635	231	.5330	756	53
8	659	272	.5320	740	52
9	683	314	.5311	724	51
10	.54708	.65355	1.5301	.83708	50
11	732	397	.5291	692	49
12	756	438	.5282	676	48
13	781	480	.5272	660	47
14	805	521	.5262	645	46
15	54829	.65563	1.5253	.83629	45
16	854	604	.5243	613	44
17	878	646	.5233	597	43
18	902	688	.5224	581	42
19	927	729	.5214	565	41
20	.54951	.65771	1.5204	.83549	40
21	975	813	.5195	533	39
22	54999	854	5185	517	38
23	.55024	896	.5175	501	37
24	048	938	.5166	485	36
25	55072	.65980	1 5156	.83469	35
26	097	.66021	5147	453	34
27	121	063	.5137	437	33
28	145	105	.5127	421	32
29	169	147	.5118	405	31
30	55194	.66189	1 5108	.83389	30
31	218	230	.5099	373	29
32	242	272	.5089	356	28
33	266	314	.5080	340	27
34	291	356	.5070	324	26
35	.55315	.66398	1 5061	.83308	25
36	339	440	.5051	292	24
37	363	482	.5042	276	23
38	388	524	.5032	260	22
39	412	566	.5023	244	21
40	55436	.66608	1.5013	.83228	20
41	460	650	.5004	212	19
42	484	692	.4994	195	18
43	509	734	.4985	179	17
44	533	776	.4975	163	16
45	.55557	.66818	1.4966	.83147	15
46	581	860	.4957	131	14
47	605	902	.4947	115	13
48	630	944	.4938	098	12
49	654	66986	.4928	082	11
50	.55678	.67028	1.4919	.83066	10
51	702	071	.4910	050	9
52	726	113	.4900	034	8
53	750	155	.4891	017	7
54	775	197	.4882	.83001	6
55	.55799	.67239	1.4872	.82985	5
56	823	282	.4863	969	4
57	847	324	.4854	953	3
58	871	366	.4844	936	2
59	895	409	.4835	920	1
60	55919	67451	1 4826	.82904	0
	Cos	Ctn	Tan	Sin	′

56°

′	Sin	Tan	Ctn	Cos	
0	55919	67451	1 4826	82904	**60**
1	943	493	.4816	887	59
2	968	536	.4807	871	58
3	55992	578	.4798	855	57
4	56016	620	4788	839	56
5	56040	67663	1 4779	82822	**55**
6	064	705	4770	806	54
7	088	748	.4761	790	53
8	112	790	4751	773	52
9	136	832	4742	757	51
10	.56160	67875	1 4733	82741	**50**
11	184	917	.4724	724	49
12	208	67960	4715	708	48
13	232	68002	.4705	692	47
14	256	045	4696	675	46
15	56280	68088	1 4687	82659	**45**
16	305	130	4678	643	44
17	329	173	4669	626	43
18	353	215	4659	610	42
19	377	258	4650	593	41
20	56401	68301	1 4641	82577	**40**
21	425	343	4632	561	39
22	449	386	4623	544	38
23	473	429	4614	528	37
24	497	471	4605	511	36
25	56521	68514	1 4596	82495	**35**
26	545	557	4586	478	34
27	569	600	4577	462	33
28	593	642	4568	446	32
29	617	685	4559	429	31
30	56641	68728	1 4550	82413	**30**
31	665	771	4541	396	29
32	689	814	4532	380	28
33	713	857	4523	363	27
34	736	900	4514	347	26
35	56760	68942	1 4505	82330	**25**
36	784	.68985	4496	314	24
37	808	69028	4487	297	23
38	832	071	4478	281	22
39	856	114	4469	264	21
40	56880	69157	1 4460	82248	**20**
41	904	200	4451	231	19
42	928	243	4442	214	18
43	952	286	4433	198	17
44	56976	329	4424	181	16
45	57000	69372	1 4415	82165	**15**
46	024	416	.4406	148	14
47	047	459	4397	132	13
48	071	502	.4388	115	12
49	095	545	4379	098	11
50	.57119	.69588	1 4370	82082	**10**
51	143	631	4361	065	9
52	167	675	.4352	048	8
53	191	718	4344	032	7
54	215	761	.4335	82015	6
55	57238	.69804	1 4326	81999	**5**
56	262	847	4317	982	4
57	286	891	4308	965	3
58	310	934	4299	949	2
59	334	69977	4290	932	1
60	57358	.70021	1 4281	81915	**0**
	Cos	**Ctn**	**Tan**	**Sin**	**′**

55°

′	Sin	Tan	Ctn	Cos	
0	57358	70021	1 4281	81915	**60**
1	381	064	4273	899	59
2	405	107	4264	882	58
3	429	151	4255	865	57
4	453	194	4246	848	56
5	57477	70238	1 4237	81832	**55**
6	501	281	4229	815	54
7	524	325	4220	798	53
8	548	368	4211	782	52
9	572	412	4202	765	51
10	57596	70455	1 4193	81748	**50**
11	619	499	4185	731	49
12	643	542	4176	714	48
13	667	586	4167	698	47
14	691	629	4158	681	46
15	57715	70673	1 4150	81664	**45**
16	738	717	4141	647	44
17	762	760	4132	631	43
18	786	804	4124	614	42
19	810	848	4115	597	41
20	57833	70891	1 4106	81580	**40**
21	857	935	4097	563	39
22	881	70979	4089	546	38
23	904	71023	4080	530	37
24	928	066	4071	513	36
25	57952	71110	1 4063	81496	**35**
26	976	154	.4054	479	34
27	57999	198	.4045	462	33
28	58023	242	4037	445	32
29	047	285	4028	428	31
30	.58070	.71329	1 4019	81412	**30**
31	094	373	4011	395	29
32	118	417	4002	378	28
33	141	461	3994	361	27
34	165	505	3985	344	26
35	58189	.71549	1 3976	81327	**25**
36	212	593	3968	310	24
37	236	637	3959	293	23
38	260	681	3951	276	22
39	283	725	3942	259	21
40	58307	.71769	1 3934	81242	**20**
41	330	813	3925	225	19
42	354	857	3916	208	18
43	378	901	.3908	191	17
44	401	946	3899	174	16
45	.58425	.71990	1 3891	81157	**15**
46	449	.72034	3882	140	14
47	472	078	3874	123	13
48	496	122	.3865	106	12
49	519	167	3857	089	11
50	.58543	.72211	1 3848	.81072	**10**
51	567	255	3840	055	9
52	590	299	.3831	038	8
53	614	344	.3823	021	7
54	637	388	3814	81004	6
55	58661	.72432	1 3806	.80987	**5**
56	684	477	.3798	970	4
57	708	521	.3789	953	3
58	731	565	.3781	936	2
59	755	610	.3772	919	1
60	58779	72654	1 3764	.80902	**0**
	Cos	**Ctn**	**Tan**	**Sin**	**′**

54°

′	Sin	Tan	Ctn	Cos	
0	58779	.72654	1.3764	.80902	60
1	802	699	3755	885	59
2	826	743	3747	867	58
3	849	788	3739	850	57
4	873	832	3730	833	56
5	58896	72877	1.3722	80816	55
6	920	921	3713	799	54
7	943	72966	3705	782	53
8	967	.73010	3697	765	52
9	58990	055	3688	748	51
10	59014	.73100	1.3680	80730	50
11	037	144	3672	713	49
12	061	189	3663	696	48
13	084	234	3655	679	47
14	108	278	3647	662	46
15	59131	.73323	1.3638	80644	45
16	154	368	3630	627	44
17	178	413	3622	610	43
18	201	457	3613	593	42
19	225	502	3605	576	41
20	59248	73547	1.3597	80558	40
21	272	592	3588	541	39
22	295	637	3580	524	38
23	318	681	3572	507	37
24	342	726	3564	489	36
25	59365	73771	1.3555	.80472	35
26	389	816	3547	455	34
27	412	861	3539	438	33
28	436	906	3531	420	32
29	459	951	3522	403	31
30	59482	73996	1.3514	80386	30
31	506	74041	3506	368	29
32	529	086	3498	351	28
33	552	131	3490	334	27
34	576	176	3481	316	26
35	59599	74221	1.3473	80299	25
36	622	267	3465	282	24
37	646	312	3457	264	23
38	669	357	3449	247	22
39	693	402	3440	230	21
40	59716	74447	1.3432	80212	20
41	739	492	3424	195	19
42	763	538	3416	178	18
43	786	583	3408	160	17
44	809	628	3400	143	16
45	59832	74674	1.3392	80125	15
46	856	719	3384	108	14
47	879	764	3375	091	13
48	902	810	3367	073	12
49	926	855	3359	056	11
50	59949	74900	1.3351	80038	10
51	972	946	3343	021	9
52	59995	74991	3335	80003	8
53	60019	75037	3327	79986	7
54	042	082	3319	968	6
55	60065	75128	1.3311	.79951	5
56	089	173	3303	934	4
57	112	219	3295	916	3
58	135	264	3287	899	2
59	158	310	3278	881	1
60	60182	75355	1.3270	79864	0
	Cos	Ctn	Tan	Sin	′

53°

′	Sin	Tan	Ctn	Cos	
0	60182	75355	1.3270	.79864	60
1	205	401	.3262	846	59
2	228	447	3254	829	58
3	251	492	3246	811	57
4	274	538	3238	793	56
5	60298	75584	1.3230	.79776	55
6	321	629	3222	758	54
7	344	675	3214	741	53
8	367	721	3206	723	52
9	390	767	3198	706	51
10	.60414	75812	1.3190	.79688	50
11	437	858	3182	671	49
12	460	904	3175	653	48
13	483	950	.3167	635	47
14	506	75996	3159	618	46
15	60529	.76042	1.3151	.79600	45
16	553	088	3143	583	44
17	576	134	3135	565	43
18	599	180	3127	547	42
19	622	226	3119	530	41
20	.60645	.76272	1.3111	.79512	40
21	668	318	3103	494	39
22	691	364	3095	477	38
23	714	410	.3087	459	37
24	738	456	3079	441	36
25	60761	.76502	1.3072	79424	35
26	784	548	3064	406	34
27	807	594	3056	388	33
28	830	640	3048	371	32
29	853	686	3040	353	31
30	60876	.76733	1.3032	79335	30
31	899	779	3024	318	29
32	922	825	3017	300	28
33	945	871	3009	282	27
34	968	918	3001	264	26
35	60991	76964	1.2993	79247	25
36	61015	77010	2985	229	24
37	038	057	2977	211	23
38	061	103	2970	193	22
39	084	149	2962	176	21
40	61107	77196	1.2954	79158	20
41	130	242	2946	140	19
42	153	289	2938	122	18
43	176	335	2931	105	17
44	199	382	2923	087	16
45	61222	77428	1.2915	.79069	15
46	245	475	2907	051	14
47	268	521	2900	033	13
48	291	568	2892	79016	12
49	314	615	2884	78998	11
50	61337	77661	1.2876	.78980	10
51	360	708	2869	962	9
52	383	754	2861	944	8
53	406	801	2853	926	7
54	429	848	2846	908	6
55	61451	77895	1.2838	78891	5
56	474	941	2830	873	4
57	497	77988	2822	855	3
58	520	78035	2815	837	2
59	543	082	2807	819	1
60	61566	78129	1.2799	78801	0
	Cos	Ctn	Tan	Sin	′

52°

′	Sin	Tan	Ctn	Cos	
0	.61566	.78129	1.2799	..78801	**60**
1	589	175	.2792	783	59
2	612	222	.2784	765	58
3	635	269	.2776	747	57
4	658	316	.2769	729	56
5	.61681	.78363	1.2761	.78711	**55**
6	704	410	.2753	694	54
7	726	457	.2746	676	53
8	749	504	.2738	658	52
9	772	551	.2731	640	51
10	.61795	.78598	1.2723	.78622	**50**
11	818	645	.2715	604	49
12	841	692	.2708	586	48
13	864	739	.2700	568	47
14	887	786	.2693	550	46
15	.61909	.78834	1.2685	.78532	**45**
16	932	881	.2677	514	44
17	955	928	.2670	496	43
18	.61978	.78975	.2662	478	42
19	.62001	.79022	.2655	460	41
20	.62024	.79070	1.2647	.78442	**40**
21	046	117	.2640	424	39
22	069	164	.2632	405	38
23	092	212	.2624	387	37
24	115	259	.2617	369	36
25	.62138	.79306	1.2609	.78351	**35**
26	160	354	.2602	333	34
27	183	401	.2594	315	33
28	206	449	.2587	297	32
29	229	496	.2579	279	31
30	.62251	.79544	1.2572	.78261	**30**
31	274	591	.2564	243	29
32	297	639	.2557	225	28
33	320	686	.2549	206	27
34	342	734	.2542	188	26
35	.62365	.79781	1.2534	.78170	**25**
36	388	829	.2527	152	24
37	411	877	.2519	134	23
38	433	924	.2512	116	22
39	456	.79972	.2504	098	21
40	.62479	.80020	1.2497	.78079	**20**
41	502	067	.2489	061	19
42	524	115	.2482	043	18
43	547	163	.2475	025	17
44	570	211	.2467	.78007	16
45	.62592	.80258	1.2460	.77988	**15**
46	615	306	.2452	970	14
47	638	354	.2445	952	13
48	660	402	.2437	934	12
49	683	450	.2430	916	11
50	.62706	.80498	1.2423	.77897	**10**
51	728	546	.2415	879	9
52	751	594	.2408	861	8
53	774	642	.2401	843	7
54	796	690	.2393	824	6
55	.62819	.80738	1.2386	.77806	**5**
56	842	786	.2378	788	4
57	864	834	.2371	769	3
58	887	882	.2364	751	2
59	909	930	.2356	733	1
60	.62932	.80978	1.2349	.77715	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

51°

′	Sin	Tan	Ctn	Cos	
0	.62932	.80978	1.2349	.77715	**60**
1	955	.81027	.2342	696	59
2	.62977	075	.2334	678	58
3	.63000	123	.2327	660	57
4	022	171	.2320	641	56
5	.63045	.81220	1.2312	.77623	**55**
6	068	268	.2305	605	54
7	090	316	.2298	586	53
8	113	364	.2290	568	52
9	135	413	.2283	550	51
10	.63158	.81461	1.2276	.77531	**50**
11	180	510	.2268	513	49
12	203	558	.2261	494	48
13	225	606	.2254	476	47
14	248	655	.2247	458	46
15	.63271	.81703	1.2239	.77439	**45**
16	293	752	.2232	421	44
17	316	800	.2225	402	43
18	338	849	.2218	384	42
19	361	898	.2210	366	41
20	.63383	.81946	1.2203	.77347	**40**
21	406	.81995	.2196	329	39
22	428	.82044	.2189	310	38
23	451	092	.2181	292	37
24	473	141	.2174	273	36
25	.63496	.82190	1.2167	.77255	**35**
26	518	238	.2160	236	34
27	540	287	.2153	218	33
28	563	336	.2145	199	32
29	585	385	.2138	181	31
30	.63608	.82434	1.2131	.77162	**30**
31	630	483	.2124	144	29
32	653	531	.2117	125	28
33	675	580	.2109	107	27
34	698	629	.2102	088	26
35	.63720	.82678	1.2095	.77070	**25**
36	742	727	.2088	051	24
37	765	776	.2081	033	23
38	787	825	.2074	.77014	22
39	810	874	.2066	.76996	21
40	.63832	.82923	1.2059	.76977	**20**
41	854	.82972	.2052	959	19
42	877	.83022	.2045	940	18
43	899	071	.2038	921	17
44	922	120	.2031	903	16
45	.63944	.83169	1.2024	.76884	**15**
46	966	218	.2017	866	14
47	.63989	268	.2009	847	13
48	.64011	317	.2002	828	12
49	033	366	.1995	810	11
50	.64056	.83415	1.1988	.76791	**10**
51	078	465	.1981	772	9
52	100	514	.1974	754	8
53	123	564	.1967	735	7
54	145	613	.1960	717	6
55	.64167	.83662	1.1953	.76698	**5**
56	190	712	.1946	679	4
57	212	761	.1939	661	3
58	234	811	.1932	642	2
59	256	860	.1925	623	1
60	.64279	.83910	1.1918	.76604	**0**
	Cos	**Ctn**	**Tan**	**Sin**	′

50°

′	Sin	Tan	Ctn	Cos	
0	.64279	.83910	1.1918	.76604	**60**
1	301	.83960	.1910	586	59
2	323	.84009	.1903	567	58
3	346	059	.1896	548	57
4	368	108	.1889	530	56
5	.64390	.84158	1.1882	.76511	**55**
6	412	208	.1875	492	54
7	435	258	.1868	473	53
8	457	307	.1861	455	52
9	479	357	.1854	436	51
10	.64501	.84407	1.1847	.76417	**50**
11	524	457	.1840	398	49
12	546	507	.1833	380	48
13	568	556	.1826	361	47
14	590	606	.1819	342	46
15	.64612	.84656	1.1812	.76323	**45**
16	635	706	.1806	304	44
17	657	756	.1799	286	43
18	679	806	.1792	267	42
19	701	856	.1785	248	41
20	.64723	.84906	1.1778	.76229	**40**
21	746	.84956	.1771	210	39
22	768	.85006	.1764	192	38
23	790	057	.1757	173	37
24	812	107	.1750	154	36
25	.64834	.85157	1.1743	.76135	**35**
26	856	207	.1736	116	34
27	878	257	.1729	097	33
28	901	308	.1722	078	32
29	923	358	.1715	059	31
30	.64945	.85408	1.1708	.76041	**30**
31	967	458	.1702	022	29
32	.64989	509	.1695	.76003	28
33	.65011	559	.1688	.75984	27
34	033	609	.1681	965	26
35	.65055	.85660	1.1674	.75946	**25**
36	077	710	.1667	927	24
37	100	761	.1660	908	23
38	122	811	.1653	889	22
39	144	862	.1647	870	21
40	.65166	.85912	1.1640	.75851	**20**
41	188	.85963	.1633	832	19
42	210	.86014	.1626	813	18
43	232	064	.1619	794	17
44	254	115	.1612	775	16
45	.65276	.86166	1.1606	.75756	**15**
46	298	216	.1599	738	14
47	320	267	.1592	719	13
48	342	318	.1585	700	12
49	364	368	.1578	680	11
50	.65386	.86419	1.1571	.75661	**10**
51	408	470	.1565	642	9
52	430	521	.1558	623	8
53	452	572	.1551	604	7
54	474	623	.1544	585	6
55	.65496	.86674	1.1538	.75566	**5**
56	518	725	.1531	547	4
57	540	776	.1524	528	3
58	562	827	.1517	509	2
59	584	878	.1510	490	1
60	.65606	.86929	1.1504	.75471	**0**
	Cos	Ctn	Tan	Sin	′

49°

′	Sin	Tan	Ctn	Cos	
0	.65606	.86929	1.1504	.75471	**60**
1	628	.86980	.1497	452	59
2	650	.87031	.1490	433	58
3	672	082	.1483	414	57
4	694	133	.1477	395	56
5	.65716	.87184	1.1470	.75375	**55**
6	738	236	.1463	356	54
7	759	287	.1456	337	53
8	781	338	.1450	318	52
9	803	389	.1443	299	51
10	.65825	.87441	1.1436	.75280	**50**
11	847	492	.1430	261	49
12	869	543	.1423	241	48
13	891	595	.1416	222	47
14	913	646	.1410	203	46
15	.65935	.87698	1.1403	.75184	**45**
16	956	749	.1396	165	44
17	.65978	801	.1389	146	43
18	.66000	852	.1383	126	42
19	022	904	.1376	107	41
20	.66044	.87955	1.1369	.75088	**40**
21	066	.88007	.1363	069	39
22	088	059	.1356	050	38
23	109	110	.1349	030	37
24	131	162	.1343	.75011	36
25	.66153	.88214	1.1336	.74992	**35**
26	175	265	.1329	973	34
27	197	317	.1323	953	33
28	218	369	.1316	934	32
29	240	421	.1310	915	31
30	.66262	.88473	1.1303	.74896	**30**
31	284	524	.1296	876	29
32	306	576	.1290	857	28
33	327	628	.1283	838	27
34	349	680	.1276	818	26
35	.66371	.88732	1.1270	.74799	**25**
36	393	784	.1263	780	24
37	414	836	.1257	760	23
38	436	888	.1250	741	22
39	458	940	.1243	722	21
40	.66480	.88992	1.1237	.74703	**20**
41	501	.89045	.1230	683	19
42	523	097	.1224	664	18
43	545	149	.1217	644	17
44	566	201	.1211	625	16
45	.66588	.89253	1.1204	.74606	**15**
46	610	306	.1197	586	14
47	632	358	.1191	567	13
48	653	410	.1184	548	12
49	675	463	.1178	528	11
50	.66697	.89515	1.1171	.74509	**10**
51	718	567	.1165	489	9
52	740	620	.1158	470	8
53	762	672	.1152	451	7
54	783	725	.1145	431	6
55	.66805	.89777	1.1139	.74412	**5**
56	827	830	.1132	392	4
57	848	883	.1126	373	3
58	870	935	.1119	353	2
59	891	.89988	.1113	334	1
60	.66913	.90040	1.1106	.74314	**0**
	Cos	Ctn	Tan	Sin	′

48°

′	Sin	Tan	Ctn	Cos	
0	.66913	.90040	1.1106	.74314	60
1	935	093	.1100	295	59
2	956	146	.1093	276	58
3	978	199	.1087	256	57
4	.66999	251	.1080	237	56
5	.67021	.90304	1.1074	.74217	55
6	043	357	.1067	198	54
7	064	410	.1061	178	53
8	086	463	.1054	159	52
9	107	516	.1048	139	51
10	.67129	.90569	1.1041	.74120	50
11	151	621	.1035	100	49
12	172	674	.1028	080	48
13	194	727	.1022	061	47
14	215	781	.1016	041	46
15	.67237	.90834	1.1009	.74022	45
16	258	887	.1003	.74002	44
17	280	940	.0996	.73983	43
18	301	.90993	.0990	963	42
19	323	.91046	.0983	944	41
20	.67344	.91099	1.0977	.73924	40
21	366	153	.0971	904	39
22	387	206	.0964	885	38
23	409	259	.0958	865	37
24	430	313	.0951	846	36
25	.67452	.91366	1.0945	.73826	35
26	473	419	.0939	806	34
27	495	473	.0932	787	33
28	516	526	.0926	767	32
29	538	580	.0919	747	31
30	.67559	.91633	1.0913	.73728	30
31	580	687	.0907	708	29
32	602	740	.0900	688	28
33	623	794	.0894	669	27
34	645	847	.0888	649	26
35	.67666	.91901	1.0881	.73629	25
36	688	.91955	.0875	610	24
37	709	.92008	.0869	590	23
38	730	062	.0862	570	22
39	752	116	.0856	551	21
40	.67773	.92170	1.0850	.73531	20
41	795	224	.0843	511	19
42	816	277	.0837	491	18
43	837	331	.0831	472	17
44	859	385	.0824	452	16
45	.67880	.92439	1.0818	.73432	15
46	901	493	.0812	413	14
47	923	547	.0805	393	13
48	944	601	.0799	373	12
49	965	655	.0793	353	11
50	.67987	.92709	1.0786	.73333	10
51	.68008	763	.0780	314	9
52	029	817	.0774	294	8
53	051	872	.0768	274	7
54	072	926	.0761	254	6
55	.68093	.92980	1.0755	.73234	5
56	115	.93034	.0749	215	4
57	136	088	.0742	195	3
58	157	143	.0736	175	2
59	179	197	.0730	155	1
60	.68200	.93252	1.0724	.73135	0
	Cos	Ctn	Tan	Sin	′

47°

′	Sin	Tan	Ctn	Cos	
0	.68200	.93252	1.0724	.73135	60
1	221	306	.0717	116	59
2	242	360	.0711	096	58
3	264	415	.0705	076	57
4	285	469	.0699	056	56
5	.68306	.93524	1.0692	.73036	55
6	327	578	.0686	.73016	54
7	349	633	.0680	.72996	53
8	370	688	.0674	976	52
9	391	742	.0668	957	51
10	.68412	.93797	1.0661	.72937	50
11	434	852	.0655	917	49
12	455	906	.0649	897	48
13	476	.93961	.0643	877	47
14	497	.94016	.0637	857	46
15	.68518	.94071	1.0630	.72837	45
16	539	125	.0624	817	44
17	561	180	.0618	797	43
18	582	235	.0612	777	42
19	603	290	.0606	757	41
20	.68624	.94345	1.0599	.72737	40
21	645	400	.0593	717	39
22	666	455	.0587	697	38
23	688	510	.0581	677	37
24	709	565	.0575	657	36
25	.68730	.94620	1.0569	.72637	35
26	751	676	.0562	617	34
27	772	731	.0556	597	33
28	793	786	.0550	577	32
29	814	841	.0544	557	31
30	.68835	.94896	1.0538	.72537	30
31	857	.94952	.0532	517	29
32	878	.95007	.0526	497	28
33	899	062	.0519	477	27
34	920	118	.0513	457	26
35	.68941	.95173	1.0507	.72437	25
36	962	229	.0501	417	24
37	.68983	284	.0495	397	23
38	.69004	340	.0489	377	22
39	025	395	.0483	357	21
40	.69046	.95451	1.0477	.72337	20
41	067	506	.0470	317	19
42	088	562	.0464	297	18
43	109	618	.0458	277	17
44	130	673	.0452	257	16
45	.69151	.95729	1.0446	.72236	15
46	172	785	.0440	216	14
47	193	841	.0434	196	13
48	214	897	.0428	176	12
49	235	.95952	.0422	156	11
50	.69256	.96008	1.0416	.72136	10
51	277	064	.0410	116	9
52	298	120	.0404	095	8
53	319	176	.0398	075	7
54	340	232	.0392	055	6
55	.69361	.96288	1.0385	.72035	5
56	382	344	.0379	.72015	4
57	403	400	.0373	.71995	3
58	424	457	.0367	974	2
59	445	513	.0361	954	1
60	.69466	.96569	1.0355	.71934	0
	Cos	Ctn	Tan	Sin	′

46°

′	Sin	Tan	Ctn	Cos	
0	.69466	.96569	1.0355	.71934	60
1	487	625	.0349	914	59
2	508	681	.0343	894	58
3	529	738	.0337	873	57
4	549	794	.0331	853	56
5	.69570	.96850	1.0325	.71833	55
6	591	907	.0319	813	54
7	612	.96963	.0313	792	53
8	633	.97020	.0307	772	52
9	654	076	.0301	752	51
10	.69675	.97133	1.0295	.71732	50
11	696	189	.0289	711	49
12	717	246	.0283	691	48
13	737	302	.0277	671	47
14	758	359	.0271	650	46
15	.69779	.97416	1.0265	.71630	45
16	800	472	.0259	610	44
17	821	529	.0253	590	43
18	842	586	.0247	569	42
19	862	643	.0241	549	41
20	.69883	.97700	1.0235	.71529	40
21	904	756	.0230	508	39
22	925	813	.0224	488	38
23	946	870	.0218	468	37
24	966	927	.0212	447	36
25	.69987	.97984	1.0206	.71427	35
26	.70008	.98041	.0200	407	34
27	029	098	.0194	386	33
28	049	155	.0188	366	32
29	070	213	.0182	345	31
30	.70091	.98270	1.0176	.71325	30
31	112	327	.0170	305	29
32	132	384	.0164	284	28
33	153	441	.0158	264	27
34	174	499	.0152	243	26
35	.70195	.98556	1.0147	.71223	25
36	215	613	.0141	203	24
37	236	671	.0135	182	23
38	257	728	.0129	162	22
39	277	786	.0123	141	21
40	.70298	.98843	1.0117	.71121	20
41	319	901	.0111	100	19
42	339	.98958	.0105	080	18
43	360	.99016	.0099	059	17
44	381	073	.0094	039	16
45	.70401	.99131	1.0088	.71019	15
46	422	189	.0082	.70998	14
47	443	247	.0076	978	13
48	463	304	.0070	957	12
49	484	362	.0064	937	11
50	.70505	.99420	1.0058	.70916	10
51	525	478	.0052	896	9
52	546	536	.0047	875	8
53	567	594	.0041	855	7
54	587	652	.0035	834	6
55	.70608	.99710	1.0029	.70813	5
56	628	768	.0023	793	4
57	649	826	.0017	772	3
58	670	884	.0012	752	2
59	690	.99942	.0006	731	1
60	.70711	1.0000	1.0000	.70711	0
	Cos	Ctn	Tan	Sin	′

45°

Degrees						Minutes		Seconds	
0°	0.00000 00	**60°**	1.04719 76	**120°**	2.09439 51	**0**	0.00000 00	**0**	0.00000 00
1	0.01745 33	61	1.06465 08	121	2.11184 84	1	0.00029 09	1	0.00000 48
2	0.03490 66	62	1.08210 41	122	2.12930 17	2	0.00058 18	2	0.00000 97
3	0.05235 99	63	1.09955 74	123	2.14675 50	3	0.00087 27	3	0.00001 45
4	0.06981 32	64	1.11701 07	124	2.16420 83	4	0.00116 36	4	0.00001 94
5	0.08726 65	**65**	1.13446 40	**125**	2.18166 16	**5**	0.00145 44	**5**	0.00002 42
6	0.10471 98	66	1.15191 73	126	2.19911 49	6	0.00174 53	6	0.00002 91
7	0.12217 30	67	1.16937 06	127	2.21656 82	7	0.00203 62	7	0.00003 39
8	0.13962 63	68	1.18682 39	128	2.23402 14	8	0.00232 71	8	0.00003 88
9	0.15707 96	69	1.20427 72	129	2.25147 47	9	0.00261 80	9	0.00004 36
10	0.17453 29	**70**	1.22173 05	**130**	2.26892 80	**10**	0.00290 89	**10**	0.00004 85
11	0.19198 62	71	1.23918 38	131	2.28638 13	11	0.00319 98	11	0.00005 33
12	0.20943 95	72	1.25663 71	132	2.30383 46	12	0.00349 07	12	0.00005 82
13	0.22689 28	73	1.27409 04	133	2.32128 79	13	0.00378 15	13	0.00006 30
14	0.24434 61	74	1.29154 36	134	2.33874 12	14	0.00407 24	14	0.00006 79
15	0.26179 94	**75**	1.30899 69	**135**	2.35619 45	**15**	0.00436 33	**15**	0.00007 27
16	0.27925 27	76	1.32645 02	136	2.37364 78	16	0.00465 42	16	0.00007 76
17	0.29670 60	77	1.34390 35	137	2.39110 11	17	0.00494 51	17	0.00008 24
18	0.31415 93	78	1.36135 68	138	2.40855 44	18	0.00523 60	18	0.00008 73
19	0.33161 26	79	1.37881 01	139	2.42600 77	19	0.00552 69	19	0.00009 21
20	0.34906 59	**80**	1.39626 34	**140**	2.44346 10	**20**	0.00581 78	**20**	0.00009 70
21	0.36651 91	81	1.41371 67	141	2.46091 42	21	0.00610 87	21	0.00010 18
22	0.38397 24	82	1.43117 00	142	2.47836 75	22	0.00639 95	22	0.00010 67
23	0.40142 57	83	1.44862 33	143	2.49582 08	23	0.00669 04	23	0.00011 15
24	0.41887 90	84	1.46607 66	144	2.51327 41	24	0.00698 13	24	0.00011 64
25	0.43633 23	**85**	1.48352 99	**145**	2.53072 74	**25**	0.00727 22	**25**	0.00012 12
26	0.45378 56	86	1.50098 32	146	2.54818 07	26	0.00756 31	26	0.00012 61
27	0.47123 89	87	1.51843 64	147	2.56563 40	27	0.00785 40	27	0.00013 09
28	0.48869 22	88	1.53588 97	148	2.58308 73	28	0.00814 49	28	0.00013 57
29	0.50614 55	89	1.55334 30	149	2.60054 06	29	0.00843 58	29	0.00014 06
30	0.52359 88	**90**	1.57079 63	**150**	2.61799 39	**30**	0.00872 66	**30**	0.00014 54
31	0.54105 21	91	1.58824 96	151	2.63544 72	31	0.00901 75	31	0.00015 03
32	0.55850 54	92	1.60570 29	152	2.65290 05	32	0.00930 84	32	0.00015 51
33	0.57595 87	93	1.62315 62	153	2.67035 38	33	0.00959 93	33	0.00016 00
34	0.59341 19	94	1.64060 95	154	2.68780 70	34	0.00989 02	34	0.00016 48
35	0.61086 52	**95**	1.65806 28	**155**	2.70526 03	**35**	0.01018 11	**35**	0.00016 97
36	0.62831 85	96	1.67551 61	156	2.72271 36	36	0.01047 20	36	0.00017 45
37	0.64577 18	97	1.69296 94	157	2.74016 69	37	0.01076 29	37	0.00017 94
38	0.66322 51	98	1.71042 27	158	2.75762 02	38	0.01105 38	38	0.00018 42
39	0.68067 84	99	1.72787 60	159	2.77507 35	39	0.01134 46	39	0.00018 91
40	0.69813 17	**100**	1.74532 93	**160**	2.79252 68	**40**	0.01163 55	**40**	0.00019 39
41	0.71558 50	101	1.76278 25	161	2.80998 01	41	0.01192 64	41	0.00019 88
42	0.73303 83	102	1.78023 58	162	2.82743 34	42	0.01221 73	42	0.00020 36
43	0.75049 16	103	1.79768 91	163	2.84488 67	43	0.01250 82	43	0.00020 85
44	0.76794 49	104	1.81514 24	164	2.86234 00	44	0.01279 91	44	0.00021 33
45	0.78539 82	**105**	1.83259 57	**165**	2.87979 33	**45**	0.01309 00	**45**	0.00021 82
46	0.80285 15	106	1.85004 90	166	2.89724 66	46	0.01338 09	46	0.00022 30
47	0.82030 47	107	1.86750 23	167	2.91469 99	47	0.01367 17	47	0.00022 79
48	0.83775 80	108	1.88495 56	168	2.93215 31	48	0.01396 26	48	0.00023 27
49	0.85521 13	109	1.90240 89	169	2.94960 64	49	0.01425 35	49	0.00023 76
50	0.87266 46	**110**	1.91986 22	**170**	2.96705 97	**50**	0.01454 44	**50**	0.00024 24
51	0.89011 79	111	1.93731 55	171	2.98451 30	51	0.01483 53	51	0.00024 73
52	0.90757 12	112	1.95476 88	172	3.00196 63	52	0.01512 62	52	0.00025 21
53	0.92502 45	113	1.97222 21	173	3.01941 96	53	0.01541 71	53	0.00025 70
54	0.94247 78	114	1.98967 53	174	3.03687 29	54	0.01570 80	54	0.00026 18
55	0.95993 11	**115**	2.00712 86	**175**	3.05432 62	**55**	0.01599 89	**55**	0.00026 66
56	0.97738 44	116	2.02458 19	176	3.07177 95	56	0.01628 97	56	0.00027 15
57	0.99483 77	117	2.04203 52	177	3.08923 28	57	0.01658 06	57	0.00027 63
58	1.01229 10	118	2.05948 85	178	3.10668 61	58	0.01687 15	58	0.00028 12
59	1.02974 43	119	2.07694 18	179	3.12413 94	59	0.01716 24	59	0.00028 60
60	1.04719 76	**120**	2.09439 51	**180**	3.14159 27	**60**	0.01745 33	**60**	0.00029 09

x Radians	Sin x	Cos x	Tan x	Equivalent of x
.00	00000	1 0000	00000	0 00′ 0
.01	01000	99995	01000	[illegible]
.02	02000	99980	02000	[illegible]
.03	03000	99955	03001	1° 43 1
.04	03999	99920	04002	2° 17′ 5
.05	04998	99875	05001	[illegible]
.06	05996	99820	06007	[illegible]
.07	06994	99755	07011	[illegible]
.08	07991	99680	08017	[illegible]
.09	.08988	99595	[illegible]	[illegible]
10	09983	99500	10033	[illegible]
.11	10978	99396	11045	[illegible]
12	.11971	99281	[illegible]	[illegible]
.13	12963	99156	13074	[illegible]
.14	13954	99022	14092	8° 01′ 3
15	14944	98877	15114	8° 35′ 7
16	15932	98723	16138	9° 10′ 0
17	16918	98558	17166	9° 44′ 4
18	17903	98384	18197	10° 18′ 8
19	18886	98200	19232	10° 53′ 2
20	19867	98007	20271	11° 27′ 5
21	20846	9780	21314	12° 01′ 9
22	21823	97590	22362	12° 36′ 3
23	22798	97367	23414	13° 10′ 7
24	23770	97134	24472	13° 45′ 1
25	24740	96891	25534	14° 19′ 4
26	.25708	96639	26602	14° 53′ 8
27	26673	96377	27676	15° 28′ 2
.28	27636	96106	28755	16° 02 6
29	28595	95824	29841	16° 36′ 9
30	29552	95534	30934	[illegible]
.31	30506	95233	32033	[illegible]
.32	31457	94924	33139	18° 20′ 1
.33	.32404	94604	34252	[illegible]
34	.33349	.94275	35374	19° 28′ 8
35	34290	93937	36503	20° 03′ 2
36	35227	93590	37640	20° 37′ 6
.37	36162	93233	38786	21° 12′ 0
.38	37092	92866	39941	21° 46′ 3
.39	.38019	92191	41105	22° 20′ 7
40	38942	92106	42279	[illegible]
41	.39861	.91712	43463	23° 29′ 5
.42	.40776	.91309	44657	24° 03′ 9
.43	41687	90897	45862	24° 38′ 2
.44	42594	90475	47078	25° 12′ 6
.45	43497	90045	48306	25° 47′ 0
.46	44395	89605	49545	26° 21′ 4
.47	.45289	89157	50797	26° 55′ 7
.48	46178	.88699	.52061	27° 30′ 1
49	47063	.88233	53339	28° 04′ 5
.50	47943	87758	54630	28° 38′ 9

x Radians	Sin x	Cos x	Tan x	Equivalent of x
50	47943	87758	4630	28° 38′ 9
51	48818	87274	[illegible]	29° 13 3
52	49688	[illegible]	[illegible]	29° 47′ 6
[illegible]	[illegible]	86281	58592	30° 22′ 0
[illegible]	51414	[illegible]	59915	30° 56′ 4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
.57	[illegible]	[illegible]	64097	32° 39′ 5
.58	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
60	6164	[illegible]	68414	34° 22′ 6
[illegible]	[illegible]	[illegible]	[illegible]	34° 57 0
.62	58104	[illegible]	[illegible]	35° 31′ 1
.63	[illegible]	[illegible]	911	[illegible]
.64	[illegible]	80210	74151	36° 40′ 2
[illegible]	[illegible]	79608	[illegible]	7° 14′ 5
66	[illegible]	[illegible]	[illegible]	37° 48′ 9
.67	62099	[illegible]	[illegible]	[illegible]
.68	[illegible]	[illegible]	80866	[illegible]
69	[illegible]	[illegible]	[illegible]	[illegible]
70	644	76484	84229	40° 06′ 4
71	65183	[illegible]	[illegible]	10° 40 8
72	[illegible]	75181	87707	41° 15′ 2
73	66687	[illegible]	89492	11° 49′ 6
74	67429	73847	91309	42° 23′ 9
75	68164	[illegible]	93160	42° 58′ 3
76	68892	72484	[illegible]	13° 32′ 7
77	69614	71791	96967	44° 07′ 1
78	[illegible]	[illegible]	[illegible]	14° 41′ 4
79	[illegible]	[illegible]	1 0092	[illegible]
80	71736	[illegible]	1 0296	[illegible]
81	72429	[illegible]	1 0505	16° 24′ 6
82	[illegible]	68222	1 0717	16° 59′ 0
83	73793	.67488	1 0934	[illegible]
.84	74464	66746	1 1156	48° 07′ 7
[illegible]	[illegible]	[illegible]	1 1383	48° 42′ 1
86	75784	[illegible]	1 1616	49° 16′ 5
87	76433	.64483	1 1853	49° 50′ 8
88	77074	63715	1 2097	[illegible]
89	77707	62941	[illegible]	[illegible]
90	78333	62161	1 2602	51° 34′ 0
91	78950	[illegible]	1 2864	52° 08′ 3
92	79560	60582	1 3133	52° 42′ 7
93	80162	59783	1 3409	53° 17′ 1
94	80756	58979	1 3692	53° 51′ 5
95	81342	58168	1 3984	54° 25′ 9
96	81919	57352	1 4284	55° 00′ 2
97	82489	56530	1 4592	55° 34′ 6
98	83050	55702	1 4910	56° 09′ 0
99	83603	54869	[illegible]	56° 43′ 4
1 00	84147	54030	1 5574	57° 17′ 7

x Radians	Sin x	Cos x	Tan x	Equivalent of x
1.00	.84147	.54030	1.5574	57° 17′.7
1.01	.84683	.53186	1.5922	57° 52′.1
1.02	.85211	.52337	1.6281	58° 26′.5
1.03	.85730	.51482	1.6652	59° 00′.9
1.04	.86240	.50622	1.7036	59° 35′.3
1.05	.86742	.49757	1.7433	60° 09′.6
1.06	.87236	.48887	1.7844	60° 44′.0
1.07	.87720	.48012	1.8270	61° 18′.4
1.08	.88196	.47133	1.8712	61° 52′.8
1.09	.88663	.46249	1.9171	62° 27′.1
1.10	.89121	.45360	1.9648	63° 01′.5
1.11	.89570	.44466	2.0143	63° 35′.9
1.12	.90010	.43568	2.0660	64° 10′.3
1.13	.90441	.42666	2.1198	64° 44′.7
1.14	.90863	.41759	2.1759	65° 19′.0
1.15	.91276	.40849	2.2345	65° 53′.4
1.16	.91680	.39934	2.2958	66° 27′.8
1.17	.92075	.39015	2.3600	67° 02′.2
1.18	.92461	.38092	2.4273	67° 36′.5
1.19	.92837	.37166	2.4979	68° 10′.9
1.20	.93204	.36236	2.5722	68° 45′.3
1.21	.93562	.35302	2.6503	69° 19′.7
1.22	.93910	.34365	2.7328	69° 54′.1
1.23	.94249	.33424	2.8198	70° 28′.4
1.24	.94578	.32480	2.9119	71° 02′.8
1.25	.94898	.31532	3.0096	71° 37′.2
1.26	.95209	.30582	3.1133	72° 11′.6
1.27	.95510	.29628	3.2236	72° 45′.9
1.28	.95802	.28672	3.3413	73° 20′.3
1.29	.96084	.27712	3.4672	73° 54′.7
1.30	.96356	.26750	3.6021	74° 29′.1

x Radians	Sin x	Cos x	Tan x	Equivalent of x
1.30	.96356	.26750	3.6021	74° 29′.1
1.31	.96618	.25785	3.7471	75° 03′.4
1.32	.96872	.24818	3.9033	75° 37′.8
1.33	.97115	.23848	4.0723	76° 12′.2
1.34	.97348	.22875	4.2556	76° 46′.6
1.35	.97572	.21901	4.4552	77° 21′.0
1.36	.97786	.20924	4.6734	77° 55′.3
1.37	.97991	.19945	4.9131	78° 29′.7
1.38	.98185	.18964	5.1774	79° 04′.1
1.39	.98370	.17981	5.4707	79° 38′.5
1.40	.98545	.16997	5.7979	80° 12′.8
1.41	.98710	.16010	6.1654	80° 47′.2
1.42	.98865	.15023	6.5811	81° 21′.6
1.43	.99010	.14033	7.0555	81° 56′.0
1.44	.99146	.13042	7.6018	82° 30′.4
1.45	.99271	.12050	8.2381	83° 04′.7
1.46	.99387	.11057	8.9886	83° 39′.1
1.47	.99492	.10063	9.8874	84° 13′.5
1.48	.99588	.09067	10.983	84° 47′.9
1.49	.99674	.08071	12.350	85° 22′.2
1.50	.99749	.07074	14.101	85° 56′.6
1.51	.99815	.06076	16.428	86° 31′.0
1.52	.99871	.05077	19.670	87° 05′.4
1.53	.99917	.04079	24.498	87° 39′.8
1.54	.99953	.03079	32.461	88° 14′.1
1.55	.99978	.02079	48.078	88° 48′.5
1.56	.99994	.01080	92.621	89° 22′.9
*1.57	*1.0000	*.00080	*1255.8	89° 57′.3
1.58	.99996	−.00920	−108.65	90° 31′.6
1.59	.99982	−.01920	−52.067	91° 06′.0
1.60	.99957	−.02920	−34.233	91° 40′.4

π radians = 180° 1 radian = 57° 17′ 44″.806 = 57.°2957795

π = 3.14159265 3600″ = 60′ = 1° = 0.01745329 radian

*1 right angle = 90° = $\pi/2$ radians = 1.5707963 radians

Table IV a — Radians to Degrees

	Radians	Tenths	Hundredths	Thousandths	Ten-thousandths
1	57°17′44″.8	5°43′46″.5	0°34′22″.6	0° 3′26″.3	0° 0′20″.6
2	114°35′29″.6	11°27′33″.0	1° 8′45″.3	0° 6′52″.5	0° 0′41″.3
3	171°53′14″.4	17°11′19″.4	1°43′07″.9	0°10′18″.8	0° 1′01″.9
4	229°10′59″.2	22°55′05″.9	2°17′30″.6	0°13′45″.1	0° 1′22″.5
5	286°28′44″.0	28°38′52″.4	2°51′53″.2	0°17′11″.3	0° 1′43″.1
6	343°46′28″.8	34°22′38″.9	3°26′15″.9	0°20′37″.6	0° 2′03″.8
7	401° 4′13″.6	40° 6′25″.4	4° 0′38″.5	0°24′03″.9	0° 2′24″.4
8	458°21′58″.4	45°50′11″.8	4°35′01″.2	0°27′30″.1	0° 2′45″.0
9	515°39′43″.3	51°33′58″.3	5° 9′23″.8	0°30′56″.4	0° 3′05″.6

AMOUNT OF ONE DOLLAR PRINCIPAL AT COMPOUND INTEREST AFTER n YEARS

n	2 %	$2\frac{1}{2}$ %	3 %	$3\frac{1}{2}$ %	4 %	$4\frac{1}{2}$ %	5 %	6 %	7 %
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2.5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

AMOUNT OF AN ANNUITY OF ONE DOLLAR PER YEAR AFTER n YEARS

n	2 %	$2\frac{1}{2}$ %	3 %	$3\frac{1}{2}$ %	4 %	$4\frac{1}{2}$ %	5 %	6 %	7 %
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0200	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2.0600	2.0700
3	3.0604	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3.1836	3.2149
4	4.1216	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4.3746	4.4399
5	5.2040	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5.6371	5.7507
6	6.3081	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6.9753	7.1533
7	7.4343	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	8.3938	8.6540
8	8.5830	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	9.8975	10.2598
9	9.7546	9.9545	10.1591	10.3685	10.5828	10.8021	11.0266	11.4913	11.9780
10	10.9497	11.2034	11.4639	11.7314	12.0061	12.2882	12.5779	13.1808	13.8164
11	12.1687	12.4835	12.8078	13.1420	13.4864	13.8412	14.2068	14.9716	15.7836
12	13.4121	13.7956	14.1920	14.6020	15.0258	15.4640	15.9171	16.8699	17.8885
13	14.6803	15.1404	15.6178	16.1130	16.6268	17.1599	17.7130	18.8821	20.1406
14	15.9739	16.5190	17.0863	17.6770	18.2919	18.9321	19.5986	21.0151	22.5505
15	17.2934	17.9319	18.5989	19.2957	20.0236	20.7841	21.5786	23.2760	25.1290
16	18.6393	19.3802	20.1569	20.9710	21.8245	22.7193	23.6575	25.6725	27.8881
17	20.0121	20.8647	21.7616	22.7050	23.6975	24.7417	25.8404	28.2129	30.8402
18	21.4123	22.3863	23.4144	24.4997	25.6454	26.8551	28.1324	30.9057	33.9990
19	22.8406	23.9460	25.1169	26.3572	27.6712	29.0636	30.5390	33.7600	37.3790
20	24.2974	25.5447	26.8704	28.2797	29.7781	31.3714	33.0660	36.7856	40.9955
21	25.7833	27.1833	28.6765	30.2695	31.9692	33.7831	35.7193	39.9927	44.8652
22	27.2990	28.8629	30.5368	32.3289	34.2480	36.3034	38.5052	43.3923	49.0057
23	28.8450	30.5844	32.4529	34.4604	36.6179	38.9370	41.4305	46.9958	53.4361
24	30.4219	32.3490	34.4265	36.6665	39.0826	41.6892	44.5020	50.8156	58.1767
25	32.0303	34.1578	36.4593	38.9499	41.6459	44.5652	47.7271	54.8645	63.2490
26	33.6709	36.0117	38.5530	41.3131	44.3117	47.5706	51.1135	59.1564	68.6765
27	35.3443	37.9120	40.7096	43.7591	47.0842	50.7113	54.6691	63.7058	74.4838
28	37.0512	39.8598	42.9309	46.2906	49.9676	53.9933	58.4026	68.5281	80.6977
29	38.7922	41.8563	45.2189	48.9108	52.9663	57.4230	62.3227	73.6398	87.3465
30	40.5681	43.9027	47.5754	51.6227	56.0849	61.0071	66.4388	79.0582	94.4608
31	42.3794	46.0003	50.0027	54.4295	59.3283	64.7524	70.7608	84.8017	102.0730
32	44.2270	48.1503	52.5028	57.3345	62.7015	68.6662	75.2988	90.8898	110.2182
33	46.1116	50.3540	55.0778	60.3412	66.2095	72.7562	80.0638	97.3432	118.9334
34	48.0338	52.6129	57.7302	63.4532	69.8579	77.0303	85.0670	104.1838	128.2588
35	49.9945	54.9282	60.4621	66.6740	73.6522	81.4966	90.3203	111.4348	138.2369
36	51.9944	57.3014	63.2759	70.0076	77.5983	86.1640	95.8363	119.1209	148.9135
37	54.0343	59.7339	66.1742	73.4579	81.7022	91.0413	101.6281	127.2681	160.3374
38	56.1149	62.2273	69.1594	77.0289	85.9703	96.1382	107.7095	135.9042	172.5610
39	58.2372	64.7830	72.2342	80.7249	90.4091	101.4644	114.0950	145.0585	185.6403
40	60.4020	67.4026	75.4013	84.5503	95.0255	107.0303	120.7998	154.7620	199.6351
41	62.6100	70.0876	78.6633	88.5095	99.8265	112.8467	127.8398	165.0477	214.6096
42	64.8622	72.8398	82.0232	92.6074	104.8196	118.9248	135.2318	175.9505	230.6322
43	67.1595	75.6608	85.4839	96.8486	110.0124	125.2764	142.9933	187.5076	247.7765
44	69.5027	78.5523	89.0484	101.2383	115.4129	131.9138	151.1430	199.7580	266.1209
45	71.8927	81.5161	92.7199	105.7817	121.0294	138.8500	159.7002	212.7435	285.7493
46	74.3306	84.5540	96.5015	110.4840	126.8706	146.0982	168.6852	226.5081	306.7518
47	76.8172	87.6679	100.3965	115.3510	132.9454	153.6726	178.1194	241.0986	329.2244
48	79.3535	90.8596	104.4084	120.3883	139.2632	161.5879	188.0254	256.5645	353.2701
49	81.9406	94.1311	108.5406	125.6018	145.8337	169.8594	198.4267	272.9584	378.9990
50	84.5794	97.4843	112.7969	130.9979	152.6671	178.5030	209.3480	290.3359	406.5289

INDEX

A

B

C

I

J

L

M

V

W